Statistical Pose Averaging with Non-Isotropic and Incomplete Relative Measurements Additional Material

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In this additional material we will prove Proposition 3.1 from the main paper. First, however, we will need to review some concepts from Riemannian geometry. We refer the reader to [1] for additional details.

1 Additional Notation and Background

Let M be a Riemannian manifold with metric $\langle \cdot, \cdot \rangle$. Let $\mathcal{X}(M)$ denote the set of smooth tangent vector fields on M, i.e., the set of mappings $x \in M \mapsto X(x) \in T_x M$. Given a local chart $(x_1, \ldots, x_d, \ldots, x_D) \mapsto x \in M$, a vector field $X \in \mathcal{X}(M)$ is locally defined as $X = \sum_d x_d \partial_d$, where ∂_d denotes the directional derivative operator along the i-th coordinate. Given $X, Y \in \mathcal{X}(M)$, the Lie bracket between two vector fields is defined (in local coordinates) as the new vector field

$$[X, Y] = \sum_d (X(y_d) - Y(x_d))\partial_d \in \mathcal{X}(M), \tag{1}$$

where $X(f)$ denotes the result of "applying" X on a smooth function f, i.e., of computing the directional derivative of $f$in the direction given by X at each point of M.

Given the metric, one can obtain the Levi-Civita connection $\nabla^M_X Y$, where $X, Y \in \mathcal{X}(M)$. By definition, the Levi-Civita connection is the unique affine symmetric connection compatible with the metric (see [1] for the precise definition of these properties). Given the Levi-Civita connection, the curvature tensor $R(X, Y)Z$ is given by

$$\mathcal{R}^M(X, Y)Z = \nabla^M_Y \nabla^M_X Z - \nabla^M_X \nabla^M_Y Z + \nabla^M_{[X,Y]} Z. \tag{2}$$

The Ricci curvature along a vector field X can be found by contracting the curvature tensor as follows:

$$\mathrm{Ric}^M(X, X) = \sum_d \langle \mathcal{R}^M(X, E_d)X, E_d \rangle, \tag{3}$$

where $E_d = \partial_d$ in some local coordinate chart. The quadratic form associated with (3) can be found using the fact that $\mathcal{R}^M$ and the metric are multilinear in their arguments, which implies the polarization identity:

$$\mathrm{Ric}^M(X, Y) = \frac{1}{4}\big(\mathrm{Ric}^M(X + Y, X + Y) - \mathrm{Ric}^M(X - Y, X - Y)\big) \tag{4}$$

The matrix form of the Ricci curvature (in some local coordinate chart) can be found by computing the i,j-th element as $\mathrm{Ric}^M(E_i, E_j)$.

We are now ready to prove Proposition 3.1.

2 Proof of Proposition 1

Consider $SE(3)$ as a Riemannian manifold with the metric defined in §2 of the main paper. As a notational convention, we decompose a vector field $X \in \mathcal{X}(SE(3))$, as $X = (X_R, X_T)$, where $X_R \in \mathcal{X}(SO(3))$ and $X_T \in \mathcal{X}(\mathbb{R}^3)$. The main idea of the proof is to explicitly compute the matrix form of $\mathrm{Ric}^{SE(3)}(X,Y)$ starting from the connection.

From [2], we have that the Levi-Civita connection on $SE(3)$ is given by

$$\nabla^{SE(3)}_X Y = (\nabla^{SO(3)}_{X_R} Y_R, \nabla^{\mathbb{R}^3}_{X_T} Y_T). \tag{5}$$

In other words, since we consider $SE(3)$ as a product manifold, the connection decomposes into the connections of the two component spaces. For $\mathbb{R}^3$, one can verify that the Levi-Civita connection is simply given by

$$\nabla^{\mathbb{R}^3}_{X_T} Y_T = \sum_d X_T(y_{Ti})\partial_d, \tag{6}$$

i.e., each component of Y_T is differentiated independently. For $SO(3)$, from [1, p. 103], we have

$$\nabla^{SO(3)}_{X_R} Y_R = \frac{1}{2}[X_R, Y_R]. \tag{7}$$

Note that, on $SO(3)$, the Lie bracket between two tangent vectors at a point $R \in SO(3)$ can be computed as

$$[X_R, Y_R]_R = R[R^T X_R, R^T Y_R]_I, \tag{8}$$

where $[A,B]_I$, the Lie bracket at the identity, is given by the simple matrix operation $[A,B]_I = XY - YX$, where $A, B \in \mathfrak{so}(3)$.

From (5), the curvature tensor on $SE(3)$ is given by

$$\mathcal{R}^{SE(3)}(X,Y)Z = \big(\mathcal{R}^{SO(3)}(X_R,Y_R)Z_R, \mathcal{R}^{\mathbb{R}^3}(X_T,Y_T)Z_T\big). \tag{9}$$

For $\mathbb{R}^3$, note that $\nabla^{\mathbb{R}^3}_{X_T}\nabla^{\mathbb{R}^3}_{Y_T} Z_T = \sum_d X\big(Y(z_d)\big)\partial_d$. Then, the curvature tensor vanishes because, from the definition of $[X,Y]$ in (1),

$$\mathcal{R}^{\mathbb{R}^3}(X_T,Y_T)Z_T = \sum_d Y\big(Z(z_d)\big)\partial_d - \sum_d X\big(Y(z_d)\big)\partial_d + \sum_d [X,Y](z_d)\partial_d = 0. \tag{10}$$

This is a simple verification of the fact that $\mathbb{R}^3$ has constant zero curvature. For $SO(3)$, again from [1, p. 103], we have

$$\mathcal{R}^{SO(3)}(X_R,Y_R)Z_R = \frac{1}{4}[[X,Y],Z] \tag{11}$$

Using (10), the Ricci curvature on $SE(3)$ reduces to

$$\mathrm{Ric}^{SE(3)}(X,X) = \mathrm{Ric}^{SO(3)}(X_R, X_R) \tag{12}$$

Combining the definition (3) with (11), we have

$$\begin{aligned}\mathrm{Ric}(X,X) &= \sum_{d_1} \frac{1}{8} \operatorname{tr}\big([[X_R, E^R_{d_1}]X_R]^T E^R_{d_1}\big) \\ &= \frac{1}{8} \sum_{d_1}\sum_{d_2}\sum_{d_3} x_{d_2} x_{d_3} \operatorname{tr}\big([[E^R_{d_2}, E^R_{d_1}]E^R_{d_3}]^T E^R_{d_1}\big),\end{aligned} \tag{13}$$

where $E^R_d = \hat{e}_d$. By direct computation, we have

$$\operatorname{tr}\big([[E^R_{d_2}, E^R_{d_1}]E^R_{d_3}]^T E^R_{d_1}\big) = \begin{cases} 2 \text{ if } d_2 = d_3, d_2 \neq d_1, \\ 0 \text{ otherwise.} \end{cases} \tag{14}$$

This implies

$$\mathrm{Ric}(E_i - E_j, E_i - E_j) = 0, \tag{15}$$

$$\mathrm{Ric}(E_i + E_j, E_i + E_j) = \begin{cases} \frac{1}{2} \text{ if } i = j, \\ 0 \text{ otherwise.} \end{cases} \tag{16}$$

Substituting into the polarization identity (4), we have

$$\mathrm{Ric}^{SO(3)}(X_R, Y_R) = \frac{1}{2} I_3. \tag{17}$$

This, together with (12), implies the claim.

2.1 Proof for Proposition 3

For simplicity, we will use the variables X and x instead of $X^{(k)}_{ij}$ and $x^{(k)}_{ij}$. Note that we will need to compute gradients of gradients. In other words, if we have a function $f : SE(3) \to \mathbb{R}$, we define $\nabla f = \operatorname{grad} f$ and we will need to compute $\operatorname{grad} \nabla f$. In order to do this, we first fix a vector $w \in T_{g_i}SE(3)$ and define $f' = \langle \nabla f, w\rangle = \dot{f}(w)$. Then, we compute $\ddot{f}(v, w) \doteq \langle \operatorname{grad} \nabla f, v\rangle = \dot{f}'(v)$. Since v and w are arbitrary, we can then extract $\operatorname{grad} \nabla f$ in a similar way to what is done for the gradient using the definition (with the difference that $\operatorname{grad} \nabla f$ is a matrix, while $\operatorname{grad} f$ is a vector). If f produces values in $\mathbb{R}^D$ instead of $\mathbb{R}$, we consider each component $e_d^T f$ separately and then proceed as before. Going back to the proof of the proposition, we first define X_c as the point X in the camera frame:

$$X_c = R_i^T (X - T_i). \tag{18}$$

Then, we compute its derivative in the direction v.

$$\begin{aligned}\dot{X}_c(v) = \dot{R}_i^T (X - T_i) + R_i^T \dot{T}_i &= \hat{v}_{Ri}^T R_i^T (X - T_i) - R_i^T v_{Ti} \\ &= \hat{X}_c v_{Ri} + R_i^T v_{Ti} \doteq J_X v\end{aligned} \tag{19}$$

For a fixed w, we then compute the d-th component of the derivative of $\dot{X}_c(w)$:

$$e_d^T \ddot{X}_c(v,w) = \frac{\mathrm{d}}{\mathrm{d}t} e_d^T \dot{X}_c(w) = e_d^T \hat{w}_{Ri}^T \hat{v}_{Ri}^T X_c - e_d^T \hat{w}_{Ri}^T R_i^T v_{Ti} - e_d^T \hat{v}_{Ri}^T R_i^T w_{Ti}$$
$$= v_{Ri}^T \hat{X}_c \hat{e}_d w_{Ri} + v_{Ti}^T R_i \hat{e}_d w_{Ri} - v_{Ri}^T \hat{e}_d R_i^T w_{Ti} \doteq v^T H_{Xd} w \quad (20)$$

We then pass to the projected image $x_p = \pi(X_c)$. Let $P = \begin{bmatrix} I_2 \; 0_{2\times 1} \end{bmatrix}$ and $\lambda_c = e_3^T X_c$. Then:

$$x_p = \frac{1}{\lambda_c} P X_c \quad (21)$$

Similarly to what we did for X_c, we compute the two derivatives (note that $x_p \in \mathbb{R}^2$ instead of $\mathbb{R}^3$):

$$\dot{x}_p(v) = \frac{1}{\lambda_c^2} P(\lambda_c I_3 - X_c e_3^T) J_X w \doteq J_x w \quad (22)$$

$$e_d'^T \ddot{x}_p(v,w) = \frac{\mathrm{d}}{\mathrm{d}t} e_d'^T \dot{x}(w) = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\lambda_c^2} e_d'^T P M \dot{X}_c(w)$$
$$= -\frac{2}{\lambda_c^3} e_3^T \dot{X}_c(v) e_d^T M \dot{X}_c(w) + \frac{1}{\lambda_c^2} e_d^T (e_3^T \dot{X}_c(v) I_3 - \dot{X}_c(v) e_3^T) \dot{X}_c(w)$$
$$+ \frac{1}{\lambda_c^2} \sum_{d'=1}^{3} e_d^T M e_{d'} e_{d'}^T \ddot{X}_c(v,w) = \frac{1}{\lambda_c^2} v^T \Bigg(\frac{1}{\lambda_c} J_X^T e_3 e_d^T M J_X$$
$$+ J_X^T (e_3 e_d^T - e_d e_3^T) J_X + \sum_{d'=1}^{3} (e_d^T M e_{d'} H_{Xd'}) \Bigg) w \doteq v^T H_{xd} w \quad (23)$$

Note that we used the fact that $e_d = P^T e_d'$ for $d = 1, 2$. We can now finally compute the gradients for our cost

$$f = \|x_p - x\|^2 \quad (24)$$

$$\dot{f}(v) = (x_p - x)^T \dot{x}_p(v) = (x_p - x)^T J_x v \quad (25)$$

$$\ddot{f}(v,w) = \langle \dot{f}(w), v \rangle = \dot{x}_p(v)^T \dot{x}_p(w) + (x_p - x)^T \ddot{x}_p(v,w)$$
$$= v^T (J_x^T J_x + \sum_{d=1}^{2} (x_p - x)^T e_d H_{xd}) w \quad (26)$$

We can also compute the gradient with respect to the measured image point x by evaluating the directional derivative along a direction $\dot{x} = v_x$ and for a fixed (R, T).

$$v_x^T \operatorname{grad} \dot{f}(w) = -\dot{x}^T \dot{x}_p(w) = -v_x^T J_x w \quad (27)$$

The claim of the proposition then follows by extracting H_{ijk} and J_{ijk} from (26) and (27), respectively.

References

1. do Carmo, M.P.: Riemannian geometry. Birkhäuser, Boston, MA (1992)
2. Ma, Y., Košecká, J., Sastry, S.: Optimization criteria and geometric algorithms for motion and structure estimation. International Journal of Computer Vision 44(3), 219–249 (2001)