Articulated and Restricted Motion Subspaces and Their Signatures

Supplemental Material

Bastien Jacquet[1], Roland Angst[12*], Marc Pollefeys[1]

ETH Zürich[1], Switzerland Stanford University[2], CA, USA

1. Video of Recovered 3D Models

The attached video shows consecutively the results obtained for the laptop example presented in Sec. 9.1 of the paper (*i.e*. one axis rotation), then the blackboard example presented in Sec. 9.2 of the paper (*i.e*. planar motion and two axes rotations), and finally the wheel presented in Sec. 9.3 of the paper (*i.e*. rotation around a translating axis).

As mentioned in the paper, our algorithm enables the visualization of smooth transitions between observed configurations (*i.e*. interpolated configuration) thanks to extracted restricted motion parameters. Generating such novel and unobserved configurations is straight-forward by adapting these motion parameters, since they represent meaningful attributes of the restricted motion.

We are using the same color coding as in the paper, but for completeness we will recall it for each dataset in the following subsection.

1.1. Hinge Joint : Laptop

We see the feature points of two parts of a laptop (screen in red, keyboard in blue) animated relatively to each other. Note that some points on the floor are considered as belonging to the keyboard part since they never moved relatively to that part. The camera poses are shown both relatively to the keyboard and the screen, thus observed configurations correspond to a camera having same pose relatively to both (*i.e*. when they overlap in space).

The animated relative motion is around the computed rotation axis, with angle varying between the minimum and maximum observed angles.

1.2. Planar Motion and Two Axes Rotations : Blackboard

We see a blackboard (feature points in pink) moving relatively to a fixed camera. The axes of rotation are shown in red, the directions of translation are shown in green.

The animated relative motion is with motion parameters corresponding to minimum and maximum observed angles and minimum and maximum observed translations.

1.3. Rotation Around a Translating Axis : Wheel

We see the wheel (feature points in pink) moving relatively to a fixed camera. The axis of rotation is shown in red, the direction of translation is shown in green and the contact line with the ground is shown in blue.

The animated relative motion is rendered using the minimum and maximum observed translations, and the corresponding angles are computed thanks to the extracted radius (as explained in Fig. 4 of the paper) with the relation :

$$\alpha_f = \frac{\tilde{t}_f}{R} \quad .$$

2. Detailed Derivations for Rotations Around Two Axes

In this section, we are going to show that even for rotations around two non intersecting axes (as considered in Sec. 6.2 of the paper), the translation subspace spanned by $\mathbf{M}_{:,10:12}$ is entirely contained in the 8 dimensional rotation subspace spanned by $\mathbf{M}_{:,1:9}$. Analogously to the other two cases in the paper in Sec. 6.1 and Sec. 6.3 this will be shown by stating a matrix $\mathbf{X}$ such that $\mathbf{M}\left[\mathbf{X}^T, \mathbf{I}_3\right]^T = \mathbf{0}$.

We recall that the motion matrix is defined as

$$\mathbf{M} = \left[\Downarrow_f \operatorname{vec}\left(\mathbf{R}_f - \mathbf{I}_3\right)^T, \mathbf{t}_f^T\right] \in \mathbb{R}^{F \times 12}.$$

*This work was done while this author was employed by ETH Zürich

According to the notation in Sec. 6.2 in the paper, the relative transformation around two axes at frame f is given by

$$\begin{bmatrix} \mathbf{R}_f & \mathbf{t}_f \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \mathbf{R}_{\mathbf{b},\beta_f}\mathbf{R}_{\mathbf{a},\alpha_f} & \mathbf{t}_b + \mathbf{R}_{\mathbf{b},\beta_f}\left(-\mathbf{t}_b + \mathbf{t}_a - \mathbf{R}_{\mathbf{a},\alpha_f}\mathbf{t}_a\right) \\ \mathbf{0}^T & 1 \end{bmatrix}. \tag{2}$$

The translation can be rewritten as

$$\begin{aligned} \mathbf{t}_f^T = &- \operatorname{vec}\left(\mathbf{R}_{\mathbf{b},\beta_f} - \mathbf{I}_3\right)^T [\mathbf{t}_b \otimes \mathbf{I}_3] \\ &- \operatorname{vec}\left(\mathbf{R}_{\mathbf{b},\beta_f}\left(\mathbf{R}_{\mathbf{a},\alpha_f} - \mathbf{I}_3\right)\right)^T [\mathbf{t}_a \otimes \mathbf{I}_3]. \end{aligned} \tag{3}$$

Interestingly for rotations around two axes, the subspace spanned by the rotational part $\operatorname{span}(\mathbf{M}_{:,1:9})$ can be decomposed into disjoint subspaces $\mathcal{A}$ and $\mathcal{B}$ spanned by the two matrices $\left[\Downarrow_f \operatorname{vec}\left(\mathbf{R}_{\mathbf{b},\beta_f}\left(\mathbf{R}_{\mathbf{a},\alpha_f} - \mathbf{I}_3\right)\right)^T\right]$ and $\left[\Downarrow_f \operatorname{vec}\left(\mathbf{R}_{\mathbf{b},\beta_f} - \mathbf{I}_3\right)^T\right]$ of rank-6 resp. rank-2, *i.e.*

$$\operatorname{span}(\mathbf{M}_{:,1:9}) = \mathcal{A} + \mathcal{B}. \tag{4}$$

This decomposition together with Eq. (3) already shows that the translation subspace $\operatorname{span}([\Downarrow_f \mathbf{t}_f])$ is entirely contained in the rotation subspace. The decomposition can be derived by plugging the axis-angle representation for $\mathbf{R}_{\mathbf{b},\beta_f}$ and $\mathbf{R}_{\mathbf{a},\alpha_f}$ into the matrices $\mathbf{R}_{\mathbf{b},\beta_f}\mathbf{R}_{\mathbf{a},\alpha_f} - \mathbf{I}_3$, vectorizing, and separating time-varying coefficients from static terms. This leads to

$$\left[\Downarrow_f \operatorname{vec}\left(\mathbf{R}_{\mathbf{b},\beta_f}\left(\mathbf{R}_{\mathbf{a},\alpha_f} - \mathbf{I}_3\right)\right)^T\right] = \tag{5}$$

$$\big[\Downarrow_f c_{\beta_f}(1 - c_{\alpha_f}), c_{\beta_f}s_{\alpha_f}, (1 - c_{\beta_f})(1 - c_{\alpha_f}), \dots \tag{6}$$

$$(1 - c_{\beta_f})s_{\alpha_f}, s_{\beta_f}(1 - c_{\alpha_f}), s_{\beta_f}s_{\alpha_f}\big]\,\mathbf{A}, \tag{7}$$

and

$$\left[\Downarrow_f \operatorname{vec}\left(\mathbf{R}_{\mathbf{b},\beta_f} - \mathbf{I}_3\right)^T\right] = \left[\Downarrow_f 1 - c_{\beta_f}, s_{\beta_f}\right]\mathbf{B} \tag{8}$$

where $c_{\beta_f} = \cos\beta_f$, $c_{\alpha_f} = \cos\alpha_f$, $s_{\beta_f} = \sin\beta_f$, $s_{\alpha_f} = \sin\alpha_f$, and

$$\mathbf{A} = \begin{bmatrix} \operatorname{vec}\left(\mathbf{a}\mathbf{a}^T - \mathbf{I}_3\right)^T \\ \operatorname{vec}\left([\mathbf{a}]_\times\right)^T \\ \operatorname{vec}\left(\mathbf{b}\mathbf{b}^T\left(\mathbf{a}\mathbf{a}^T - \mathbf{I}_3\right)\right)^T \\ \operatorname{vec}\left(\mathbf{b}\mathbf{b}^T[\mathbf{a}]_\times\right)^T \\ \operatorname{vec}\left([\mathbf{b}]_\times\left(\mathbf{a}\mathbf{a}^T - \mathbf{I}_3\right)\right)^T \\ \operatorname{vec}\left([\mathbf{b}]_\times[\mathbf{a}]_\times\right)^T \end{bmatrix} \in \mathbb{R}^{6\times 9} \tag{9}$$

$$\mathbf{B} = \begin{bmatrix} \operatorname{vec}\left(\mathbf{b}\mathbf{b}^T - \mathbf{I}_3\right)^T \\ \operatorname{vec}\left([\mathbf{b}]_\times\right)^T \end{bmatrix} \in \mathbb{R}^{2\times 9}. \tag{10}$$

Therefore, we make the following choice for $\mathbf{X}$

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}^\dagger \left[\begin{bmatrix} \mathbf{I}_6 \\ \mathbf{0}_{2\times 6} \end{bmatrix}\mathbf{A}[\mathbf{t}_a \otimes \mathbf{I}_3] + \begin{bmatrix} \mathbf{0}_{6\times 2} \\ \mathbf{I}_2 \end{bmatrix}\mathbf{B}[\mathbf{t}_b \otimes \mathbf{I}_3]\right], \tag{11}$$

where $\mathbf{A}^\dagger$ denotes the Moore-Penrose pseudo-inverse. This choice will fulfill the Eq. 5 of the paper : $\mathbf{M}\left[\mathbf{X}^T, \mathbf{I}_3\right]^T = \mathbf{0}$.