Second order cross-derivative of the approximation of the Hausdorff distance used in *Shape statistics for image segmentation with priors*

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Abstract

We need the literal expression of the second order derivative of the approximation of the Hausdorff distance with respect to the curves in order to include into the usual image segmentation framework an a priori term involving second-order statistics of shapes [2]. More exactly, we need the expression of $\partial_B \partial_A d_H(A, B)$*. This is the subject of the four following pages.*

This approximation of the Hausdorff distance and its gradient were introduced in [1].

References

[1] G. Charpiat, O. Faugeras, and R. Keriven. Approximations of shape metrics and application to shape warping and empirical shape statistics. *Foundations of Computational Mathematics*, 5(1):1–58, February 2005.

[2] G. Charpiat, O. Faugeras, and R. Keriven. Shape statistics for image segmentation with prior. In *Proc. of the IEEE Int. Conf. of Computer Vision and Pattern Recognition (CVPR'07)*, 2007.

1 Notations

A, B	two planar curves
$d(x,y)$	usual Euclidean distance between points x and y
ϕ	a strictly decreasing application $\mathbb{R}^+ \mapsto \mathbb{R}^+$
ψ, Ψ	strictly increasing applications $\mathbb{R}^+ \mapsto \mathbb{R}^+$
Φ	$\dfrac{\psi'}{\phi'}$
Θ	$\dfrac{\Psi'}{\psi'}$
$\langle f(x) \rangle_{x \in A}$, $\langle f(\cdot) \rangle_A$	$\dfrac{1}{\lvert A \rvert} \displaystyle\int_A f(x)\, dx$ (mean of application f on curve A)
$\sigma_A(f(\cdot), y)$	$\langle f \rangle_A - f(y)$ (deviation of application f at point y of curve A)
$\phi \circ d(\cdot)$	$\phi\,(d(\cdot))$
$\langle d \rangle_A^\phi$	$\phi^{-1}\left(\langle \phi \circ d(\cdot) \rangle_B\right)$ mean on A of application $d(\cdot)$ in the sense of function ϕ
$\langle a, b \rangle^\Psi$	$\Psi^{-1}\left(\dfrac{1}{2}\Psi(a) + \dfrac{1}{2}\Psi(b)\right)$ mean of two real values in the sense of function Ψ
$\Phi_{B\phi}(y)$	$\Phi\left(\langle d(\cdot, y) \rangle_B^\phi\right)$
$\Theta_{B\phi A\psi}$	$\Theta\left(\left\langle \langle d(\cdot,\cdot) \rangle_B^\phi \right\rangle_A^\psi\right) = \Theta\left(\left\langle \langle d(x,y) \rangle_{x \in B}^\phi \right\rangle_{y \in A}^\psi\right)$
$\overrightarrow{D}(y,x)$	$\dfrac{y-x}{d(x,y)}\ \phi'(d(x,y))$
$d_H(A,B)$	$\left\langle \left\langle \langle d(\cdot,\cdot) \rangle_A^\phi \right\rangle_B^\psi , \left\langle \langle d(\cdot,\cdot) \rangle_B^\phi \right\rangle_A^\psi \right\rangle^\Psi$ smooth approximation of the Hausdorff distance
$\overrightarrow{n}(y)$	(unit vector) normal to the current curve at point y
$\kappa(y)$	curvature of the current curve at point y
$\xi(a)$	$1 - a\,\dfrac{\Psi''(a)}{\Psi'(a)}$

$$
\begin{aligned}
\mathbf{U}(y,x)(\delta y)(\delta x) \quad &= D_x\left[\overrightarrow{D}(y,x) \cdot \overrightarrow{n}(y)\ \ \overrightarrow{n}(y) \cdot \delta y\right] \cdot \delta x \quad \text{(symmetric)} \\
&= \phi''(d(x,y))\left(\frac{x-y}{d(x,y)} \cdot \delta x\right)\left(\frac{y-x}{d(x,y)} \cdot \delta y\right) \\
&\qquad + \frac{\phi'(d(x,y))}{d(x,y)}\left[-\delta x \cdot \delta y \ + \left(\frac{x-y}{d(x,y)} \cdot \delta x\right)\left(\frac{y-x}{d(x,y)} \cdot \delta y\right)\right]
\end{aligned}
$$

2 First order derivative

The first order derivative of a functional with respect to a curve A is a vector field defined on this curve, that is, an application that associates to each point y of curve A a vector. Note we consider here the derivative of the square of d_H. We recall its expression:

$$
\begin{aligned}
\nabla_A d_H^2(A,B) \quad = \quad & \frac{1}{2|A|}\frac{d_H(A,B)}{\Psi'(d_H(A,B))} \Big[\ \Theta_{B\phi A\psi} \ \ \sigma_A\left(\psi_{B\phi}(\cdot), \ y\right) \ \kappa(y)\overrightarrow{n}(y) \\
& + \ \Big\langle \left(\Phi_{B\phi}(y)\, \Theta_{B\phi A\psi} \ + \ \Phi_{A\phi}(x)\, \Theta_{A\phi B\psi}\right) \ \overrightarrow{D}(y,x)\cdot\overrightarrow{n}(y)\Big\rangle_{x\in B} \overrightarrow{n}(y) \\
& + \ \Theta_{A\phi B\psi} \ \langle \ \Phi_{A\phi}(x) \ \ \sigma_A\left(\phi\circ d(x,\cdot), \ y\right) \ \rangle_{x\in B} \ \kappa(y)\overrightarrow{n}(y) \ \Big]
\end{aligned}
$$

3 Second order derivative

The second order cross-derivative $\partial_B\left[\partial_A\, d_H^2(A,B)\right](\delta A)(\delta B)$ is an application that associates a real value to any couple of two fields δB and δA defined respectively on B and A. This application is supposed to be symmetric ($\partial_A\partial_B = \partial_B\partial_A$ and $d_H(A,B)$ is symmetric).

It can be rewritten as:

$$
\partial_B\left[\left\langle \nabla_A\, d_H^2(A,B)(y)\middle|\, \delta A(y)\right\rangle_{y\in A}\right](\delta B)
$$

hence

$$
\partial_B\left[\nabla_A\, d_H^2(A,B)\right](y)
$$

is an application that associates to any field δB a field defined on A.

With a slight abuse of notations:

$$
\nabla_B\left[\nabla_A\, d_H^2(A,B)\right](y)(z)
$$

is a kind of super-matrix which for any two points y on A and z on B is a 2×2 matrix. For two fields δA and δB,

$$
\nabla_B\left[\nabla_A d_H^2(A,B)\right](y)(z)\,(\delta A(y))\,(\delta B(z))
$$

is a real value for each (y,z). In the sequel, $\nabla_B\left[\nabla_A\, d_H^2(A,B)\right](y)(z)$ will be proportional to the tensor product $\overrightarrow{n}(y)\ \overrightarrow{n}(z)$, where $\overrightarrow{n}(y)$ is to be thought as $\overrightarrow{n}(y)\cdot\delta A(y)$, and $\overrightarrow{n}(z)$ as $\overrightarrow{n}(z)\cdot\delta B(z)$.

Theorem 1. *With the previous notations, the second order cross-derivative of the approximation of the Hausdorff distance is:*

$$
\begin{aligned}
&\nabla_B \left[\nabla_A \, d_H^2(A,B)\right](y)(z) \\
&\quad = \\
&\xi(d_H(A,B)) \quad \nabla_B \left[d_H(A,B)\right](z) \quad \nabla_A \left[d_H(A,B)\right](y) \\
&+ \quad \frac{1}{2|A|\;|B|} \, \frac{d_H(A,B)}{\Psi'(d_H(A,B))} \times \Bigg[\\
&\quad \Theta_{A\phi B\psi} \; \Phi_{A\phi}(z) \;\; \sigma_A\left(\overrightarrow{D}(z,\cdot), \; y\right) \cdot \overrightarrow{n}(z) \; \overrightarrow{n}(z) \quad \kappa(y) \overrightarrow{n}(y) \\
&\quad + \; \Theta_{A\phi B\psi} \;\; \sigma_B\left(\Phi_{A\phi}(\cdot) \, \sigma_A\left(\phi \circ d(\cdot,\cdot), \; y\right), \; z\right) \; \kappa(z) \overrightarrow{n}(z) \quad \kappa(y) \overrightarrow{n}(y) \\
&\quad + \; \Theta_{A\phi B\psi} \left(\frac{\Phi'}{\phi'}\right)_{A\phi} (z) \quad \sigma_A\left(\phi \circ d(z,\cdot), \; y\right) \; \left\langle \overrightarrow{\mathcal{D}}(z,\cdot) \right\rangle_A \cdot \overrightarrow{n}(z) \; \overrightarrow{n}(z) \quad \kappa(y) \overrightarrow{n}(y) \\
&\quad + \; \left(\frac{\Theta'}{\psi'}\right)_{A\phi B\psi} \Phi_{A\phi}(z) \; \left\langle \Phi_{A\phi}(\cdot) \;\; \sigma_A\left(\phi \circ d(\cdot,\cdot), \; y\right)\right\rangle_B \; \left\langle \overrightarrow{\mathcal{D}}(z,\cdot) \right\rangle_A \cdot \overrightarrow{n}(z) \; \overrightarrow{n}(z) \quad \kappa(y) \overrightarrow{n}(y) \\
&\quad + \; \left(\frac{\Theta'}{\psi'}\right)_{A\phi B\psi} \;\; \sigma_B\left(\psi_{A\phi}(\cdot), \; z\right) \; \left\langle \Phi_{A\phi}(\cdot) \;\; \sigma_A\left(\phi \circ d(\cdot,\cdot), \; y\right)\right\rangle_B \;\; \kappa(z) \overrightarrow{n}(z) \quad \kappa(y) \overrightarrow{n}(y) \\
&\quad + \; \Theta_{A\phi B\psi} \left(\frac{\Phi'}{\phi'}\right)_{A\phi} (z) \quad \left\langle \overrightarrow{\mathcal{D}}(z,\cdot) \right\rangle_A \cdot \overrightarrow{n}(z) \; \overrightarrow{n}(z) \quad \overrightarrow{\mathcal{D}}(y,z) \cdot \overrightarrow{n}(y) \; \overrightarrow{n}(y) \\
&\quad + \; \Theta_{A\phi B\psi} \;\; \sigma_B\left(\Phi_{A\phi}(\cdot) \; \overrightarrow{\mathcal{D}}(y,\cdot), \; z\right) \cdot \overrightarrow{n}(y) \; \overrightarrow{n}(y) \;\; \kappa(z) \overrightarrow{n}(z) \\
&\quad + \; \left(\frac{\Theta'}{\psi'}\right)_{A\phi B\psi} \Phi_{A\phi}(z) \; \left\langle \Phi_{A\phi}(\cdot) \;\; \overrightarrow{\mathcal{D}}(y,\cdot) \right\rangle_B \cdot \overrightarrow{n}(y) \; \overrightarrow{n}(y) \;\; \left\langle \overrightarrow{\mathcal{D}}(z,\cdot) \right\rangle_A \cdot \overrightarrow{n}(z) \; \overrightarrow{n}(z) \\
&\quad + \; \left(\frac{\Theta'}{\psi'}\right)_{A\phi B\psi} \;\; \sigma_B\left(\psi_{A\phi}(\cdot), \; z\right) \; \left\langle \Phi_{A\phi}(\cdot) \;\; \overrightarrow{\mathcal{D}}(y,\cdot) \right\rangle_B \cdot \overrightarrow{n}(y) \; \overrightarrow{n}(y) \;\; \kappa(z) \overrightarrow{n}(z) \\
&\quad + \; \Theta_{A\phi B\psi} \; \Phi_{A\phi}(z) \; \mathbf{U}(y,z) \left(\overrightarrow{n}(y)\right) \left(\overrightarrow{n}(z)\right) \\
&\; + \; \textbf{symmetric term} \; (A \mapsto B, \quad y \mapsto z) \\
&\Bigg].
\end{aligned}
$$

4 Calculi

In order to compute $\nabla_B \left(\nabla_A d_H^2(A,B)(y)\right)(z)$, where z is a point of the curve B, the following derivatives have been used:

$$\nabla_B \left(\frac{1}{2|A|} \frac{d_H(A,H)}{\Psi'(d_H(A,B))} \right) = \frac{1}{2|A|} \frac{\Psi'(d_H) - \Psi''(d_H)}{\Psi'(d_H)^2} \;\; \nabla_B d_H(A,B)$$

$$\nabla_B \left(\langle f \rangle_B \right)(y) = \frac{1}{|B|} \left[\sigma_B(f,y)\kappa(y) \; + \; f'(y) \cdot \mathbf{n}(y) \right] \; \mathbf{n}(y)$$

$$\nabla_B \left(\langle f(B,\cdot) \rangle_B \right)(y) = \langle \nabla_1 f(B,x)(y) \rangle_{x \in B} \; + \; \frac{1}{|B|} \left[\sigma_B(f(B,\cdot),y)\kappa(y) \; + \; \nabla_2 f(B,y) \cdot \mathbf{n}(y) \right] \; \mathbf{n}(y)$$

$$\nabla_B \left(\langle d(\cdot,y) \rangle_{B\phi} \right)(z) = \frac{1}{|B| \phi'(\langle d(\cdot,y) \rangle_{B\phi})} \left[\sigma_B \left(\phi \circ d(\cdot,y), z \right) \kappa(z) + \mathbf{U}(z,y) \cdot \mathbf{n}(z) \right] \; \mathbf{n}(z)$$

$$\nabla_B \left(\Phi_{B\phi}(y) \right)(z) = \Phi' \left(\langle d(\cdot,y) \rangle_{B\phi} \right) \;\; \nabla_B \left(\langle d(\cdot,y) \rangle_{B\phi} \right)(z)$$

$$\begin{aligned} \nabla_B \left(\Theta_{B\phi A\psi} \right)(z) \;\; = \;\; & \frac{\Theta'}{\psi'} \left(\left\langle \langle d \rangle_{B\phi} \right\rangle_{A\psi} \right) \;\; \times \\ & \left\langle \frac{1}{|B|} \Phi_{B\phi}(y) \;\left[\; \sigma_B \left(\phi \circ d(\cdot,y), z \right) \kappa(z) \; + \; \mathbf{U}(z,y) \cdot \mathbf{n}(z) \;\right] \right\rangle_{y \in A} \;\; \mathbf{n}(z) \end{aligned}$$

$$\begin{aligned} \nabla_B \left(\Theta_{A\phi B\psi} \right)(z) \;\; = \;\; & \frac{\Theta'}{\psi'} \left(\left\langle \langle d \rangle_{A\phi} \right\rangle_{B\psi} \right) \;\; \times \\ & \frac{1}{|B|} \left[\sigma_B \left(\psi \left(\langle d(\cdot,\cdot) \rangle_{A\phi} \right), z \right) \kappa(z) \; + \; \Phi_{A\phi}(z) \left\langle \mathbf{U}(z,y) \cdot \mathbf{n}(z) \right\rangle_{y \in A} \right] \mathbf{n}(z) \end{aligned}$$

$\mathbf{U}$ is a derivative: $\mathbf{U}(y,x) \cdot (\delta y) = \nabla_y \left(\phi(d(y,x)) \right)(\delta y)$ so $\nabla_x \left(\mathbf{U}(y,x) \right)$ is a second order (symmetric) derivative:

$$\begin{aligned} \nabla_x \left(\mathbf{U}(y,x) \right)(\delta x)(\delta y) \;\; = \;\; & \phi''(d(x,y)) \left(\frac{x-y}{d(x,y)} \cdot \delta x \right) \left(\frac{y-x}{d(x,y)} \cdot \delta y \right) \\ & + \frac{\phi'(d(x,y))}{d(x,y)} \left[-\delta x \cdot \delta y \; + \; \left(\frac{x-y}{d(x,y)} \cdot \delta x \right) \left(\frac{y-x}{d(x,y)} \cdot \delta y \right) \right] \end{aligned}$$

$$\nabla_x \left(\sigma_A \left(\phi(d(x,\cdot)), y \right) \right) = \langle \mathbf{U}(x,z) \rangle_{z \in A} - \mathbf{U}(x,y) = \sigma_A \left(\mathbf{U}(x,\cdot), y \right)$$