Unnatural L_0 Sparse Representation for Natural Image Deblurring Supplementary Material

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http://www.cse.cuhk.edu.hk/leojia/projects/l0deblur/

New Sparsity Function

In this supplementary file, we provide more details about the new measure that approximates L_0 sparsity during optimization.

Given an input image z, the new sparsity measure is applied to image gradient vectors $\partial_* z$ to regularize the high frequency part, where $* \in \{h, v\}$ denoting two directions. The function is

$$\phi_0(\partial_* z) = \sum_i \phi(\partial_* z_i), \tag{1}$$

where

$$\phi(\partial_* z_i; \epsilon) = \begin{cases} \frac{1}{\epsilon^2}|\partial_* z_i|^2, & \text{if } |\partial_* z_i| \le \epsilon \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

$\phi(\cdot)$ is a concatenation of two functions – one is a quadratic penalty and the other is a constant. i indexes pixels. One example of the penalty function is shown in Fig. 1(a), with its shape very well approximating L_0 penalty when ϵ is small.

During optimization, we use another form of Eq. (2), which is defined as

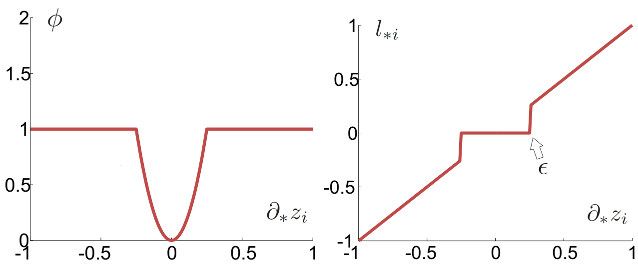
$$\phi(\partial_* z_i; \epsilon) = \min_{l_{*i}} \left\{ |l_{*i}|^0 + \frac{1}{\epsilon^2}(\partial_* z_i - l_{*i})^2 \right\}, \tag{3}$$

where $* \in \{h, v\}$. Each $l_{*i} \in \mathbb{R}$ and each $|l_{*i}|^0$ is a number with the zero power – that is, $|l_{*i}|^0 = 1$ if $l_{*i} \neq 0$ and $|l_{*i}|^0 = 0$ otherwise.

We give the closed-form solution to the problem defined in Eq. (3) in what follows and also show the equivalence between Eqs. (2) and (3).

Claim 1. *The function defined in Eq. (3) taking the form $f(l_{*i}) = |l_{*i}|^0 + 1/\epsilon^2(\partial_* z_i - l_{*i})^2$ has a closed-form solution through hard thresholding as*

$$l_{*i} = \begin{cases} 0, & |\partial_* z_i| \le \epsilon; \\ \partial_* z_i, & \textit{otherwise} \end{cases} \tag{4}$$



(a) (b)

Figure 1. Plots of new sparsity function (a) and the hard thresholding (b).

Proof. If $|\partial_* z_i| \le \epsilon$, we compare the output from $|l_{*i}|^0$ and $\frac{1}{\epsilon^2}(\partial_* z_i - l_{*i})^2$. If l_{*i} is not 0, it must hold that

$$|l_{*i}|^0 + \frac{1}{\epsilon^2}(\partial_* z_i - l_{*i})^2 > 1.$$

If $l_{*i} = 0$,

$$|l_{*i}|^0 + \frac{1}{\epsilon^2}(\partial_* z_i - l_{*i})^2 = \frac{1}{\epsilon^2}(\partial_* z_i)^2 < 1.$$

So the minimum is reached with $l_{*i} = 0$.

Similarly, if $|\partial_* z_i| > \epsilon$, we compare the output from $|l_{*i}|^0$ and $\frac{1}{\epsilon^2}(\partial_* z_i - l_{*i})^2$. If l_{*i} is not 0, it must hold that

$$\min_{l_{*i}} |l_{*i}|^0 + \frac{1}{\epsilon^2}(\partial_* z_i - l_{*i})^2 = 1,$$

when $\partial_* z_i = l_{*i}$. If $l_{*i} = 0$,

$$|l_{*i}|^0 + \frac{1}{\epsilon^2}(\partial_* z_i - l_{*i})^2 = \frac{1}{\epsilon^2}(\partial_* z_i)^2 > 1.$$

So the minimum is reached with $\partial_* z_i = l_{*i}$ in this case.

Combining the two situations, the final closed-form solution is given by Eq. (4). □

The relationship between l_{*i} and image gradient $\partial_* z_i$ is illustrated in Fig. 1(b).

Claim 2. *With the optimal l_{*i}, the penalty function w.r.t. $\partial_* z_i$ defined in Eq. (3) is equivalent to the function in Eq. (2).*

Proof. With the optimal value of l_{*i} yielded by the hard thresholding in Eq. (4), $\phi(\partial_* z_i; \epsilon)$ output from Eq. (3) is determined by one of the two segments (functions). Specifically, if $|\partial_* z_i| \leq \epsilon$, l_{*i} has been proved to be zero to reach the minimum in Eq. (3). Taking it into Eq. (2), we get the simplified function $\frac{1}{\epsilon^2}|\partial_* z_i|^2$. When $|\partial_* z_i| > \epsilon$, $l_{*i} = \partial_* z_i$ makes the function in (3) also be simplified to (2). □

In our algorithm, we use a family of loss functions by varying ϵ and start from $\epsilon = 1$, which makes the loss function quadratic, taking the fact into consideration that each normalized $|\partial_* z_i|$ is always smaller than or equal to 1. In optimization, the penalty function evolves by decreasing ϵ, gradually but steadily heading towards the L_0 sparsity function realization. **It is a really algorithmically practical, effective and useful technique whenever L_0 sparsity is required.**