Supplemental Material for "SRA: Fast Removal of General Multipath for ToF Sensors"

1 Proof of Theorem 1

Proof: x^* solves the optimization

$$\min_{x\geq 0} \|x\|_1 \quad \text{subject to} \quad (\Phi x - v)^T C^{-1}(\Phi x - v) \leq \epsilon^2 \|v\|^2$$

Note that $x^*_{s,\Delta}$ solves the same optimization, but with $F_{s,\Delta}v$ replacing v and $F_{s,\Delta}\Phi$ replacing Φ. In the above form of the minimization, the quantities are all real; but $F_{s,\Delta}$ is complex. Rather than convert it to its real pieces, we use the following argument.

Let $z \in \mathbb{R}^{2m}$ be an abitrary real $2m$-dimensional vector, and let $\bar{z} \in \mathbb{C}^m$ be the complex version gotten from "unstacking" z:

$$\bar{z}_j = z_j + iz_{j+m}$$

Then for a diagonal real matrix $A \in \mathbb{R}^{2m\times 2m}$ satisfying $A_{jj} = A_{j+m,j+m}$, we have that

$$z^T Az = \sum_{j=1}^{2m} A_{jj} z_j^2 = \sum_{j=1}^{m} A_{jj}(z_j^2 + z_{j+m}^2) = \sum_{j=1}^{m} A_{jj}|\bar{z}_j|^2$$

Now, if we take $A = C^{-1}$ and $z = \Phi x - v$, we have that the constraint

$$(\Phi x - v)^T C^{-1}(\Phi x - v) \leq \epsilon^2 \|v\|^2$$

becomes

$$\sum_{j=1}^{m} \frac{1}{C_{jj}} |(\overline{\Phi x - v})_j|^2 \leq \epsilon^2 \|v\|^2$$

which is in turn equivalent to

$$\sum_{j=1}^{m} \frac{1}{C_{jj}} |(\bar{\Phi} x - \bar{v})_j|^2 \leq \epsilon^2 \sum_{j=1}^{m} |\bar{v}_j|^2$$

where $\bar{\Phi}$ is the "unstacked" version of the matrix Φ.

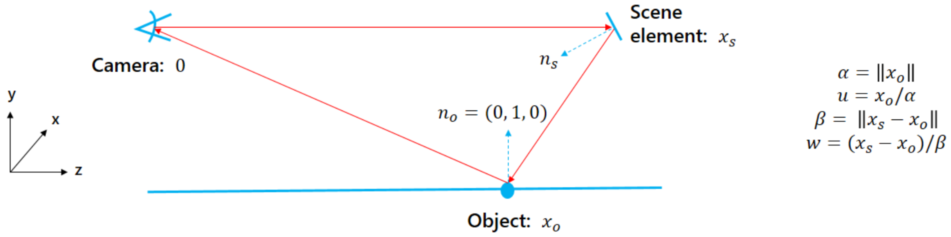

Figure 1: Specular Multipath. See description in the text.

Now, we have that $F_{s,\Delta}\bar{v}$ replaces $\bar{v}$ and $F_{s,\Delta}\bar{\Phi}$ replaces $\bar{\Phi}$. Thus, the constraint becomes

$$
\begin{aligned}
& \sum_{j=1}^{m} \frac{1}{C_{jj}} |(F_{s,\Delta}\bar{\Phi}x - F_{s,\Delta}\bar{v})_j|^2 \leq \epsilon^2 \sum_{j=1}^{m} |F_{s,\Delta}\bar{v}_j|^2 \\
\Leftrightarrow \quad & \sum_{j=1}^{m} \frac{1}{C_{jj}} |[F_{s,\Delta}(\bar{\Phi}x - \bar{v})]_j|^2 \leq \epsilon^2 \sum_{j=1}^{m} |F_{s,\Delta}\bar{v}_j|^2 \\
\Leftrightarrow \quad & \sum_{j=1}^{m} \frac{1}{C_{jj}} |se^{-2\pi i\Delta/\lambda_j}(\bar{\Phi}x - \bar{v})_j|^2 \leq \epsilon^2 \sum_{j=1}^{m} |se^{-2\pi i\Delta/\lambda_j}\bar{v}_j|^2 \\
\Leftrightarrow \quad & \sum_{j=1}^{m} \frac{s^2}{C_{jj}} |(\bar{\Phi}x - \bar{v})_j|^2 \leq \epsilon^2 \sum_{j=1}^{m} s^2|\bar{v}_j|^2 \\
\Leftrightarrow \quad & \sum_{j=1}^{m} \frac{1}{C_{jj}} |(\bar{\Phi}x - \bar{v})_j|^2 \leq \epsilon^2 \sum_{j=1}^{m} |\bar{v}_j|^2
\end{aligned}
$$

which is exactly the original constraint.

Since only this constraint changes (the objective function remains the same), the two optimization problems are identical, and our proof is complete. ■

2 Generation of Three Path Interference

Specular multipath with three or more paths results naturally from simple scene geometries. In Figure 1, the object is lying on a Lambertian surface, while the scene element is taken to be purely specular. The interfering path is shown in red; note that the angle of incidence to the scene element equals the angle of reflection, as is required for a specular surface. Indeed, for any fixed position x_s of the scene element, we can compute an appropriate normal n_s such that we generate multipath to given fixed object position x_o as follows: (1) orient n_s so that it lies in the same plane as x_s and $x_o - x_s$, and

(2) place n_s within that plane so as to bisect the angle between $x_o - x_s$ and $-x_s$. We note that using a very similar analysis, one can fix the normal of the scene element n_s, and choose its position x_s to generate an appropriate interfering path.

Let us now analyze the amplitudes and distances of the two paths, direct and interfering. First, let us introduce some useful notation (matching that shown in Figure 1): let $\alpha = \|x_o\|$ and $u = x_o/\alpha$. Then the direct path has distance α. Its amplitude can be computed using the fact that the surface is Lambertian: the returning ray's power is proportional to the cosine of the angle between the ray and the surface normal, which is just $|u_2|$. The analysis for the interfering path is similar. If we let $\beta = \|x_s - x_o\|$, and $w = (x_s - x_o)/\beta$, then the interfering path has distance given by $(\|\alpha u + \beta w\| + \alpha + \beta)/2$, and its amplitude is given $|w_2|$.

It is not hard to see that by varying the parameters α, β, u, and w, we can generate any relative amplitudes we wish between the direct and interfering paths, as well as any set of path distances. The above analysis was for two path multipath, but three or more paths come naturally from adding in more specular surfaces to the scene.