Systematic Design of an Ultrasonic Horn Profile for High Displacement Amplification

Iman Khalaji, Student Member, IEEE, Rajni V. Patel, Fellow, IEEE and Michael D. Naish, Member, IEEE

Abstract—A new analytical-numerical technique is proposed to design a high displacement amplification ultrasonic horn. The profile of the horn is a NURBS curve whose parameters can be optimized to attain a high displacement amplification factor. Compared to the traditional finite element-based design approach of ultrasonic horns, which is case dependent, our approach formulates and solves the optimal horn profile problem in nondimensional space. Therefore, both the working frequency and the length of the system can be readily adjusted. Depending on the number of control points and their associated weights, the NURBS profile can be simplified to a Bezier, a rational Bezier or a B-spline curve. The optimum profile for each type of curve is then sought and the corresponding amplification factor, natural frequency and maximum strain are verified using finite element analysis. Considering a horn with a cross-sectional ratio of 16, amplification factors of 11.55, 12.12, 13.55 and 15.97 are obtained for the optimum profiles of Bezier, rational Bezier, B-spline and NURBS curves, respectively.

I. INTRODUCTION

High-power (10–1000 W/cm²) low-frequency (20–100 kHz) ultrasound has a wide range of applications in medical procedures. So far, high-power ultrasound has been used in various clinical applications such as dentistry, phacoemulsification, maxillofacial surgery, neurosurgery, tissue dissection and tumor fragmentation, as well as vascular plaque ablation [1]. The required ultrasonic vibration amplitude for tissue fragmentation is different for various tissue textures and ranges from 20 µm in vascular occlusion therapy to about 300 µm in neurosurgery and spinal surgery [1].

Traditionally, ultrasonically activated scalpels consist of a vibration generator (often a Langevin piezoceramic) attached to a vibration amplifier (horn) and a blade. Since piezoceramic transducers can only generate a few microns of vibration amplitude, the device is tuned to resonate at a certain ultrasonic frequency. Therefore, its length must be a half-wavelength or a multiple of the half-wavelength at the driving frequency. Moreover, the horn profile must be carefully designed to give enough vibration amplitude at the blade tip [2]. This, however, causes high stress values along the horn and consequently, the resulting device is bulky. A long and bulky tool leads to more vibration attenuation and heat generation along the tool.

While miniaturizing the ultrasonic scalpels may enhance their flexibility and efficiency, proper design of the horn profile is the bottleneck of such miniaturization. Various ultrasonic horn profiles have been investigated and developed by many researchers over the past 60 years. Analytical approaches have been extensively employed to analyze and design several generic profiles such as conical, exponential, catenoidal, cosine, stepped [3]–[5] and Gaussian [6], where 1 or 2 parameters are enough to control the shape of the profile (considering known length and end-surface diameters). Should the geometry of the horn profile become more complex, finding a closed-form solution would be unlikely.

Structural shape optimization using finite element (FE) analysis, on the other hand, has been shown to be more practical in constraint optimization of complex geometries. There is, however, always a trade-off between the number of design variables and complexity of the model. Therefore, it is inefficient to consider finite element nodes as the design variables, as this leads to a jagged and irregular boundary. Alternatively, low-order splines, i.e., nonuniform rational B-splines (NURBS), can be employed for shape parametrization in structural optimization [7]. This leads to fewer design variables and the optimization procedure can be directly linked to a computer aided design (CAD) system.

The recent advent of NURBS-based isogeometric analysis [8] has made it even more beneficial to perform structural optimization using NURBS-based shape parametrization, since both the geometry and the physical field are directly related to the control points. Moreover, easily controlled continuity (Cn continuity) and exact geometry representation, as well as a fewer number of NURBS-based elements (patches) make isogeometric shape optimization superior over traditional FE-based optimization [9], [10]. There is, however, always a problem associated with structural optimization using either finite element or isogeometric analysis, as they are case dependent. In other words, when using finite element method, all of the governing differential equations are always solved in the physical domain and therefore the obtained results cannot be generalized to other similar cases.

Recently, Wang et al. [11] developed a cubic Bezier...
curve horn profile with high displacement amplitude. They employed a multi-objective genetic algorithm to maximize the displacement amplification of the profile at a particular ultrasonic working frequency. Although the resulting profile exhibits high displacement amplification (about 60% of an equivalent stepped horn) and low stress distribution along the horn, the design is semi-optimal with respect to two predetermined conflicting factors, i.e., the working frequency and the length. Moreover, should a new working frequency and/or dimension be required, a new optimization procedure must be performed, which may lead to a dissimilar geometry.

In this work, a new robust analytical-numerical technique is proposed to design spline-based horn profiles. Starting from the 1-D equation of the longitudinal wave propagation in a bar, a normalized ordinary differential equation (ODE) based on spline basis functions will be obtained. Using a two-stage optimization scheme, the geometry of the profile will then be updated while the resulting ODE is numerically solved at each iteration. The obtained normalized optimum result can then be applied to any length or working frequency. Bezier, rational Bezier, B-spline and NURBS profiles are used to illustrate the capabilities of the approach. Finite element analyses are then carried out to evaluate the mechanical properties of the resulting designs.

II. INTRODUCTION TO NURBS

This section provides a very brief introduction to NURBS geometry. It introduces some notation that will be used in deriving the governing ODE in the following sections. For more details, refer to [12].

A $p$th-degree NURBS curve is defined by

$$C(\xi) = \sum_{i=0}^{n} N_{i,p}(\xi)w_i \mathbf{P}_i,$$  \hspace{1cm} (1)

where $\{\mathbf{P}_i\}$ are the control points, $\{w_i\}$ are the positive weights, and $\{N_{i,p}(\xi)\}$ are the $p$th-degree B-spline basis functions defined on the nonuniform knot vector

$$\Xi = \{0, \ldots, 0, \xi_{p+1}, \ldots, \xi_{m-p-1}, 1, \ldots, 1\}.$$  \hspace{1cm} (2)

The $i$-th B-spline basis function can be defined recursively:

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi \leq \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i, p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1, p-1}(\xi).$$

Setting

$$B_{i,p}(\xi) = \frac{N_{i,p}(\xi)w_i}{\sum_{j=0}^{n} N_{j,p}(\xi)w_j}$$

enables us to rewrite (1) as:

$$C(\xi) = \sum_{i=0}^{n} B_{i,p}(\xi)\mathbf{P}_i.$$  \hspace{1cm} (4)

$\{B_{i,p}(\xi)\}$ are the rational basis functions on $0 \leq \xi \leq 1$. A number of important properties of $B_{i,p}(\xi)$ that will be used in the following sections are [12]:

P.1 Partition of unity: $\sum_{i=0}^{n} B_{i,p}(\xi)=1$ for all $\xi \in [0, 1]$;

P.2 $C(0) = \mathbf{P}_0$ and $C(1) = \mathbf{P}_n$;

P.3 Strong convex hull property: if $\xi \in [u_i, u_{i+1})$, then $C(\xi)$ lies within the convex hull of the control points $\mathbf{P}_{i-p}, \ldots, \mathbf{P}_i$;

P.4 $C(\xi)$ is infinitely differentiable on the interior of the knot spans and is $p-k$ times differentiable at a knot of multiplicity $k$;

P.5 For any $a \neq 0$, if $w_i = a$ for all $i$, then $B_{i,p}(\xi) = N_{i,p}(\xi)$;

P.6 A NURBS curve with no interior knots is a rational Bezier curve. This, together with P.5, implies that B-splines and rational and nonrational Bezier curves are special cases of NURBS curves;

P.7 Local approximation: if the control point $\mathbf{P}_i$ is moved, or the weight $w_i$ is changed, it affects only the portion of the curve on the interval $\xi \in [\xi_i, \xi_{i+p+1})$.

III. HORN DESIGN

A. Longitudinal Wave Propagation

Consider a bar with length $l$ and cross-section $S_x = S(x)$. Let $a(x, t)$ represent the displacement of the cross-section $x$ from its position. Consider a harmonic force $F(t) = F_0 e^{j\omega t}$ applied to the bar at $x = 0$. As a result, a harmonic vibration will be generated along the bar:

$$a(x, t) = \tilde{u}_x e^{j\omega t} = u_x e^{j(\omega t + \phi_x)}.$$  \hspace{1cm} (5)

Here, $\tilde{u}_x = \tilde{u}(x)$ is the complex amplitude, and $u_x$ and $\phi_x$ are the amplitude and the phase, respectively. Considering an isotropic material, the complex modulus of elasticity can be described as:

$$\tilde{E} = E \left(1 + \frac{j\psi}{2\pi}\right),$$  \hspace{1cm} (6)

where $E$ is the Young’s modulus and $\psi$ is the coefficient of absorption, which characterizes the internal friction of the material [5].

The equation of 1-dimensional longitudinal wave propagation within a tapered rod is:

$$\frac{\partial}{\partial x} \left[ \tilde{E} S_x \frac{\partial a(x, t)}{\partial x} \right] = \rho S_x \frac{\partial^2 a(x, t)}{\partial t^2},$$  \hspace{1cm} (7)

where $\rho$ is the density of the material. Substituting (5) into (7) gives a time-invariant equation:

$$\tilde{E} \left[ \tilde{u}_x'' + \frac{S_x'}{S_x} \tilde{u}_x' \right] + \rho \omega^2 \tilde{u}_x = 0.$$  \hspace{1cm} (8)

Note that in deriving (7), it is assumed that the vibrating system is linear and stationary. Now, consider an arrangement where one of the bar’s ends is free (no strain) and the other end is subjected to a harmonic force $F(t) = F_0 e^{j\omega t}$. Therefore, the Neumann boundary conditions are:

$$\tilde{u}_x'\big|_{x=0} = 0, \tilde{E} S_0 \tilde{u}_x'\big|_{x=0} = -F_0.$$  \hspace{1cm} (9)
Equation (8) together with (9) determines the governing differential equation of the vibration within a rod. The final solution to this two-point boundary value problem depends on the waveguide’s form and is determined by the cross-sectional function \( S_x \). The closed form solution to this problem is provided for several typical profiles in the literature [3]–[6].

B. NURBS-based Horn Profile

Now consider an axisymmetric NURBS-based horn profile whose geometry is described by (4). Knowing all of the control points \( P_i \) and the weights \( w_i \), it is desired to solve (8) for the complex amplitude \( \hat{u}_x \). Once the solution is known, an optimization scheme can be used to maximize the displacement amplification coefficient \( K = \frac{\hat{u}_x}{u_0} \) and find the optimal values of \( P_i \), \( w_i \), \( l \) and \( \omega \).

We speculate that a NURBS profile satisfies most of the requirements for a smooth and high displacement amplification horn:

- In order to avoid stress concentrations in a vibrating bar, the ratio \( \frac{S}{S_x} \) in (8) must exist and be bounded. Therefore, \( S_x \) should be at least \( C^1 \) continuous. On the other hand, according to P.4, a NURBS curve is at least \( C^{p-k} \) continuous at a knot of multiplicity \( k \). Therefore, as long as \( p > k + 1 \), there would be no stress concentrations within the bar.

- The strong convex hull property of a NURBS curve (P.3) confines the horn profile within the design space. In other words, the horn profile will always stay inside the design space as long as the control points lie within the same space.

- The local approximation property of a NURBS curve (P.7) helps in optimization of the profile, where adjusting the control point \( P_i \) or the weight \( w_i \) only affects the curve locally.

- For a specified \( S_0 \) and \( S_I \), the first and the last control points will be fixed at \( P_0 \) and \( P_I \), respectively. This follows directly from P.2.

Under these assumptions, an explicit equation for \( S_x \) must be derived to be used in (8). This approach is, however, infeasible as it involves tedious mathematical analyses. Alternatively, (8) can be slightly altered to suit our speculation.

C. Longitudinal Wave Equation Revisited

Assuming a \( p \)-th-degree NURBS curve as an axisymmetric horn profile, the \( x \) and \( r \)-components of the curve can be written using (4) as:

\[
x(\xi) = \sum_{i=0}^{n} B_{i,p}(\xi)x_i
\]

\[
r(\xi) = \sum_{i=0}^{n} B_{i,p}(\xi)r_i,
\]

where

\[
P_i = \{x_i, r_i\}
\]

is the \( i \)-th control point. A normalized form of (10) can also be derived using property P.1:

\[
X(\xi) = \frac{\hat{u}(x(\xi))}{l} = \sum_{i=0}^{n} B_{i,p}(\xi)X_i
\]

\[
R(\xi) = \frac{\hat{u}(r(\xi))}{r_0 - r_n} = \sum_{i=0}^{n} B_{i,p}(\xi)R_i,
\]

where

\[
Q_i = \begin{bmatrix} X_i \\ R_i \end{bmatrix} = \begin{bmatrix} \frac{x_i - x_0}{x_n - x_0} \\ \frac{r_i - r_0}{r_n - r_0} \end{bmatrix}
\]

is the \( i \)-th normalized control point. Note that in deriving (11) it was assumed that the cross-sectional ratio \( \frac{S_I}{S_0} > 1 \) is specified and therefore:

\[
0 = X_0 \leq X_1, ..., X_{n-1} \leq X_n = 1
\]

\[
1 = R_0 \geq R_1, ..., R_{n-1} \geq R_n = 0.
\]

Equation (8) can also be described in a nondimensional form by defining:

\[
l = x_n - x_0 = \text{length of the rod}
\]

\[
\hat{U}(\xi) = \frac{\hat{u}(x(\xi))}{l}
\]

\[
b = \frac{r_n}{r_0 - r_n}
\]

\[
\lambda = j\omega l = \frac{j\omega l}{c\sqrt{1 + j\psi/2\pi}}
\]

where \( c \) is the wave speed within the waveguide. The normalized wave equation within the horn can then be derived as:

\[
\hat{\ddot{U}}_\xi + \hat{\dddot{U}}_\xi \left( \frac{2h\hat{U}}{\pi c^2 + \pi} - \frac{X'_\xi}{X'_n} \right) - \lambda^2 (X'_\xi)^2 \hat{U}_\xi = 0
\]

\[
\hat{U}_{\xi} \big|_{\xi=1} = 0, \hat{U}_{\xi} \big|_{\xi=0} = -\frac{F_0}{E\xi_0} X'_\xi = -\xi_0 X'_\xi,
\]

where \( R_\xi = R(\xi) \) and \( X_\xi = X(\xi) \) are the \( r \) and \( x \) components of the normalized NURBS curve in (11) and \( \xi_0 = \xi(0) \) is the complex strain caused by \( F_0 \) at the input. It is worth mentioning that although (14) is derived for a NURBS-based profile, its use is not limited to such curves.

D. Solving the Normalized Wave Equation

Ultrasonic horns are usually made of low damping materials in order to minimize energy waste and maximize the power transferred between the blade and the specimen under action. It is, therefore, safe to assume that the absorption coefficient \( \psi \) is small (\( \psi << 1 \)). Expanding the expression in (6) using small power series and neglecting the higher order terms, (13d) becomes:

\[
\lambda = \left( j + \frac{\psi}{4\pi} \right) \frac{\omega l}{c}.
\]

Neither (8) nor (14) are analytically solvable in their general forms. Numerical solutions can, however, be sought provided that all of the parameters are known. For a system close to resonance and under the assumption of small dissipation levels, the solution to (14) can be written in the form [5].
\[ \tilde{U}(\xi) = \varepsilon_0 \left[ T(\xi) + jW(\xi) \right], \]  
where \( \varepsilon_0 \) is the input strain amplitude and \( T(\xi) \) and \( W(\xi) \) characterize the elastic and dissipative properties of the vibration system, respectively. At certain \( \lambda(\omega_n, l_n) = \lambda^* \), the system will vibrate at its first longitudinal mode shape (two antinodes at both ends, one node in between), and hence, \( T^*(0) \) and \( T^*(1) \) will be maximized with opposite signs.

Putting (15) and (16) back into (14), a system of two coupled ordinary differential equations and four boundary conditions will be obtained. Numerical boundary value problem algorithm can then be used to solve this system of ODEs.

### E. Optimization Scheme

Assuming a \( p \)-th degree NURBS curve with \( n \) control points, a knot vector is chosen as in (2). For simplicity and to make the optimization process more stable, knot values are not considered as design variables. The profile of the horn is optimized by allowing the normalized control points \( Q_i \) to move within the design space enclosed by the dashed square in Fig. 1. In the case of a rational Bezier or NURBS curve, the weights \( w_i \) will also be considered as design variables during the optimization process.

In order to avoid singularities in (14) and to make the resulting horn profile realistic, additional constraints must be considered during the optimization procedure:

\[
X'(\xi) > 0 \quad \forall \xi \in [0, 1] \quad (17a)
\]

\[
R'(\xi) < 0 \quad \forall \xi \in [0, 1]. \quad (17b)
\]

Equation (17b) ensures that no wavy profile will be generated during the process.

As the horn profile evolves during the optimization process, \( \lambda \) must be changed to put the system back into resonance. For this reason, a two-stage optimization procedure must be iterated upon. In the first step, a multi-variable optimization scheme is used to find the optimum horn profile by adjusting the normalized control points \( Q_i \) and weights \( w_i \). In the second step, the corresponding \( \lambda^* \) is determined using a single variable optimization scheme and the optimization procedure restarts with this new value for \( \lambda^* \). The optimization stops when no change is observed in \( Q_i^*, w_i^* \) and \( \lambda^* \).

The only constant parameter that must be specified in (14) is \( b \), which is directly related to the cross-sectional ratio of the input to the output:

\[
\frac{S_0}{S_I} = \left( \frac{r_0}{r_1} \right)^2 = (1 + b)^2. \quad (18)
\]

While \( b \) depends on the horn’s geometric constraints, \( \psi \) and \( \varepsilon_0 \) can be chosen arbitrarily, since the displacement amplification factor is independent of the material \( [5] \) and the actuator interaction. They must, however, be small enough to avoid violating the assumptions made earlier.

The following outlines the constraint optimization procedure employed in this work:

1) Initialization: considering predetermined values for \( b, \psi \) and \( \varepsilon_0 \), choose some initial values for \( Q_i^0, w_i^0 \) and \( \lambda^0 \). In order to accelerate the optimization process, the initial values must be feasible; i.e., the initial values should satisfy all of the constraints.

2) Set the current values of the parameters:

\[
\begin{align*}
Q_i^{cur} &= Q_i^0 \\
w_i^{cur} &= w_i^0 \\
\lambda^{cur} &= \lambda^0.
\end{align*}
\]

3) First stage constraint multi-variable optimization: (14) is solved at each optimization iteration to find the optimum horn profile:

\[
\min_{Q_i, w_i} \frac{T_0}{T_1} \quad \text{s.t.} \quad \begin{cases}
0 \leq X_i \leq 1 & i = 1, \ldots, n - 1 \\
0 \leq R_i \leq 1 & i = 1, \ldots, n - 1 \\
0 \leq w_i \leq w_{ib} & i = 0, \ldots, n \\
X'(\xi) > 0 & 0 \leq \xi \leq 1 \\
R'(\xi) < 0 & 0 \leq \xi \leq 1,
\end{cases}
\]

where \( w_{ib} \) is an arbitrary upper bound for the weights. Update the current values of \( Q_i^{cur} \) and \( w_i^{cur} \):

\[
\begin{align*}
Q_i^{cur} &= Q_i^* \\
w_i^{cur} &= w_i^*.
\end{align*}
\]

4) Second stage single-variable optimization: calculating the optimum horn profile, (14) is once again solved at each optimization iteration to find the corresponding resonant \( \lambda^* \),

\[
\min_{\lambda} \left| \frac{1}{T_1 - T_0} \right|.
\]

Note that, the subtraction operator in this equation ensures that \( T(1) \) and \( T(0) \) have opposite signs; i.e., the system is at its first mode of vibration. Update the value of \( \lambda^{cur} \):

\[
\lambda^{cur} = \lambda^*.
\]

5) Stopping criteria: If \( Q_i^{cur}, w_i^{cur} \) and \( \lambda^{cur} \) have not been changed from their previous values, terminate the process; otherwise go to Step 3.
IV. ANALYSES

A. Optimization Results

In this section, the optimum horn profiles that are obtained using the proposed approach are presented. Starting from a Bezier curve, it will be shown that a slightly higher amplification can be obtained compared to [11]. Switching to rational Bezier curves, the optimum amplification factor improves further. As B-spline profiles come into play, higher amplification will be achieved at the expense of higher strain values. Finally, it will be shown that a NURBS curve converges safely to a stepped horn with an amplification factor close to that of a stepped horn.

The Sequential Quadratic Programming (SQP) algorithm is utilized to perform the two-stage optimization process in Matlab. As the gradient vectors and Hessian matrices of the objective functions are almost impossible to calculate, Matlab uses finite difference to estimate them. Nonconvexity of the objective functions will also cause the problem to have local minima. Therefore, in each case, various initial values will be used to find several local minima. The optimization results of each profile are tabulated in Table I.

1) Bezier profile: Four normalized control points \( \{Q_0, Q_1, Q_2, Q_3\} \) are used to construct a 3-degree Bezier profile horn. Considering \( \frac{S_0}{S_l} = 16 \), the optimization process is repeated for different initial values of \( Q_1 \) and \( Q_2 \) (\( Q_0 = (0, 1) \) and \( Q_3 = (1, 0) \) are fixed). The amplification factor of the obtained optimized profiles ranges from 10.88 to 11.55 (Table I) which is greater than 10.5, as reported in [11]. As mentioned earlier, the main limitation of Wang’s work [11] is the choice of a fixed working frequency and a fixed length, which makes their result suboptimal.

2) Rational Bezier profile: In addition to the normalized control points, their corresponding weights \( w_j \) can also be considered among the design variables. The resulting profile is now called a rational Bezier curve. The main advantage of rational Bezier curves over Bezier curves is their ability to exactly approximate conic sections. A 3-degree rational Bezier curve is formed using four normalized control points. Considering all of the unknown parameters \( \{Q_1, Q_2, w_0, w_1, w_2, w_3\} \) as the design variables, several local minima are sought using different initial values. The obtained amplification factors range from 10.21 to 11.05 which are less than the amplification range of the Bezier curves obtained earlier. In order to make the optimization procedure more robust, the values of \( \{Q_1, Q_2\} \) were kept constant at the optimum results obtained earlier and the weights were considered as the only design parameters. Under this assumption, the new amplification factor reached to the value of 12.12. A summary of the results obtained at this stage is reported in Table I. In addition, Fig. 2 shows an example of an optimum rational Bezier profile obtained using the optimization procedure.

3) B-spline profile: In order to have a 3-degree B-spline curve, at least 5 control points are required. Choosing a constant knot vector \( \Xi = \{0, 0, 0, 0.5, 1, 1, 1, 1\} \), three normalized control points, i.e., \( \{Q_1, Q_2, Q_3\} \) need to be adjusted using the optimization procedure. Various local minima are obtained for the optimization problem where the amplification factor varies between 11.48 to 13.55. Table I summarizes the results obtained using the B-spline profile.

4) NURBS profile: Including control point weights \( \{w_0, w_1, w_2, w_3, w_4\} \) into the 3-degree B-spline curve leads to a NURBS curve. Similar to the rational Bezier curves, two optimization trials are sought here: 1. optimization of all unknown parameters and 2. optimization of the weights solely. As is shown in Figure 3, the resulting optimized profile in both cases looks much like a stepped horn with an amplification factor close to 16. This, in fact, is true, as the upper bound of the amplification factor of a horn is \( \frac{S_0}{S_l} \) and can be achieved using a stepped horn [5].

B. Finite Element Verification

Obtaining the optimized normalized profiles, the corresponding control points are calculated using (11), considering horn size specification in [11]:

\[
\frac{l}{S_0} = \frac{l}{S_l} = \left( \frac{10 \text{ mm}}{2.5 \text{ mm}} \right)^2 = 16.
\]

The Solidworks API is then used to construct the associated curve of each profile by directly manipulating the control points, weights and knot vector. The resulting horn profiles are then analyzed within Abaqus. Stainless Steel (SS 41) is chosen for the horn material properties as in [11]. In addition, Rayleigh structural damping (\( \alpha = 0, \beta = 10^{-7} \)) is also considered to make the responses bounded. 2D axisymmetric modal analysis is then performed to find the first longitudinal natural frequency and the frequency response function of each profile by considering a sinusoidal forcing function at the base. The amplification factor, \( M \), and resonance frequency, \( f \), of each case is then compared to the corresponding values obtained by our method. As can be seen from Table I, very good agreement is obtained among the results from both methods.
TABLE I

Comparison of the amplification factor, $M$, and resonance frequency, $f$, of various spline-based horn profiles, obtained from the proposed approach and finite element method (FEM)

<table>
<thead>
<tr>
<th>Profile ID</th>
<th>$f_{\text{optimum}}$ (Hz)</th>
<th>$M_{\text{optimum}}$</th>
<th>$f_{\text{R}}$ (Hz)</th>
<th>$M_{\text{R}}$</th>
<th>$\xi_{\text{optimum}}$</th>
<th>$\xi_{\text{R}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bezier #1</td>
<td>30230</td>
<td>10.8759</td>
<td>30580</td>
<td>10.5622</td>
<td>0.0350</td>
<td></td>
</tr>
<tr>
<td>Bezier #2</td>
<td>31607</td>
<td>11.47</td>
<td>31223</td>
<td>11.025</td>
<td>0.0362</td>
<td></td>
</tr>
<tr>
<td>Bezier #3</td>
<td>30999</td>
<td>11.5497</td>
<td>31386</td>
<td>11.0854</td>
<td>0.0362</td>
<td></td>
</tr>
<tr>
<td>Bezier #4</td>
<td>30574</td>
<td>10.4661</td>
<td>30362</td>
<td>10.3771</td>
<td>0.0371</td>
<td></td>
</tr>
<tr>
<td>Rational Bezier #1</td>
<td>33694</td>
<td>10.2121</td>
<td>33854</td>
<td>9.8145</td>
<td>0.0312</td>
<td></td>
</tr>
<tr>
<td>Rational Bezier #2</td>
<td>30380</td>
<td>11.0493</td>
<td>30823</td>
<td>10.6688</td>
<td>0.0363</td>
<td></td>
</tr>
<tr>
<td>Rational Bezier #3</td>
<td>29667</td>
<td>12.1180</td>
<td>29940</td>
<td>11.6280</td>
<td>0.0370</td>
<td></td>
</tr>
<tr>
<td>Rational Bezier #4</td>
<td>27606</td>
<td>11.9716</td>
<td>27932</td>
<td>11.4951</td>
<td>0.0387</td>
<td></td>
</tr>
<tr>
<td>B-spline #1</td>
<td>29022</td>
<td>13.55</td>
<td>29189</td>
<td>13.2295</td>
<td>0.0325</td>
<td></td>
</tr>
<tr>
<td>B-spline #2</td>
<td>30834</td>
<td>13.0007</td>
<td>31056</td>
<td>12.5359</td>
<td>0.0326</td>
<td></td>
</tr>
<tr>
<td>B-spline #3</td>
<td>29816</td>
<td>12.5646</td>
<td>30155</td>
<td>12.1637</td>
<td>0.0329</td>
<td></td>
</tr>
<tr>
<td>B-spline #4</td>
<td>31735</td>
<td>11.7430</td>
<td>32031</td>
<td>11.2992</td>
<td>0.0344</td>
<td></td>
</tr>
<tr>
<td>B-spline #5</td>
<td>26618</td>
<td>11.4797</td>
<td>26954</td>
<td>11.0565</td>
<td>0.0395</td>
<td></td>
</tr>
<tr>
<td>NURBS #1</td>
<td>27361</td>
<td>15.971</td>
<td>27372</td>
<td>15.5074</td>
<td>0.0267</td>
<td></td>
</tr>
<tr>
<td>NURBS #2</td>
<td>28148</td>
<td>15.8663</td>
<td>27002</td>
<td>15.0127</td>
<td>0.0255</td>
<td></td>
</tr>
<tr>
<td>Stepped</td>
<td>-</td>
<td>-</td>
<td>26658</td>
<td>15.6324</td>
<td>0.0230</td>
<td></td>
</tr>
</tbody>
</table>

The last column in Table I indicates the ratio between $|\varepsilon_x|$, the vibration amplitude of the horn at the output, and $\varepsilon_{\text{max}}$, the maximum strain observed along the bar. The higher this ratio, the larger the vibration amplitude generated at the output per unit maximum strain along the bar. As long as the system works in the linear regime and the horn has negligible influence on the actuator, this condition holds. This ratio is especially useful when a horn works close to its fatigue strength. This number together with the amplification factor, $M$, are in fact, figures of merit of a designed resonator. Considering these performance measures, B-spline #5 followed by rational Bezier #5 and #4 exhibit the best performance according to Table I.

This proves our speculation that employing spline-based profiles can improve the displacement amplification of ultrasonic horns. The immediate outcome of this finding is that ultrasonic knives can further be miniaturized, as a lower ratio of $\varepsilon_{\text{optimum}}/\varepsilon_{\text{max}}$ is required to attain the required vibration displacement at the blade tip. In addition, the proposed method introduces a new systematic approach for designing an ultrasonic horn profile. The only design parameter that is required is $\varepsilon_{\text{optimum}}$ and the method finds the optimum profile and its normalized working frequency. The result can then be adapted for any length or working frequency.

V. CONCLUSIONS AND FUTURE WORK

A new analytical-numerical method has been proposed to systematically design a high amplification ultrasonic horn profile. The 1-D wave equation was first normalized in the design space, where the $x$ and $r$-components of the horn profile were assumed to be independent and parametric. Due to the generality of NURBS curves in defining complex geometries and their compact representation, NURBS basis functions were employed to explicitly define the horn profile. A two-stage optimization method was then employed to find the optimum profile that results in the highest displacement amplification ratio. The obtained amplification factor and natural frequencies were verified via finite element analysis.

As the optimization problem of a horn design is nonconvex, there could be many local minima where the algorithm may become trapped. Proper initialization of the design parameters is, therefore, important. It is expected that by employing a global optimization approach such as simulated annealing or a genetic algorithm, a global optimum can be found. It is, however, important to note that there is no unique representation of a certain geometry using NURBS curves. This means that NURBS-based structural optimization problems can have multiple global minima.

REFERENCES