UPnP: An optimal $O(n)$ solution to the absolute pose problem with universal applicability

Supplemental material document

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Abstract. The present document provides supplemental material in respect to the anonymous ECCV 2014 submission number 93. It outlines the derivation of the block-matrix inversion of $(\mathbf{A}^T\mathbf{A})$ in case the coefficients emanate from the prime field $\mathbb{Z}_p$, and furthermore outlines the detailed derivation of the Hessian of $\mathbf{E}$ which is needed in order to verify second-order optimality.

1 Block-matrix inversion in $\mathbb{Z}_p$

We recall that—for our computations in $\mathbb{Z}_p$—the coefficients of the polynomials should be derived in a geometrically consistent way, and not just chosen randomly. As outlined in the paper, this requires to first chose a random pose for the camera system and random world points in $\mathbb{Z}_p$, and then—among other operations—chase those measurements through our block-matrix inversion in order to obtain the corresponding coefficients of the final polynomials. A crucial condition for the block-matrix inversion in the paper to work consists of $\|\mathbf{f}_i\| = 1$. In other words, we need to apply the square root in order to enforce unit-norm of our bearing vectors. Computing the square root in $\mathbb{Z}_p$ requires special techniques. Our solution consists of deriving an alternative variant of the block-matrix inversion that accepts an arbitrary norm for our measurement vectors.

The original matrix we want to invert is given by

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{f}_1^T & & \\ & \dots & \\ & & \mathbf{f}_n^T \\ -\mathbf{I} & \dots & -\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{f}_1 & & & -\mathbf{I} \\ & \dots & & \vdots \\ & & \mathbf{f}_n & -\mathbf{I} \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \mathbf{f}_1^T\mathbf{f}_1 & & & -\mathbf{f}_1^T \\ & \dots & & \vdots \\ & & \mathbf{f}_n^T\mathbf{f}_n & -\mathbf{f}_n^T \\ -\mathbf{f}_1 & \dots & -\mathbf{f}_n & n\mathbf{I} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \tag{3}$$

We thus realized that $(\mathbf{A}^T\mathbf{A})$ admits a block-structure. Using the Schur-complement and block-matrix inversion, we obtain

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{bmatrix}, \tag{4}$$

with

$$\begin{aligned}
\mathcal{H} &= \left(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)^{-1} && (5)\\
&= \left(n\mathbf{I} - \begin{bmatrix} -\mathbf{f}_1 & \dots & -\mathbf{f}_n \end{bmatrix} \begin{bmatrix} \frac{1}{\mathbf{f}_1^T\mathbf{f}_1} & & \\ & \dots & \\ & & \frac{1}{\mathbf{f}_n^T\mathbf{f}_n} \end{bmatrix} \begin{bmatrix} -\mathbf{f}_1^T \\ \vdots \\ -\mathbf{f}_n^T \end{bmatrix} \right)^{-1} && (6)\\
&= \left(n\mathbf{I} - \sum_{i=1}^{n} \frac{\mathbf{f}_i\mathbf{f}_i^T}{\mathbf{f}_i^T\mathbf{f}_i} \right)^{-1} && (7)\\
\mathcal{E} &= \mathcal{A}^{-1} + \mathcal{A}^{-1}\mathcal{B}\left(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)^{-1}\mathcal{C}\mathcal{A}^{-1} && (8)\\
&= \begin{bmatrix} \frac{1}{\mathbf{f}_1^T\mathbf{f}_1} & & \\ & \dots & \\ & & \frac{1}{\mathbf{f}_n^T\mathbf{f}_n} \end{bmatrix} + \begin{bmatrix} \frac{\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} \\ \vdots \\ \frac{\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n} \end{bmatrix} \mathcal{H} \begin{bmatrix} \frac{\mathbf{f}_1}{\mathbf{f}_1^T\mathbf{f}_1} & \cdots & \frac{\mathbf{f}_n}{\mathbf{f}_n^T\mathbf{f}_n} \end{bmatrix} && (9)\\
\mathcal{F} &= -\mathcal{A}^{-1}\mathcal{B}\left(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)^{-1} && (10)\\
&= \begin{bmatrix} \frac{\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} \\ \vdots \\ \frac{\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n} \end{bmatrix} \mathcal{H} && (11)\\
\mathcal{G} &= -\left(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)^{-1}\mathcal{C}\mathcal{A}^{-1} && (12)\\
&= \mathcal{H} \begin{bmatrix} \frac{\mathbf{f}_1}{\mathbf{f}_1^T\mathbf{f}_1} & \cdots & \frac{\mathbf{f}_n}{\mathbf{f}_n^T\mathbf{f}_n} \end{bmatrix}. && (13)
\end{aligned}$$

We finally obtain

$$\begin{aligned}
\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} &= \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T && (14)\\
&= \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^T & & \\ & \dots & \\ & & \mathbf{f}_n^T \\ -\mathbf{I} & \dots & -\mathbf{I} \end{bmatrix}, && (15)
\end{aligned}$$

and therefore

$$\mathbf{V} = \mathcal{H}\left[\frac{\mathbf{f}_1\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} - \mathbf{I} \dots \frac{\mathbf{f}_n\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n} - \mathbf{I}\right] \quad (16)$$

$$\mathbf{U} = \mathcal{E}\begin{bmatrix}\mathbf{f}_1^T & & \\ & \dots & \\ & & \mathbf{f}_n^T\end{bmatrix} + \begin{bmatrix}\frac{\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} \\ \vdots \\ \frac{\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n}\end{bmatrix}\mathcal{H}\left[-\mathbf{I} \dots -\mathbf{I}\right] \quad (17)$$

$$= \begin{bmatrix}\frac{\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} & & \\ & \dots & \\ & & \frac{\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n}\end{bmatrix} + \begin{bmatrix}\frac{\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} \\ \vdots \\ \frac{\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n}\end{bmatrix}\mathcal{H}\left[\frac{\mathbf{f}_1\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} - \mathbf{I} \dots \frac{\mathbf{f}_n\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n} - \mathbf{I}\right] \quad (18)$$

$$= \begin{bmatrix}\frac{\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} & & \\ & \dots & \\ & & \frac{\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n}\end{bmatrix} + \begin{bmatrix}\frac{\mathbf{f}_1^T}{\mathbf{f}_1^T\mathbf{f}_1} \\ \vdots \\ \frac{\mathbf{f}_n^T}{\mathbf{f}_n^T\mathbf{f}_n}\end{bmatrix}\mathbf{V}. \quad (19)$$

Computing $\mathcal{H}$, $\mathbf{U}$, and $\mathbf{V}$ in this way allows us to use bearing vectors with arbitrary norm. Using $\mathbf{f}_i^T\mathbf{f}_i = 1$ and $\mathcal{H} = \mathbf{H}$, we finally get back to the result in the paper.

2 Derivation of the Hessian of E

We start by reciting the expression for the first-order derivatives of E (i.e. the Jacobian J_E), which is needed for the first-order optimality conditions:

$$J_E = \begin{bmatrix}2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_1} \\ 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_2} \\ 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_3}\end{bmatrix}, \quad (20)$$

where $\tilde{\mathbf{s}} = \begin{bmatrix}\mathbf{s}^T & 1\end{bmatrix}^T$ and $\frac{\partial\tilde{\mathbf{s}}}{\partial c_j} = \begin{bmatrix}\frac{\partial\mathbf{s}^T}{\partial c_j} & 0\end{bmatrix}^T$. We have used the fact that we already compensated for a solution for our rotation, and are now operating in the Cayley-space of rotations around that solution. $\mathbf{s}$ as a function of the Cayley-parameters $\mathbf{c} = \begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}^T$ is given by

$$\mathbf{s} = \frac{1}{\Delta}[1, c_1^2, c_2^2, c_3^2, c_1, c_2, c_3, c_1c_2, c_1c_3, c_2c_3]^T, \quad (21)$$

where $\Delta = 1 + c_1^2 + c_2^2 + c_3^2$. The Hessian of our energy term becomes

$$H_E = \begin{bmatrix}2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_1\partial c_1} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_1}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_1}, & 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_1\partial c_2} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_2}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_1}, & 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_1\partial c_3} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_3}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_1} \\ 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_2\partial c_1} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_1}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_2}, & 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_2\partial c_2} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_2}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_2}, & 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_2\partial c_3} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_3}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_2} \\ 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_3\partial c_1} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_1}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_3}, & 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_3\partial c_2} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_2}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_3}, & 2\tilde{\mathbf{s}}^T\mathbf{M} \cdot \frac{\partial^2\tilde{\mathbf{s}}}{\partial c_3\partial c_3} + 2\frac{\partial\tilde{\mathbf{s}}^T}{\partial c_3}\mathbf{M} \cdot \frac{\partial\tilde{\mathbf{s}}}{\partial c_3}\end{bmatrix}, \quad (22)$$

where $\frac{\partial^2 \tilde{\mathbf{s}}}{\partial c_j \partial c_k} = \left[\frac{\partial^2 \mathbf{s}^T}{\partial c_j \partial c_k}\; 0\right]^T$.

The second-order optimality is now analyzed by evaluating H_E around $\mathbf{c} = \mathbf{0}$. $H_E|_{\mathbf{c}=\mathbf{0}}$ is easily constructed by in turn evaluating $\mathbf{s}$, $\frac{\partial \mathbf{s}}{\partial c_j}$, and $\frac{\partial^2 \mathbf{s}}{\partial c_j \partial c_k}$ around $\mathbf{c} = \mathbf{0}$. The derivations are straightforward, and we only give the final result here:

$$\begin{aligned}
\mathbf{s}|_{\mathbf{c}=\mathbf{0}} &= [1\,0\,0\,0\,0\,0\,0\,0\,0\,0]^T \\
\frac{\partial \mathbf{s}}{\partial c_1}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,1\,0\,0\,0\,0\,0]^T \\
\frac{\partial \mathbf{s}}{\partial c_2}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,1\,0\,0\,0\,0]^T \\
\frac{\partial \mathbf{s}}{\partial c_3}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,0\,1\,0\,0\,0]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_1 \partial c_1}|_{\mathbf{c}=\mathbf{0}} &= [-2\,2\,0\,0\,0\,0\,0\,0\,0\,0]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_1 \partial c_2}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,0\,0\,1\,0\,0]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_1 \partial c_3}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,0\,0\,0\,1\,0]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_2 \partial c_1}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,0\,0\,1\,0\,0]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_2 \partial c_2}|_{\mathbf{c}=\mathbf{0}} &= [-2\,0\,2\,0\,0\,0\,0\,0\,0\,0]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_2 \partial c_3}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,0\,0\,0\,0\,1]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_3 \partial c_1}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,0\,0\,0\,1\,0]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_3 \partial c_2}|_{\mathbf{c}=\mathbf{0}} &= [0\,0\,0\,0\,0\,0\,0\,0\,0\,1]^T \\
\frac{\partial^2 \mathbf{s}}{\partial c_3 \partial c_3}|_{\mathbf{c}=\mathbf{0}} &= [-2\,0\,0\,2\,0\,0\,0\,0\,0\,0]^T
\end{aligned}$$

Back-substitution into (22) leads to

$$H_E|_{\mathbf{c}=\mathbf{0}} = 2 \left[\begin{matrix} M_{5,5} + 2M_{1,2} - 2M_{1,1} + 2M_{2,11} - 2M_{1,11} & \cdots \\ M_{5,6} + M_{1,8} + M_{8,11} & \cdots \\ M_{5,7} + M_{1,9} + M_{9,11} & \cdots \end{matrix} \right. \tag{23}$$

$$\left. \begin{matrix} \cdots & M_{5,6} + M_{1,8} + M_{8,11} & M_{5,7} + M_{1,9} + M_{9,11} \\ \cdots & M_{6,6} + 2M_{1,3} - 2M_{1,1} + 2M_{3,11} - 2M_{1,11} & M_{6,7} + M_{1,10} + M_{10,11} \\ \cdots & M_{6,7} + M_{1,10} + M_{10,11} & M_{7,7} + 2M_{1,4} - 2M_{1,1} + 2M_{4,11} - 2M_{1,11} \end{matrix} \right], \tag{24}$$

where we used the fact that $\mathbf{M}$ is a symmetric matrix, and $M_{i,j}$ represents the element in row i and column j of $\mathbf{M}$. Second-order optimality is given if $H_E|_{\mathbf{c}=\mathbf{0}}$

is positive-definite, which we verify by computing the sign of the Eigenvalues of $H_E|_{\mathbf{c}=\mathbf{0}}$.

The root polishing step depends on the Jacobian J_E around $\mathbf{c} = \mathbf{0}$ as well, which is why we also need to construct $J_E|_{\mathbf{c}=\mathbf{0}}$. Back-substituting the above elements into (20) finally leads to

$$J_E|_{\mathbf{c}=\mathbf{0}} = 2 \begin{bmatrix} M_{1,5} + M_{5,11} \\ M_{1,6} + M_{6,11} \\ M_{1,7} + M_{7,11} \end{bmatrix} . \tag{25}$$

We can see that the computation of the Jacobian and the Hessian in Cayley space becomes very compact once the matrix $\mathbf{M}$ is established around the candidate rotation.