Supplementary Material of "Efficient k-Support Matrix Pursuit"

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In this supplementary material, we prove the two lemma in the main paper.

Lemma 1. *Suppose that l satisfies $z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} \geq z_{l+1}$. Let $low, high, mid$ are variables generated by the Algorithm 3, Then*
(1) If $z_{mid} > \frac{T_{r,mid}}{mid-k+r+1+\beta(r+1)}$ then $mid \leq l$.
(2) If $z_{mid} \leq \frac{T_{r,mid}}{mid-k+r+1+\beta(r+1)}$ then $l \leq mid-1$.

And

Lemma 2. *For any r such that $r \in [0, k-1]$. If $z_{k-r} > 0$, it definitely exists a l satisfies $z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} \geq z_{l+1}$ in the range of $[k-r, nd]$. If $z_{k-r} = 0$, we can return $l = k-r$.*

Before we prove the Lemma 1, we need the following two lemmas.

Lemma 3. *If $z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)}$, then $z_{l-1} > \frac{T_{r,l-1}}{l-1-k+r+1+\beta(r+1)}$, where $T_{r,l} = \sum_{i=k-r}^{l} z_i$.*

Proof.

$$\begin{aligned}
& z_{l-1} * (l-1-k+rL+1+\beta(r+1)) \\
=\ & z_{l-1} * (l-k+r+1+\beta(r+1)) - z_{l-1} \\
\geq\ & z_l * (l-k+r+1+\beta(r+1)) - z_{l-1} \\
>\ & T_{r,l} - z_{l-1} \\
=\ & T_{r,l-1}
\end{aligned} \tag{1}$$

where the first inequality follows $z_{l-1} \geq z_l$ and the second inequality is the assumption. ∎

Lemma 4. *If $z_l \leq \frac{T_{r,l}}{l-k+r+1+\beta(r+1)}$, then $z_{l+1} \leq \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)}$.*

Proof.

$$\begin{aligned}
& z_{l+1} * (l+1-k+r+1+\beta(r+1)) \\
=\ & z_{l+1} * (l-k+r+1+\beta(r+1)) + z_{l+1} \\
\leq\ & z_l * (l-k+r+1+\beta(r+1)) + z_{l+1} \\
\leq\ & T_{r,l} + z_{l+1} \\
=\ & T_{r,l+1}
\end{aligned} \tag{2}$$

where the first inequality follows $z_l \geq z_{l+1}$ and the second inequality is the assumption. ∎

According to Lemma 3 and Lemma 4, we can proof the Lemma 1.

Proof. (1) Assumed that the claim does not hold. Thus, we have $l+1 \leq mid$. According to $z_{mid} > \frac{T_{r,mid}}{mid-k+r+1+\beta(r+1)}$ and Lemma 3, we know $z_{l+1} > \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)}$. Hence

$$\begin{aligned} & z_{l+1} > \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)} \\ & \rightarrow z_{l+1}(l-k+r+1+\beta(r+1)) + z_{l+1} > T_{r,l+1} \\ & \rightarrow z_{l+1}(l-k+r+1+\beta(r+1)) > T_{r,l} \\ & \rightarrow z_{l+1} > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} \end{aligned} \tag{3}$$

This is a contradiction with $\frac{T_{r,l}}{l-k+r+1+\beta(r+1)} \geq z_{l+1}$.

(2) Assumed that the claim does not hold. Thus, we have $l \geq mid$. According to $z_{mid} \leq \frac{T_{r,mid}}{mid-k+r+1+\beta(r+1)}$ and Lemma 4, we know $z_l \leq \frac{T_{r,l}}{l-k+r+1+\beta(r+1)}$. This is a contradiction with $z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)}$. ■

Now, we proof the Lemma 2.

Proof. First, when $z_{k-r} = 0$, it means that $z_{k-r+1} = ... = z_{nd} = 0$. In such case, there is not exist a l satisfies $z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)}$. Any $l \in [k-r, nd]$ can be returned and not influence the result. Hence, we can simply let $l = k-r$.

Now, we consider the case of $z_{k-r} > 0$. To proof $l \in [k-r, nd]$, we only need to show (1) $z_{k-r} > \frac{T_{r,k-r}}{k-r-k+r+1+\beta(r+1)}$ and (2) $z_{nd+1} \leq \frac{T_{r,nd+l}}{nd+1-k+r+1+\beta(r+1)}$. If both (1) and (2) are satisfied, according to Lemma 1, we have $l \geq k-r$ and $l \leq nd$. Hence $l \in [k-r, nd]$.

It is easy to verify that both (1) and (2) are true. Since $z_{k-r} - \frac{T_{r,k-r}}{k-r-k+r+1+\beta(r+1)} = z_{k-r} - \frac{z_{k-r}}{1+\beta(r+1)} > 0$, we have the (1). Since $z_{d+1} = -\infty$, it is less than or equal to any value, hence we have the (2).

Now, we show which $l \in [k-r, nd]$ satisfies the inequalities. Since both (1) and (2) are true, this indicates that we can find at least a $l \in [k-r, nd]$ satisfies $z_l - \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} > 0$ and $z_{l+1} - \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)} \leq 0$.

We have $z_{l+1} - \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)} \leq 0 \Rightarrow z_{l+1} \leq \frac{T_{r,l}+z_{l+1}}{l+1-k+r+1+\beta(r+1)} \Rightarrow (l+1-k+r+1+\beta(r+1))z_{l+1} \leq T_{r,l} + z_{l+1} \Rightarrow (l-k+r+1+\beta(r+1))z_{l+1} \leq T_{r,l} \Rightarrow z_{l+1} \leq \frac{T_{r,l}}{l-k+r+1+\beta(r+1)}$.

Hence,$z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} > z_{l+1}$. we find the l. ■

Lemma 5. *If $z_{k-r} > 0$, there is an unique l satisfies $z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} \geq z_{l+1}$ in the range of $[k-r, d]$.*

Proof. Since $z_l > \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} \geq z_{l+1}$, we have (1) $z_l - \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} > 0$ and (2) $z_{l+1} - \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)} \leq 0$. (According to the Lemma 2).

We show there is an unique $l \in [k-r, d]$ satisfies the inequalities. Assumed the claim does not hold. Thus, there exists l and $\hat{l}$ for which $l < \hat{l}$, we have

$$\begin{cases} z_l - \frac{T_{r,l}}{l-k+r+1+\beta(r+1)} > 0 \\ z_{l+1} - \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)} \leq 0 \end{cases}$$

and

$$\begin{cases} z_{\hat{l}} - \frac{T_{r,\hat{l}}}{\hat{l}-k+r+1+\beta(r+1)} > 0 \\ z_{\hat{l}+1} - \frac{T_{r,\hat{l}+1}}{\hat{l}+1-k+r+1+\beta(r+1)} \leq 0 \end{cases}$$

Since $z_{l+1} - \frac{T_{r,l+1}}{l+1-k+r+1+\beta(r+1)} \leq 0$ and $\hat{l} \geq l+1$, according to Lemma 4, we have $z_{\hat{l}} - \frac{T_{r,\hat{l}}}{\hat{l}-k+r+1+\beta(r+1)} \leq 0$. This is a contradiction with $z_{\hat{l}} - \frac{T_{r,\hat{l}}}{\hat{l}-k+r+1+\beta(r+1)} > 0$. ■

1 Computation of the proximity operator

$$\min_w \frac{\beta}{2}||w-v||_F^2 + \frac{1}{2}(||w||_k^{sp})^2 \tag{4}$$

Argyriou *et al.* [8] showed that this computation of the *proximity operator* can be done in $O(nd(k + log(nd)))$ steps. Here we include the derivation for self-containedness.

Before we present the solution, we firstly give the following two lemmas. Lemma 6 indicates that the each component of the optimal solution w^* has the same sign of its counterpart in v. Lemma 7 shows that if $|v_i|$ is the jth largest element of $|v|$, then $|w_i|$ is the jth largest element of $|w^*|$.

Lemma 6. *Let w^* be the optimal solution to the minimization problem given in Eq. (4). Then $w_i^* v_i \geq 0$ for all $i = 1, ..., nd$.*

Proof. Assume that the claim does not hold. Thus, there exists i for which $w_i^* v_i < 0$. Let w be a vector such that $w_k = w_k^*$ for all $k \neq i$ and $w_i = 0$. It is easy to verify that (1) $||w^*||_k^{sp} \geq ||w||_k^{sp}$ and (2) $||w^* - v||^2 > ||w - v||^2$. We thus find a solution w which attains an objective value smaller than that of w^*. This is a contradiction with w^* is the optimal solution. ■

Lemma 7. *Let w^* be the optimal solution to the minimization problem given in Eq. (4). Then for any i, j, if $|v_i| \geq |v_j|$, we also have $|w_i| \geq |w_j|$.*

Proof. Assume that the claim does not hold. Thus, there exists i, j for which $|v_i| \geq |v_j|$ and $|w_i^*| < |w_j^*|$. Let w be a vector such that $w_k = w_k^*$ for all $k \neq i, k \neq j$ and $w_i = sign(v_i)|w_j^*|, w_j = sign(v_j)|w_i^*|$. Therefore, $\frac{\lambda}{2}(||w^*||_k^{sp})^2 + \frac{1}{2}||w^* - v||^2 - \frac{\lambda}{2}(||w||_k^{sp})^2 - \frac{1}{2}||w - v||^2 = \frac{1}{2}((w_i^* - v_i)^2 + (w_j^* - v_j)^2 - (sign(v_i)|w_j^*| - v_i)^2 - (sign(v_j)|w_i^*| - v_j)^2) = -|w_i^*||v_i| - |w_j^*||v_j| + |w_j^*||v_i| + |w_i^*||v_j| = (|w_j^*| - |w_i^*|)(|v_i| - |v_j|) \geq 0$.

Hence, w attains an objective value less than or equal to that of w^*. This is a contradiction. ■

Based on the lemma 6 and lemma 7, we can rewrite the optimization problem as

$$\min_q \frac{1}{2\beta}(\sum_{i=1}^{k-r-1} q_i^2 + \frac{1}{r+1}(\sum_{i=k-r}^{nd} |q_i|)^2) + \frac{1}{2}||q-z||^2$$
$$s.t.\ \ q_1 \geq q_2 \geq ... \geq q_{nd} \geq 0 \tag{5}$$
$$q_{k-r-1} > \frac{1}{r+1}(\sum_{i=k-r}^{nd} q_i) \geq q_{k-r}$$

where z denotes the vector obtained by sorting the absolute value of v in a descending order, $z_1 \geq z_2 \geq ... \geq z_{nd} \geq 0$. Let s be denoted as the corresponding index, $|v_{s_i}| = z_i$. Once we obtain the optimal solution of Eq. (5), we can construct the solution of Eq. (4) by setting

$$w_{s_i} = sign(v_{s_i})q_i. \tag{6}$$

Now, we consider to solve the Eq. (5). Without the constrains, Eq.(5) can be rewrite as the following two sub problems:

$$\min_{q_1,...,q_{k-r-1}} \frac{1}{2}\sum_{i=1}^{k-r-1}(q_i^2/\beta + (q_i - z_i)^2) \tag{7}$$

$$\min_{q_{k-r},...,q_{nd}} \frac{1}{2\beta(r+1)}(\sum_{i=k-r}^{nd} |q_i|)^2 + \frac{1}{2}\sum_{i=k-r}^{nd}(q_i - z_i)^2 \tag{8}$$

Eq (7) is a simple problem. The optimal solution is

$$q_i = \frac{\beta}{\beta+1} z_i \quad for\ i = 1, ..., k-r-1 \tag{9}$$

We take the derivative of Eq.(8) with respect to q_j to zero, where $j = k-r, ..., nd$. we obtain

$$\frac{1}{\beta(r+1)}(\sum_{i=k-r}^{nd} |q_i|)\nabla|q_j| + (q_j - z_j) = 0 \tag{10}$$

where $\nabla|q_j|$ is the sub-gradient of $|q_j|$. Since $q_j \geq 0$, we have

$$\nabla|q_j| = \begin{cases} \{c_j \in R | 0 \leq c_j \leq 1\} & \text{if } q_j = 0 \\ 1 & \text{if } q_j > 0 \end{cases}$$

Hence, we need to discuss the two cases for finding the solution of Eq.(8). Suppose that $q_{k-r} \geq ... \geq q_l > 0$ and $q_{l+1} = ... = q_{nd} = 0$. Substitution it into

Eq. (10), we have

$$\begin{aligned} &\frac{1}{\beta(r+1)}(\sum_{i=k-r}^{l} q_i) + (q_{k-r} - z_{k-r}) = 0 \\ &\cdots \\ &\frac{1}{\beta(r+1)}(\sum_{i=k-r}^{l} q_i) + (q_l - z_l) = 0 \\ &\frac{1}{\beta(r+1)}(\sum_{i=k-r}^{l} q_i) \geq z_{l+1} \end{aligned} \tag{11}$$

Hence, the optimal solution of Eq (8) is

$$q_i = \begin{cases} z_i - \frac{\sum_{i=k-r}^{l} z_i}{l-k+r+1+\beta(r+1)} & \text{if } i = k-r, ..., l \\ 0 & \text{if } i = l+1, ..., nd \end{cases} \tag{12}$$

Substitution the solution into Eq. (11), we have l satisfies

$$z_l > \frac{\sum_{i=k-r}^{l} z_i}{\beta(r+1) + l - k + r + 1} \geq z_{l+1}. \tag{13}$$

Now, we consider the constrain $q_{k-r-1} > \frac{1}{r+1}(\sum_{i=k-r}^{nd} q_i)^2 \geq q_{k-r}$. Substitution the solutions of Eq. (8) and Eq. (7) into it, we have

$$\begin{aligned} &q_{k-r-1} > \frac{1}{r+1} \sum_{i=k-r}^{nd} q_i \geq q_{k-r} \\ \Rightarrow &\frac{\beta}{\beta+1} z_{k-r-1} > \beta(\frac{\sum_{i=k-r}^{l} z_i}{\beta(r+1) + l - k + r + 1}) \\ &\geq z_{k-r} - \frac{\sum_{i=k-r}^{l} z_i}{\beta(r+1) + l - k + r + 1} \\ \Rightarrow &\frac{1}{\beta+1} z_{k-r-1} > \frac{\sum_{i=k-r}^{l} z_i}{\beta(r+1) + l - k + r + 1} \geq \frac{1}{\beta+1} z_{k-r} \end{aligned} \tag{14}$$

Hence, the solution of Eq. (5) is

$$q_i = \begin{cases} \frac{\beta}{\beta+1} z_i & \text{if } i = 1, ..., k-r-1 \\ z_i - \frac{\sum_{i=k-r}^{l} z_i}{l-k+r+1+\beta(r+1)} & \text{if } i = k-r, ..., l \\ 0 & \text{if } i = l+1, ..., nd \end{cases} \tag{15}$$

where r and l satisfy that

$$\begin{cases} \frac{1}{\beta+1} z_{k-r-1} > \frac{\sum_{i=k-r}^{l} z_i}{\beta(r+1)+l-k+r+1} \geq \frac{1}{\beta+1} z_{k-r} \\ z_l > \frac{\sum_{i=k-r}^{l} z_i}{\beta(r+1)+l-k+r+1} \geq z_{l+1} \end{cases} \tag{16}$$