Supplementary note for
“Discrete MRF Inference of Marginal Densities for
Non-uniformly Discretized Variable Space”

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Detailed derivation of the MF and BP algorithms

We show here detailed derivation of the new MF and BP algorithms. In our main paper, we present simplified derivation which needs an assumption that the number \( S_i \) of mixtures is the same for all sites, i.e., \( S_i = S \) for any \( i \). In what follows, we present complete derivation of the two algorithms which do not require this assumption; several equations that are omitted in the main paper are also given. Our derivation follows that of conventional MF and BP algorithms in [2].

Derivation of the new MF algorithm

As mentioned in our main paper, MF and BP algorithms find \( P \) that minimizes the following free energy:

\[
F[P] = \langle E \rangle_P - S[P],
\]

(30)

where the first term is the expectation defined as \( \langle E \rangle_P = \int P(x)E(x)dx \), and the second term is the entropy of \( P \), i.e., \( S[P] = -\int P(x)\ln P(x)dx \).

The derivation of MF algorithms start with assuming that the variable of each site \( i \) is independent of that of any other site:

\[
P(x) = \prod_i p_i(x_i).
\]

(31)

Our approach is to represent \( p_i(x_i) \) by a mixture of \( S_i \) rectangular densities as

\[
p_i(x_i) = \sum_{s=1}^{S_i} a_i^s h_i^s(x_i),
\]

(32)

where \( a_i^s \) is a mixing weight and \( h_i^s(x_i) \) is a component rectangular density; the former is defined such that for any \( i \)

\[
\sum_{s=1}^{S_i} a_i^s = 1,
\]

(33)

and the latter is defined such that the supports of any pair of the component densities do not overlap in the variable space, which is expressed as \( X_i^f \cap X_i^r = \emptyset \) using the notations in our main paper.

Using Eqs.(31) and (32), \( \langle E \rangle_P \) reduces as follows:

\[
\langle E \rangle_P = \sum_i \int p_i(x_i)f_i(x_i)dx_i + \sum_{(i,j) \in E} \int \int p_i(x_i)p_j(x_j)f_{ij}(x_i,x_j)dx_idx_j
\]

\[
= \sum_i \sum_s a_i^s \int f_i(x_i)h_i^s(x_i)dx_i + \sum_{(i,j) \in E} \sum_{s,s'} \sum_{t \in F_i} a_i^s a_j^{s'} f_{ij}^{t,s'}
\]

\[
= \sum_i \sum_s a_i^s f_i^s + \sum_{(i,j) \in E} \sum_{s,s'} \sum_{t \in F_i} a_i^s a_j^{s'} f_{ij}^{t,s'}.
\]

(34)

Although the calculation of the entropy of a mixture density is in general intractable, owing to the introduced constraint \( X_i^f \cap X_i^r = \emptyset \), we can reduce \( S[P] \) as follows:

\[
S[P] = -\sum_i \sum_s a_i^s \int h_i^s(x_i) \ln \left( \sum_{s'} a_i^{s'} h_i^{s'}(x_i) \right) dx_i
\]

\[
= -\sum_i \sum_s a_i^s \int h_i^s(x_i) \ln \left( a_i^s h_i^s(x_i) \right) dx_i
\]

\[
= -\sum_i \sum_s a_i^s \ln a_i^s - \sum_i \sum_s a_i^s \int h_i^s(x_i) \ln h_i^s(x_i) dx_i
\]

\[
= -\sum_i \sum_s a_i^s \ln a_i^s + \sum_i \sum_s a_i^s B_i^s.
\]

(35)

Using Eqs.(34) and (35), Eq.(30) is rewritten as

\[
F_{MF}[^a] = \sum_i \sum_s a_i^s f_i^s + \sum_{(i,j) \in E} \sum_{s,s'} \sum_{t \in F_i} a_i^s a_j^{s'} f_{ij}^{t,s'}
\]

\[
+ \sum_{(i,j) \in E} \sum_{s,s'} \sum_{t \in F_i} a_i^s a_j^{s'} f_{ij}^{t,s'} + \sum_i \sum_s a_i^s \ln a_i^s.
\]
We then wish to find $P$ minimizing this free energy. The minimization is performed with respect to $\alpha_i$’s under the constraint of Eq.(33). Thus, introducing Lagrange multipliers $\gamma_i$, we consider minimizing

$$J_{MF} = F_{MF}[\alpha] + \sum_i \gamma_i \left(1 - \sum_s \alpha_i^s\right).$$

(36)

The calculation of $\partial J_{MF}/\partial \alpha_i^s = 0$ followed by elimination of $\gamma_i$’s yields the following fixed point equation:

$$\alpha_i^s \propto \exp\left[-\left(\langle f_i^s \rangle - B_i^s\right) + \sum_{s'\neq s} \gamma_s \sum_{j} \langle f_{ij}^{s',s} \rangle\right].$$

(37)

Our MF algorithm iteratively updates $\alpha_i^s$’s according to this equation.

**Derivation of the new BP algorithm**

The derivation of BP algorithms starts with assuming

$$P(x) = \frac{\prod_i p_i(x_i, x_j)}{\prod_j p_j(x_j)^{N_j}}.$$  

(38)

Using this, $\langle E \rangle_P$ reduces to

$$\langle E \rangle_P = \sum_i \int p_i(x_i)f_i(x_i)dx_i + \sum_{(i,j)\in E} \int \int p_i(x_i, x_j)f_{ij}(x_i, x_j)dx_idx_j.$$  

(39)

Similarly, $S[P]$ reduces to

$$S[P] = \sum_i (z_i - 1) \int p_i(x_i) \ln p_i(x_i)dx_i - \sum_{(i,j)\in E} \int \int p_i(x_i, x_j) \ln p_{ij}(x_i, x_j)dx_idx_j.$$  

(40)

(The free energy consisting of Eqs.(39) and (40) is called the Bethe Free Energy.)

Our approach is to represent $p_i$ and $p_{ij}$ as

$$p_i(x_i) = \sum_{s=1}^{S_i} \alpha_i^s h_i^s(x_i),$$

$$p_{ij}(x_i, x_j) = \sum_{s=1}^{S_i} \sum_{s'\in E_{j}} \alpha_{ij}^{s,s'} h_i^s(x_i) h_j^{s'}(x_j),$$

(41a, 41b)

where $\alpha_i^s$ and $\alpha_{ij}^{s,s'}$ are the parameters to be determined such that they satisfy the following three constraints:

$$\sum_s \alpha_i^s = 1,$$

$$\sum_{s,j} \alpha_{ij}^{s,s'} = 1,$$

$$\sum_j \alpha_{ij}^{s,s'} = \alpha_i^s.$$  

(42a, 42b, 42c)

By substituting Eqs.(41) into Eq.(39), we have

$$\langle E \rangle_P = \sum_i \sum_s \alpha_i^s \int f_i(x_i) h_i^s(x_i) dx_i + \sum_{(i,j)\in E} \sum_{s,s'} \alpha_{ij}^{s,s'} \int \int f_{ij}(x_i, x_j) h_i^s(x_i) h_j^{s'}(x_j) dx_i dx_j,$$

$$= \sum_i \sum_s \alpha_i^s f_i^s + \sum_{(i,j)\in E} \sum_{s,s'} \alpha_{ij}^{s,s'} f_{ij}^{s,s'}.$$  

(43)

To derive the entropy $S[P]$, we calculate the entropies of $p_i$ and $p_{ij}$, respectively. Using Eq.(41a), the (negative) entropy of $p_i$ is written as

$$\int p_i(x_i) \ln p_i(x_i) dx_i = \sum_s \alpha_i^s \ln \alpha_i^s - \sum_s \alpha_i^s B_i^s,$$

(44)

where we use $B_i^s$ defined in Eq.(16) in our main paper. Similarly, Using Eq.(41b), that of $p_{ij}$ is written as

$$\sum_{i,j} \alpha_{ij}^{s,s'} \int \int h_i^s(x_i) h_j^{s'}(x_j) \ln \left( \sum_{s,s''} \alpha_{ij}^{s,s''} h_i^s(x_i) h_j^{s''}(x_j) \right) dx_i dx_j,$$

$$= \sum_{i,j} \sum_{s,s'} \alpha_{ij}^{s,s'} \int \int h_i^s(x_i) h_j^{s'}(x_j) \ln \left( \alpha_{ij}^{s,s'} h_i^s(x_i) h_j^{s'}(x_j) \right) dx_i dx_j,$$

$$= \sum_{i,j} \alpha_{ij}^{s,s'} \ln \alpha_{ij}^{s,s'} - \sum_{i,j} \alpha_{ij}^{s,s'} (B_i^s + B_j^{s'}).$$

(45)

Using these, the entire entropy $S[P]$ is given as

$$S[p] = \sum_i (z_i - 1) \sum_s \alpha_i^s \ln \alpha_i^s - \sum_i (z_i - 1) \sum_s \alpha_i^s B_i^s - \sum_{(i,j)\in E} \sum_s \alpha_{ij}^{s,s'} \ln \alpha_{ij}^{s,s'} + \sum_{(i,j)\in E} \sum_s \alpha_{ij}^{s,s'} (B_i^s + B_j^{s'}).$$

(46)

Then, using Eq.(43) and Eq.(46), $F[P]$ is rewritten as

$$F_{BP}[\alpha] = \sum_i \sum_s \alpha_i^s (f_i^s + (z_i - 1)B_i^s) + \sum_{(i,j)\in E} \sum_{s,s'} \alpha_{ij}^{s,s'} (f_{ij}^{s,s'} - B_i^s - B_j^{s'}).$$

(47)

To simplify the expression, we define

$$\hat{E}_{ij}^{s,s'} \equiv \langle f_{ij}^{s,s'} \rangle - B_i^s - B_j^{s'},$$

$$E_i^s \equiv f_i^s + (z_i - 1)B_i^s,$$

(48a, 48b)

The substitution of them into Eq.(47) yields

$$F_{BP}[\alpha] = \sum_{(i,j)\in E} \sum_s \alpha_{ij}^{s,s'} (\hat{E}_{ij}^{s,s'} + \ln \alpha_{ij}^{s,s'}) - \sum_i (z_i - 1) \sum_s \alpha_i^s (E_i^s + \ln \alpha_i^s).$$

(49)
We then minimize Eq.(49) with respect to $\alpha^s_i$’s and $\alpha^t_{ij}$’s under the three constraints of Eqs.(42). By introducing Lagrange multipliers $\lambda^t_{ij}$, $\gamma_i$, and $\gamma_{ij}$, we consider minimizing

$$J_{BP} \equiv F_{BP}[\alpha]$$

$$+ \sum_{(i,j) \in E} \sum_{t} \lambda^t_{ij} \left( \alpha^t_{ij} - \sum_s \alpha^s_{ij} \right)$$

$$+ \sum_t \gamma_t \left( 1 - \sum_s \alpha^s_{ij} \right) + \sum_{(i,j) \in E} \sum_s \gamma_{ij} \left( 1 - \sum_t \alpha^t_{ij} \right)$$

(50)

The calculation of $\partial J_{BP}/\partial \alpha^t_{ij} = 0$ yields

$$\ln \alpha^t_{ij} = -\mathcal{E}^t_{ij} + \lambda^t_{ij} + \lambda^s_{ji} + \gamma_{ij} - 1.$$  \hspace{1cm} (51)

Similarly, the calculation of $\partial J_{BP}/\partial \alpha^s_i = 0$ yields

$$(z_i - 1)(\ln \alpha^s_i + 1) = -(z_i - 1)\mathcal{E}^s_i + \lambda^s_{ji} + \gamma_i.$$  \hspace{1cm} (52)

Introducing messages $m^t_{kj}$, are redefined the Lagrange multiplier $\lambda^t_{ij}$ as

$$\lambda^t_{ij} = \ln \prod_{k \in N \setminus i} m^t_{kj}.$$  \hspace{1cm} (53)

Then, by substituting Eq.(53) into Eqs.(51) and (52), we have

$$\alpha^s_i \propto \psi^s_i \phi^t_i \prod_{k \in N \setminus i} m^s_{ki} \prod_{j \in N \setminus i} m^t_{ij},$$  \hspace{1cm} (54a)

$$\alpha^t_{ij} \propto \phi^t_i \prod_{k \in N} m^t_{ki}.$$  \hspace{1cm} (54b)

Substituting Eqs.(54a) and (54b) into Eq.(42c), we have the following updating rule of BP:

$$m^t_{ij} \leftarrow \sum_s \phi^t_s \psi^t_s \prod_{k \in N \setminus i} m^s_{ki}.$$  

Additional experimental results

We show here additional results of stereo matching obtained by the proposed method. Choosing three images from the Middlebury MRF library, Cloth1, Rocks1, and Flowerpots, we applied the proposed method to them in a similar manner to Aloe shown in the main paper. Fig.6 shows their input images (including Aloe) along with their ground truths; the results obtained by the MAP inference (the $\alpha$-expansion algorithm [1]) are also shown for comparison with our results.

Figs.7–12 show the results formatted in the same way as Aloe in our main paper. Similar to Aloe, it is seen that the mixture densities depict the marginal densities in a finer way with the increasing number of blocks; as compared with the results of the fixed discretization, those of the dynamic discretization draw much finer details even for smaller number of blocks. It is seen from the estimated disparity maps that those of the dynamic discretization tend to be smoother than the fixed discretization, which well agrees with the observation on the estimated marginal densities.

References


Figure 6. Four datasets used in our experiments: Aloe, Cloth1, Rocks1, and Flowerpots. Upper row: Input left images. Middle row: Ground truths. Lower row: Disparity maps estimated by the $\alpha$-expansion algorithm.

Figure 7. Results for Cloth1 of the MF algorithm with the dynamic discretization. Upper row: Disparity maps. Lower row: The mixture of rectangular densities approximating the marginal density at the site of the image pixel (100, 100).

Figure 8. Results for Cloth1 of the BP algorithm with the dynamic discretization. Upper row: Disparity maps. Lower row: The mixture of rectangular densities approximating the marginal density at the site of the image pixel (100, 100).
Figure 9. Results for Rocks1 of the MF algorithm with the dynamic discretization. Upper row: Disparity maps. Lower row: The mixture of rectangular densities approximating the marginal density at the site of the image pixel (100, 100).

Figure 10. Results for Rocks1 of the BP algorithm with the dynamic discretization. Upper row: Disparity maps. Lower row: The mixture of rectangular densities approximating the marginal density at the site of the image pixel (100, 100).

Figure 11. Results for Flowerpots of the MF algorithm with the dynamic discretization. Upper row: Disparity maps. Lower row: The mixture of rectangular densities approximating the marginal density at the site of the image pixel (100, 100).
Figure 12. Results for *Flowerpots* of the BP algorithm with the dynamic discretization. Upper row: Disparity maps. Lower row: The mixture of rectangular densities approximating the marginal density at the site of the image pixel (100, 100).