A geometric approach to the Uncontrolled Manifold analysis

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Abstract—In this work we extend the classical Uncontrolled Manifold (UCM) analysis to the case of micro-manipulation via hand-held tools. A major contribution of this paper is that our algorithm is independent of the choice of coordinates. This is made possible by the explicit choice of a geometric structure which is suggested by the specific application, i.e. manipulation via hand-held tools. Based on the properties of rigid body motions, available from the robotics literature, we implemented the crucial steps of UCM analysis based on the intrinsic notions of geodesics. In particular, this allowed defining a mean posture and deviations from it which are coordinate independent.

We applied our geometric, coordinate-independent UCM analysis to experimental data acquired from healthy subjects performing static pointing tasks with a hand-held tool. All subjects display much larger variability on the nullspace of the Jacobian, i.e. the UCM. In other words, variability is tremor and subjects seem to project more tremor along motions which do not directly affect the task. We thus have a tool to analyze the control of accurate motion, e.g. where/how older and experienced surgeons perform better than young novices, despite larger tremor.

keywords: Micro-Manipulation, Lie Groups, Rigid Body Motions, Tremor, Surgery

I. INTRODUCTION

Although highly stereotyped, human movements performed with the same intention are never exactly the same and consecutive trials have large variability. Rather than just ‘biological noise’, many studies have pointed out how variability may in fact provide important clues on neural strategies. Analysis of structure in variability, and its changes, has therefore become an important tool for researcher in neuromotor control and learning, especially in presence of redundancy [7].

The problem of variability in redundant motor tasks was formulated by Bernstein [1] who studied the kinematics of skilled movements performed by professional blacksmiths while striking a chisel with a hammer. Bernstein observed how the variability of the trajectory of the hammer, at its tip, was in fact smaller than the variability of each of the joints of the arm holding the hammer. This suggests that the individual joints are not controlled independently from each other and the brain might exploit the kinematic redundancy to compensate for inaccuracies at each joint.

Redundancy and motion variability are important not only for blacksmiths but characterize virtually every daily activity, from grasping a cup to signing off a letter, where we typically have many more degrees-of-freedom (dof) than necessary to fulfill the task. We are particularly interested in tasks involving hand-held tools such as microsurgery, where noise induced by tremor is amplified by the visual magnification provided by the optical microscope [4].

We specifically considered static pointing tasks, such as the one in Fig. 1, where a subject is asked to keep the tip of a pen-like tool, e.g. a surgical scalpel, in contact with another object. In other words, the position of the tip, characterized by $m = 3$ dof of mobility, is prescribed while the subject is free to choose among different postures, which include positioning of the torso, joints angle of the arm (i.e. shoulder, elbow and wrist) as well as grasping pattern of the hand. For sake of simplicity, if we strap the torso and ‘freeze’ the hand (e.g. with a splint), the arm accounts for $n = 7$ dof mobility. For postures away from biomechanical limits, there exists a $n - m = 4$ dof ‘task-equivalent’ space, or manifold, consisting of distinct postures which do not affect the task.

Scholz and Schoner [11] hypothesized that movement variance, as measured across tasks repetitions, might project differently onto the task-equivalent space than it does onto its ‘orthogonal’ complement (which directly affects the task). A larger variance projected in the task-equivalent space (or nullspace) is indicative of neural control. Therefore the task-equivalent space, where larger variance is expected for skilled movements, was named uncontrolled manifold (UCM).

Despite its appeal, the computational procedures behind UCM (and principal component analysis in general) have been recently criticized for being coordinate-sensitive [12]. For example, one of the steps involved in the UCM approach (see next section) is a definition of an average (or reference) posture.

For an intrinsic definition of average posture, i.e. a definition which does not depend on coordinates1, an inner product is required. An inner product is also needed to define the orthogonal space (another crucial step of the UCM approach, 1Coordinates are often an arbitrary choice. For example, results might vary if we parameterize postures via angles between adjacent body segments or via angles between body segment and the vertical direction. More fundamentally, we do not know which coordinates are used by the human central nervous system.

Fig. 1. Static pointing task performed with a hand-held tool. The tool is grasped at a fix position, at distance $\lambda$ from the tool tip. The pose of the hand-held tool with respect to a fixed frame $S$, can be specified as the position and orientation of a moving frame $B$ attached to the tool at the gripping point.
is identified which maps an \( n \)-dimensional configuration space \( \{ \theta = [\theta_1, \ldots, \theta_n]^T \} \) to an hypothesized \( m \)-dimensional control variable \( \{ y = [y_1, \ldots, y_m]^T \} \);

ii. a mean configuration

\[ \theta_0 := \frac{1}{N} \sum_{i=1}^{N} \theta_i \]

is computed across measurements (\( N \) is the number of measurements);

iii. a linearization

\[ y \approx f(\theta_0) + \frac{\partial f}{\partial \theta} \bigg|_{\theta_0} (\theta - \theta_0) \]

of the forward kinematics is performed around the mean configuration and the relative \( m \times n \) Jacobian \( J := \frac{\partial f}{\partial \theta} \bigg|_{\theta_0} \) is evaluated.

iv. the nullspace (\( \mathcal{N} \))

\[ \mathcal{N} = \text{span}\{n_1, \ldots, n_{n-m}\} \]

of the Jacobian (i.e. \( J n_i = 0 \) for \( i = 1, \ldots, n - m \)) is used to define the UCM as well as its orthogonal complement (\( \mathcal{O} := \mathcal{N}^\perp \))

\[ \mathcal{O} = \text{span}\{o_1, \ldots, o_m\} \];

v. the projection of the deviations \( \Delta_i := \theta - \theta_0 \) onto the UCM and its orthogonal complement are evaluated, respectively, as

\[ \Delta_i^{\text{null}} = \sum_{j=1}^{n-m} (n_i^T \Delta_i) n_j, \quad \Delta_i^{\text{orth}} = \sum_{j=1}^{m} (o_j^T \Delta_i) o_j \]

vi. variance per dof of the motion in the nullspace and its orthogonal complement is evaluated, respectively, as

\[ \sigma^2_{\text{null}} = \frac{1}{(n - m)} \frac{1}{N} \sum_{i=1}^{N} \| \Delta_i^{\text{null}} \|^2, \]

\[ \sigma^2_{\text{orth}} = \frac{1}{m} \frac{1}{N} \sum_{i=1}^{N} \| \Delta_i^{\text{orth}} \|^2. \]

Finally, statistical analysis is performed to test the hypothesis that variance along UCM is significantly greater than variance along its orthogonal.

B. Forward kinematics for the static pointing task

With reference to Fig. 1, consider a space-fixed frame \( \mathcal{S} \) and a moving frame \( \mathcal{B} \) attached to the hand-held tool at the gripping point. Let \( P_{\text{grip}} \) represent the 3D coordinates of the gripping point in space coordinates. The orientation of \( \mathcal{B} \) with respect to \( \mathcal{S} \) is determined by the coordinate axes \( e_1, e_2, e_3 \), the latter being aligned with the tool major axis, pointing away from the tip. At all times, the orientation of \( \mathcal{B} \) relative to \( \mathcal{S} \) can be represented via a \( 3 \times 3 \) rotation matrix \( R \) whose first, second and third columns represent, respectively, the space-fixed coordinates of the axes \( e_1, e_2, e_3 \). For computational purposes, the pose \( g = \{ R, P_{\text{grip}} \} \) of the hand-tool can be
conveniently represented via a $4 \times 4$ homogeneous matrix notation (21) in Appendix.

We assume that the gripping point of the hand tool is at a constant distance $\lambda$ from the tip, therefore

$$P_{\text{tip}} = P_{\text{grip}} - \lambda e_3 = P_{\text{grip}} - \lambda \hat{R}$$  \hspace{1cm} (1)

Differentiating with respect to time leads to

$$V_{\text{tip}} = V_{\text{grip}} - \lambda \dot{\hat{R}}$$  \hspace{1cm} (2)

where $V_{\text{tip}} := \dot{P}_{\text{tip}}$ and $V_{\text{grip}} := \dot{P}_{\text{grip}}$. To express everything in body coordinates, we can left-multiply all terms by $\hat{R}^T$. Expressing the velocities in body coordinates $v_{\text{tip}}$ and $v_{\text{grip}}$, via the transformations $V_{\text{tip}} = \hat{R} v_{\text{tip}}$ and $V_{\text{grip}} = \hat{R} v_{\text{grip}}$, we get

$$v_{\text{tip}} = v_{\text{grip}} - \lambda \omega \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$  \hspace{1cm} (3)

where $\omega := R^T \dot{R}$ is the body angular velocity, see (19) in Appendix.

By introducing the generalized velocity $[v_{\text{grip}} \omega]^T$, see (22) in Appendix, and the Jacobian matrix

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (4)

we can rewrite (3) simply as

$$v_{\text{tip}} = J \begin{bmatrix} v_{\text{grip}} \\ \omega \end{bmatrix}$$  \hspace{1cm} (5)

An advantage of expressing the problem from the moving frame perspective is that, despite the complexity of the problem, the relation (5) is linear, without the need of any linearization, and time-invariant, in contrast to eq. (1)–(2). This a direct consequence of the so-called left-invariance of the problem, i.e. it is independent of the space frame $S$.

In the kinematic relation (5), the Jacobian projects the 6 dof generalized velocity of the hand-held tool into the 3 dof velocity of the task. This is the starting point to analyze redundancy as the Jacobian (4) has clearly rank 3 and therefore its kernel is three-dimensional.

C. An inner product for hand-held tools motions

In general, an inner product can be defined on the (linear) space of a rigid body generalized velocities $[v \omega]^T$ as a symmetric, positive definite, bilinear form $\langle \cdot , \cdot \rangle$ which acts on any pair of possible velocities $[v_1 \omega_1]^T$ and $[v_2 \omega_2]^T$ to return a scalar and can be expressed as

$$\langle \begin{bmatrix} v_1 \\ \omega_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ \omega_2 \end{bmatrix} \rangle := \begin{bmatrix} v_1 & \omega_1 \end{bmatrix} B \begin{bmatrix} v_2 \\ \omega_2 \end{bmatrix}$$  \hspace{1cm} (6)

where $B$ is a symmetric, positive definite $6 \times 6$ matrix. Park [9] showed that the only left-invariant definition is in the form:

$$B = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix}$$  \hspace{1cm} (7)

where $I$ is the $3 \times 3$ identity matrix and $\alpha$ and $\beta$ are scalars. As highlighted by Park [9] there is no natural choice for these scalar coefficients and a selection will depend on the application.

In our case, any scaling factor common to both $\alpha$ and $\beta$ would not change the orthogonality conditions nor the ratio of variances, so we can fix $\alpha = 1$ and just define a value for $\beta$. From (5) it is evident that, for example, an infinitesimal rotation $\delta \theta$ about $e_2$ would cause a tip displacement $\lambda \delta \theta$ along $e_1$. This suggests setting $\beta = \lambda^2$, i.e.

$$B = \begin{bmatrix} I & 0 \\ 0 & \lambda^2 I \end{bmatrix}$$  \hspace{1cm} (8)

and the inner product reduces to

$$\langle \begin{bmatrix} v_1 \\ \omega_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ \omega_2 \end{bmatrix} \rangle = v_1^T v_2 + \lambda^2 \omega_1^T \omega_2$$  \hspace{1cm} (9)

D. Null and Orthogonal Spaces

Thanks to the left-invariance of the problem, we can define the nullspace of the Jacobian and its orthogonal complement even before knowing what the reference (mean) position will be.

The nullspace $\mathcal{N}$ is the sub-space of generalized velocities $[v \omega]^T$ of the tool which produce no motion at the end-tip, i.e. $0 = J [v \omega]^T$. It is straightforward verifying that the following generalized velocities

$$\{ n_1 := \frac{1}{\sqrt{3}} \begin{bmatrix} e_1 \\ \lambda^{-1} e_2 \end{bmatrix}^T; \quad n_2 := \frac{1}{\sqrt{2}} \begin{bmatrix} -e_2 \\ \lambda^{-1} e_1 \end{bmatrix}^T; \quad n_3 := \begin{bmatrix} 0 \\ \lambda^{-1} e_3 \end{bmatrix}^T \}$$  \hspace{1cm} (10)

have the following properties:

i) produce zero end-tip velocity, i.e. $J n_i = 0$ for $i = 1, 2, 3$;

ii) are mutually orthogonal, i.e. $\langle n_i, n_j \rangle = 0$ whenever $i \neq j$;

iii) have unitary length, i.e. $\langle n_i, n_i \rangle = 1$ for $i = 1, 2, 3$.

Since the nullspace is of rank 3, then

$$\mathcal{N} = \text{span} \{ n_1; n_2; n_3 \}.$$  

We now wish to find the orthogonal space $\mathcal{O}$ and in particular an orthonormal base $\{ o_1; o_2; o_3 \}$ such that

$$\mathcal{O} := \mathcal{N}^\perp = \text{span} \{ o_1; o_2; o_3 \}$$

This can be done via the inner product (9) and the classical the Gram-Schmidt orthogonalization procedure. Nevertheless it is straightforward verifying that the following vectors do form
an orthonormal basis and that each of them is orthogonal to the nullspace:
\[
\begin{align*}
\mathbf{o}_1 &:= \frac{1}{\sqrt{2}} \begin{bmatrix} e_1 & -\lambda^{-1} e_2 \end{bmatrix}^T; \\
\mathbf{o}_2 &:= \frac{1}{\sqrt{2}} \begin{bmatrix} e_2 & \lambda^{-1} e_1 \end{bmatrix}^T; \\
\mathbf{o}_3 &:= \begin{bmatrix} e_3 & 0 \end{bmatrix}^T.
\end{align*}
\] (11)

E. A reference position for hand-held tools

Given a set of \( N \) rigid body poses \( \mathbf{g}_i = \{ R_i, \mathbf{P}_i \} \), where \( i = 1, 2, \ldots, N \), an intrinsic definition of mean pose, see [2], [6] and reference therein, is
\[
g_0 := \arg\min_g \sum_{i=1}^{N} d(g, \mathbf{g}_i)
\]
where \( d(g, \mathbf{g}_i) \) is the distance between the poses \( g \) and \( \mathbf{g}_i \) as defined in (24). It can be shown that the intrinsic mean pose can be expressed as \( g_0 = \{ R_0, \mathbf{P}_0 \} \), where
\[
\begin{align*}
\mathbf{P}_0 &:= N^{-1} \sum_{i=1}^{N} \mathbf{P}_i, \\
R_0 &:= \arg\min \sum_{i=1}^{N} \| \log_\ast(R_i^T R) \|^2
\end{align*}
\] (12)

For further details, the reader is referred to [2], [6].

F. Analysis of Variance via Principal Geodesic Analysis

In the classical UCM approach [11] [14], where postures are typically parameterized via a vector of \( n \) joint angles \( \theta = [\theta_1, \ldots, \theta_n]^T \), deviations from the reference position \( \theta_0 \) are directly computed as differences between vectors \( (\theta - \theta_0) \) and projected onto the nullspace of the Jacobian and onto its orthogonal complement.

For nonlinear spaces as for rigid body motions, this is not possible and we will extend the classical UCM approach with the concept of geodesics, as proposed by Fletcher et al. [2]. Starting from a given point (e.g. a reference position), geodesic curves are completely specified once the initial velocity is given and allow connecting sufficiently close points via minimal-paths. Therefore, geodesics are a natural way to define the deviation of a point \( \mathbf{B} \) from a point \( \mathbf{A} \) as the initial velocity for a geodesic curve to start in \( \mathbf{A} \) and reach \( \mathbf{B} \) in a unit time.

For rigid body motions, a geodesic curve \( g(t) \) connecting two poses can be computed as in (23), see [9] [6]. The deviation of a pose \( \mathbf{g}_i \) from a reference pose \( g_0 \) is defined from the initial body velocity \( g^{-1} \dot{g} \) at the initial time. In terms of generalized velocities (22), the corresponding deviation of a pose \( \mathbf{g}_i \) from the reference position \( g_0 \) is
\[
\Delta_i := \begin{bmatrix} R_0^T (P_i - P_0) \\ \log_\ast (R_i^T P_i) \end{bmatrix}
\] (13)

which can be projected onto the UCM and its orthogonal complement via the available inner product (9):
\[
\begin{align*}
\Delta_{null}^i &:= \langle \Delta_i, \mathbf{n}_1 \rangle \mathbf{n}_1 + \langle \Delta_i, \mathbf{n}_2 \rangle \mathbf{n}_2 + \langle \Delta_i, \mathbf{n}_3 \rangle \mathbf{n}_3 \\
\Delta_{\text{orth}}^i &:= \langle \Delta_i, \mathbf{o}_1 \rangle \mathbf{o}_1 + \langle \Delta_i, \mathbf{o}_2 \rangle \mathbf{o}_2 + \langle \Delta_i, \mathbf{o}_3 \rangle \mathbf{o}_3
\end{align*}
\] (14)

As in the classical UCM approach, variance-per-dof\(^4\) can be computed as
\[
\begin{align*}
\sigma_{null}^2 &= \frac{1}{N} \sum_{i=1}^{N} \| \Delta_{null}^i \|^2 \\
\sigma_{\text{orth}}^2 &= \frac{3}{N} \sum_{i=1}^{N} \| \Delta_{\text{orth}}^i \|^2
\end{align*}
\] (15)
where \( N \) is the number of measurements and 3 is the dimension for both the nullspace and its orthogonal.

This section first introduced the general steps involved in the classical UCM approach and then, specifically for the case of static pointing with hand-held tools, derives an intrinsic definition of each of these steps. In particular, an intrinsic definition of deviations (13) from an average pose (12) is made possible by the use of geodesics. In this way, the variance (15) of these deviations on nullspace and on its orthogonal complement can be carried out independently on the choice of coordinates.

III. EXPERIMENTS

A. Static pointing experiment

Experiments were conducted with 4 healthy subjects with no known history of neuromuscular impairment. All of them declared to be right-handed and gave their informed consent prior to the experiment. Each subject was asked to hold a sensorized stylus of a Polhemus LibertyTM system (38μm and 0.0012° resolution within 30cm range) at a specific gripping point, at a fixed distance \( \lambda = 23 \) cm from the tip, onto which a hypodermic needle with luer connector (Terumo® 0.40×13mm) attached. The subject was then asked to touch the tip of a similar needle, firmly attached to a wooden table in a vertical position, with the tip of the stylus, as shown in Fig. 1. The position of the tip \( \mathbf{P}_{\text{tip}} \) and the orientation of the stylus \( R \), which are related to the position of the gripping point \( \mathbf{P}_{\text{grip}} \) via (1), were acquired at 240 Hz via the Polhemus Liberty system and recorded onto a local PC for off-line data analysis. For both needles, only 1 mm of the tip is exposed, while the remaining part is isolated with tapes. The setup is such that, when electrical contact between the exposed tips of the two needles occurs, a beeping sound is produced.

B. Experimental protocol

The experimental protocol consisted of 4 consecutive trials where, in each trial, the subject was asked to make a 15 seconds, steady contact between stylus and target tips, 5 consecutive times, each time separated by a large movement (moving the stylus approximately 20 cm away from the body). Only the inner most 10 seconds between two large movements were analyzed. Firstly, the furthest positions away from the target were detected then a midpoint was calculated. For data analysis we considered only the data points within 5 seconds before or after the midpoint.

One minute rest was given before the next trial was carried out. No visual magnification was provided to the subject.

\(^4\)The definition in (15) corresponds to the geometric framework proposed by Fletcher et al. [2] where, based on the early work of Frechet [3], the variance of a random variable in a metric space is defined as the expected value of the squared distance from the mean.
The protocol was performed in two different experimental conditions. In the first condition (Exp I) the elbow of the right arm was supported on the table, while in the second condition (Exp II) the whole arm was not supported, resulting in different noise conditions [10]. Both types of experiments were carried out in the same day for every subject, with Exp I preceding Exp II and one hour rest in between.

C. Data reduction

For each trial, the inner most 10 seconds (2400 samples) of steady contact were analyzed and the relative UCM components $\sigma_{null}^2$ and $\sigma_{orth}^2$ were derived as in (15). The logarithm of their ratio, referred to as ‘UCM ratio’, was computed

$$\rho_{ucm} = \ln \left( \frac{\sigma_{null}^2}{\sigma_{orth}^2} \right)$$

(16)

The UCM components for all trials of a representative subject are shown in Fig. 2.

![Fig. 2. Variance per dof in the nullspace and its orthogonal complement for all 20 trials of one representative subject.](image)

Similarly to the UCM components and their ratio, also the variances of $P_{tip}$ and $P_{grip}$ and their ratio

$$\rho_{pos} = \ln \left( \frac{\sigma_{grip}^2}{\sigma_{tip}^2} \right)$$

(17)

are computed for each trial.

D. Statistical analysis

According to the UCM theory [11], a larger variance in the nullspace ($\sigma_{null}^2$) than in its orthogonal complement ($\sigma_{orth}^2$) indicates that the position of the stylus tip is a variable directly under neural control. Therefore, we hypothesized that the UCM ratio ($\rho_{ucm}$) will be significantly greater than zero. We also tested the influence of the experimental conditions (Exp I and Exp II) on the UCM ratio. Similar analysis was conducted for the variances of $\sigma_{tip}^2$, $\sigma_{grip}^2$ and their ratio (17).

To test whether the average UCM ratio (16) is significantly different from zero, a t-test was run for the each subject on the $\rho_{ucm}$ values derived for every trial, separately for the two experimental conditions Exp I and Exp II. Similarly was done for $\rho_{pos}$.

To test the effect of the experimental condition and of UCM components on the variance-per-dof, three-ways and two-ways repeated-measures ANOVA analysis was conducted based, respectively, on the MATLAB implementations RMAOV33 [16] and RMAOV2 [17].

IV. Results

For both experimental conditions, there was significantly more variability in the UCM subspace than in the orthogonal subspace ($\sigma_{null}^2 > \sigma_{orth}^2$). For each subject and for each experimental condition, t-tests were conducted on $\rho_{ucm}$ relative to all trials, indicating in all cases that the mean is significantly different from zero ($p < 0.0019$). A three-way repeated measures ANOVA (experimental condition $\times$ UCM component $\times$ trials) conducted on the variance-per-dof indicated that there was a UCM component effect ($p = 0.0004$), but no experimental condition effect ($p = 0.2385$) nor trials effect ($p = 0.6659$). Also, no significant interactions were found ($p > 0.25$ for all possible interactions).

V. Conclusion

In this work, we presented a geometric approach to the UCM analysis for the specific case of static pointing with hand-held tools. The main contribution of this paper is that, unlike the classical UCM analysis, our approach is does not depend on the choice of coordinates. This is made possible by the specific application, i.e. manipulation via hand-held tools, which suggests the use of a left-invariant inner product (9), well known in the literature for rigid body motions. Among other things, an inner product leads to the definition geodesics which allow to intrinsically define i) mean (or reference) posture and ii) deviations from such a reference posture.

We applied our geometric UCM analysis to experimental data acquired from healthy subjects performing static pointing tasks. The major finding is that all subjects display much larger variability on the nullspace of the Jacobian, i.e. the UCM. In other words, variability is tremor and subjects seem to project more tremor along motions which do not directly affect the task. This is in line with typical findings reported in UCM-related literature but, in our case, these results are coordinate-independent, i.e. every researcher would come to the same conclusion independently of the choice of coordinates.

Although our method and experiments specifically targeted static pointing tasks, our approach can be extended to general manipulation via hand-held tools. UCM analysis is all about kinematic redundancy. Understanding how humans manage such redundancy would be valuable, for example, in the design of robots assisting surgeons.

APPENDIX: NOTATIONS

Rigid Body Rotations

The 3D orientation of a rigid body can be described by means of a $3 \times 3$ rotation matrix $R$ (satisfying ortho-normality $R^T R = I$ and ‘right-handedness’ $\det R = +1$). A rotation is physically determined once the rotation axis $n$ ($|n| = 1$) and the rotation angle $\theta$ are known, thus can be described by a rotation vector $r = \theta n$. 

\footnote{Instead of the ratio of the UCM components, its logarithm is used to correct for non-normal distribution [5], [13].}
The rotation matrix $R$ corresponding to a rotation vector $r$ can be computed via the Rodrigues’ formula [8]:

$$ R = \exp(\hat{r}) = I + \sin|r| \frac{\hat{r}}{|r|} + (1 - \cos|r|) \frac{\hat{r}^2}{|r|^2}, $$

(18)

where the skew-symmetric matrix $\hat{r}$ is defined through:

$$ \hat{r} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} $$

(19)

Conversely, for a given rotation matrix $R$, the corresponding rotation vector can be computed via the logarithmic map:

$$ r = \log_{\vee}(R) = \frac{\theta}{2 \sin \theta} \begin{bmatrix} R_{3,2} - R_{2,3} \\ R_{1,3} - R_{3,1} \\ R_{2,1} - R_{1,2} \end{bmatrix} $$

(20)

where $\theta = \arccos((\text{trace}(R) - 1)/2)$, valid for $\theta < \pi$.

Rigid Body Motions

The pose of a rigid body, e.g. the hand-help tool in Fig. 1, is fully specified by the relative position $P$ and orientation $R$ of the moving frame $\mathcal{B}$ with respect to the space-fixed frame $\mathcal{S}$. The pose $g = \{R, P\}$ is computationally represented as a $4 \times 4$ homogenous matrix

$$ g = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} $$

(21)

The velocity $\dot{g}$ of the rigid body from the perspective of the moving frame $(g^{-1}\dot{g})$ can be concisely written as a 6D generalized velocity vector

$$ g^{-1}\dot{g} = \begin{bmatrix} R^T \dot{R} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\omega} \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} \omega \\ 0 \end{bmatrix} $$

(22)

which contains the body linear velocity $v$ and body angular velocity $\omega$.

As briefly shown in Sec. II-C, an inner product can be defined on the space of generalized velocities. Once an inner product is in place, important concepts such as distance and geodesics can be defined. As shown by Park [9], [6], based on the metric (8), the geodesic curve $g(t)$ connecting, in a time unit, an initial pose $g(0) = \{R_0, P_0\}$ to a final pose $g(1) = \{R_1, P_1\}$ is

$$ g(t) = \begin{bmatrix} R_0 \exp(\hat{\Omega} t) (1-t)P_0 + tP_1 \\ 0 \end{bmatrix} $$

(23)

where $\Omega = \log_{\vee}(R_0^T R_1)$ and the distance $d(g_0, g_1)$ is

$$ d(g_0, g_1) := \sqrt{||P_0 - P_1||^2 + \lambda^2 ||\log_{\vee}(R_0^T R_1)||^2} $$

(24)

The classical definition of logarithm ($\log$) of a rotation matrix returns a skew-symmetric matrix, which is associated with a unique 3D vector via the natural isomorphism (19). The map $\log_{\vee}$ simply combines the logarithm and the isomorphism, returning directly a rotation vector.