Supplementary Material for Paper Titled
A Theoretical Analysis of Linear and Multi-linear Models of Image Appearance
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CVPR 2008

Derivation of (17) in the main paper

As

$$\mathcal{N} = \frac{\frac{\partial \mathcal{C}}{\partial u} \times \frac{\partial \mathcal{C}}{\partial v}}{\|\frac{\partial \mathcal{C}}{\partial u} \times \frac{\partial \mathcal{C}}{\partial v}\|} = \frac{\mathcal{C}_u \times \mathcal{C}_v}{\|\mathcal{C}_u \times \mathcal{C}_v\|} = \frac{\hat{\mathcal{C}}_u \mathcal{C}_v}{\sqrt{\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v}}, \tag{1}$$

where $\hat{\mathcal{C}}$ denote the skew symmetric matrix with entries $\begin{pmatrix} 0 & -\mathcal{C}^{(3)} & \mathcal{C}^{(2)} \\ \mathcal{C}^{(3)} & 0 & -\mathcal{C}^{(1)} \\ -\mathcal{C}^{(2)} & \mathcal{C}^{(1)} & 0 \end{pmatrix}$, and the superscript $\mathcal{C}^{(1)}$ indicates the first dimension of the vector. Taking the partial derivative of $\mathcal{N}$ with respect to t, we have

$$\frac{\partial \mathcal{N}}{\partial t} = \frac{\frac{\partial \hat{\mathcal{C}}_u}{\partial t}\mathcal{C}_v + \hat{\mathcal{C}}_u \frac{\partial \mathcal{C}_v}{\partial t}}{\sqrt{\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v}} - \frac{\hat{\mathcal{C}}_u \mathcal{C}_v \frac{\partial(\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v)}{\partial t}}{2(\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v)^{\frac{3}{2}}}. \tag{2}$$

Taking the partial derivative of (4) in the main paper with respect to u and v, and assuming $\frac{\partial^2 \mathcal{C}}{\partial u \partial t}$ and $\frac{\partial^2 \mathcal{C}}{\partial t \partial u}$ exist and are smooth (which is assumption **A3**), we have

$$\begin{aligned} \frac{\partial^2 \mathcal{C}}{\partial u \partial t} &= \frac{\partial \beta}{\partial u}\mathcal{N} + \beta \frac{\partial \mathcal{N}}{\partial u} = \beta_u \mathcal{N} + \beta \mathcal{N}_u = \frac{\partial \mathcal{C}_u}{\partial t}, \\ \frac{\partial^2 \mathcal{C}}{\partial v \partial t} &= \frac{\partial \beta}{\partial v}\mathcal{N} + \beta \frac{\partial \mathcal{N}}{\partial v} = \beta_v \mathcal{N} + \beta \mathcal{N}_v = \frac{\partial \mathcal{C}_v}{\partial t}. \end{aligned} \tag{3}$$

As the skew symmetric matrix $\hat{\mathcal{C}}$ is linear with respect to the original vector $\mathcal{C}$, we have

$$\begin{aligned} \frac{\partial \hat{\mathcal{C}}_u}{\partial t} &= \beta_u \hat{\mathcal{N}} + \beta \hat{\mathcal{N}}_u, \\ \frac{\partial \hat{\mathcal{C}}_v}{\partial t} &= \beta_v \hat{\mathcal{N}} + \beta \hat{\mathcal{N}}_v. \end{aligned} \tag{4}$$

Substitute (3) and (4) back into the numerator of the first term in the right hand side of (2), we have

$$\begin{aligned} \frac{\partial \hat{\mathcal{C}}_u}{\partial t}\mathcal{C}_v + \hat{\mathcal{C}}_u \frac{\partial \mathcal{C}_v}{\partial t} &= (\beta_u \hat{\mathcal{N}} + \beta \hat{\mathcal{N}}_u)\mathcal{C}_v + \hat{\mathcal{C}}_u(\beta_v \mathcal{N} + \beta \mathcal{N}_v) \\ &= \beta_u \hat{\mathcal{N}} \mathcal{C}_v + \beta \hat{\mathcal{N}}_u \mathcal{C}_v + \hat{\mathcal{C}}_u \beta_v \mathcal{N} + \hat{\mathcal{C}}_u \beta \mathcal{N}_v \\ &= \beta_u \mathcal{N} \times \mathcal{C}_v + \beta \mathcal{N}_u \times \mathcal{C}_v + \beta_v \mathcal{C}_u \times \mathcal{N} + \beta \mathcal{C}_u \times \mathcal{N}_v. \end{aligned} \tag{5}$$

Similarly, the numerator of the second term in the right hand side of (2) can be simplified as

$$\begin{aligned} \frac{\partial(\mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v)}{\partial t} &= (\beta_v \mathcal{N}^{\mathbf{T}} + \beta \mathcal{N}_v^{\mathbf{T}})\hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{C}_v + (\beta_u \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{N}}^{\mathbf{T}} + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{N}}_u^{\mathbf{T}})\hat{\mathcal{C}}_u \mathcal{C}_v + (\beta_u \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{N}} + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{N}}_u)\mathcal{C}_v \\ &\quad + (\beta_v \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{N} + \beta \mathcal{C}_v^{\mathbf{T}} \hat{\mathcal{C}}_u^{\mathbf{T}} \hat{\mathcal{C}}_u \mathcal{N}_v). \end{aligned} \tag{6}$$

Note that

$$\mathcal{N}^{\mathbf{T}}\hat{\mathcal{C}}_u^{\mathbf{T}}\hat{\mathcal{C}}_u\mathcal{C}_v = (\mathcal{C}_u \times \mathcal{N})^{\mathbf{T}}(\mathcal{C}_u \times \mathcal{C}_v). \tag{7}$$

Because $\mathcal{C}_u \parallel \mathcal{T}_u$ and $\mathcal{C}_v \parallel \mathcal{T}_v$, thus $(\mathcal{C}_u \times \mathcal{N})\perp\mathcal{N}$ while $(\mathcal{C}_u \times \mathcal{C}_v) \parallel \mathcal{N}$. Consequently, the inner product between the two terms in (7) is zero. Similarly, we have

$$\mathcal{C}_v^{\mathbf{T}}\hat{\mathcal{N}}^{\mathbf{T}}\hat{\mathcal{C}}_u\mathcal{C}_v = \mathcal{C}_v^{\mathbf{T}}\hat{\mathcal{C}}_u^{\mathbf{T}}\hat{\mathcal{N}}\mathcal{C}_v = \mathcal{C}_v^{\mathbf{T}}\hat{\mathcal{C}}_u^{\mathbf{T}}\hat{\mathcal{C}}_u\mathcal{N} = 0. \tag{8}$$

Thus, (6) can be simplified as

$$\begin{aligned}
& \beta\mathcal{N}_v^{\mathbf{T}}\hat{\mathcal{C}}_u^{\mathbf{T}}\hat{\mathcal{C}}_u\mathcal{C}_v + \beta\mathcal{C}_v^{\mathbf{T}}\hat{\mathcal{N}}_u^{\mathbf{T}}\hat{\mathcal{C}}_u\mathcal{C}_v + \beta\mathcal{C}_v^{\mathbf{T}}\hat{\mathcal{C}}_u^{\mathbf{T}}\hat{\mathcal{N}}_u\mathcal{C}_v + \beta\mathcal{C}_v^{\mathbf{T}}\hat{\mathcal{C}}_u^{\mathbf{T}}\hat{\mathcal{C}}_u\mathcal{N}_v \\
= \; & \beta(\mathcal{C}_u \times \mathcal{N}_v)^{\mathbf{T}}(\mathcal{C}_u \times \mathcal{C}_v) + \beta(\mathcal{N}_u \times \mathcal{C}_v)^{\mathbf{T}}(\mathcal{C}_u \times \mathcal{C}_v) + \beta(\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}}(\mathcal{N}_u \times \mathcal{C}_v) + \beta(\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}}(\mathcal{C}_u \times \mathcal{N}_v)) \\
= \; & 2\beta(\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}}(\mathcal{C}_u \times \mathcal{N}_v + \mathcal{N}_u \times \mathcal{C}_v).
\end{aligned} \tag{9}$$

Thus, substituting (5) and (9) back into (2), we have

$$\frac{\partial\mathcal{N}}{\partial t} = \frac{\beta_u\mathcal{N} \times \mathcal{C}_v + \beta_v\mathcal{C}_u \times \mathcal{N}}{\|\mathcal{C}_u \times \mathcal{C}_v\|} + \beta\frac{\|\mathcal{C}_u \times \mathcal{C}_v\|^2\mathbf{I} - (\mathcal{C}_u \times \mathcal{C}_v)(\mathcal{C}_u \times \mathcal{C}_v)^{\mathbf{T}}}{\|\mathcal{C}_u \times \mathcal{C}_v\|^3}\left(\mathcal{C}_u \times \mathcal{N}_v + \mathcal{N}_u \times \mathcal{C}_v\right). \tag{10}$$

Because $\mathcal{N}_u \parallel \mathcal{C}_u$ and $\mathcal{N}_v \parallel \mathcal{C}_v$, thus $(\mathcal{C}_u \times \mathcal{N}_v) \parallel (\mathcal{N}_u \times \mathcal{C}_v) \parallel \mathcal{N}$. Let $\mathcal{C}_u \times \mathcal{N}_v + \mathcal{N}_u \times \mathcal{C}_v = p\mathcal{N}$ and $\mathcal{C}_u \times \mathcal{C}_v = q\mathcal{N}$, where p and q are scalars. Thus the second term in the right hand side of (10) becomes

$$\beta\frac{q^2p\mathcal{N} - q^2\mathcal{N}\mathcal{N}^{\mathbf{T}}p\mathcal{N}}{q^3} = \beta\frac{q^2p\mathcal{N} - q^2p\mathcal{N}}{q^3} = 0. \tag{11}$$

Thus, (10) can be simplified as

$$\frac{\partial\mathcal{N}}{\partial t} = \frac{\beta_u\mathcal{N} \times \mathcal{C}_v + \beta_v\mathcal{C}_u \times \mathcal{N}}{\|\mathcal{C}_u \times \mathcal{C}_v\|}. \tag{12}$$

Thus, if $\beta_u = 0$ and $\beta_v = 0$, the surface evolve isotropically, and the norm does not change over deformation. By choosing proper parameters u and v, we can let $\|\mathcal{C}_u\| = 1$, $\|\mathcal{C}_v\| = 1$, and $\mathcal{C}_u \perp \mathcal{C}_v$. Use this set of parameterization and assume the right hand coordinate system to be $(u \times v) \parallel \mathcal{N}$, (12) can be simplified as

$$\frac{\partial\mathcal{N}}{\partial t} = -(\beta_u\mathcal{C}_u + \beta_v\mathcal{C}_v). \tag{13}$$

Thus, the second term in the right hand side of (8) in the main paper, i.e., temporal change of norm due to the deformation, can be simplified as

$$\begin{aligned}
\frac{\partial\mathcal{N}}{\partial t}|_{u_2,v_2,t_1}\mathbf{\Delta}t &= -(\mathbf{b}_d^{\mathbf{T}}\Phi_u\mathcal{C}_u + \mathbf{b}_d^{\mathbf{T}}\Phi_v\mathcal{C}_v)|_{u_2,v_2,t_1} \\
&= -(\mathcal{C}_u, \mathcal{C}_v)|_{u_2,v_2,t_1}\begin{pmatrix} \Phi_u^{\mathbf{T}} \\ \Phi_v^{\mathbf{T}} \end{pmatrix}|_{u_2,v_2,t_1}\mathbf{b}_d \\
&= -\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))|_{(u_2,v_2,t_1)}\mathbf{J}_{\mathcal{N}}(\Phi|(u,v))|_{(u_2,v_2,t_1)}^{\mathbf{T}}\mathbf{b}_d.
\end{aligned} \tag{14}$$

Substituting

$$\begin{aligned}
\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))|_{(u_2,v_2,t_1)} &= \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))|_{(u_1,v_1,t_1)} + \frac{\partial\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))}{\partial(u,v)}|_{(u_1,v_1,t_1)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix}, \\
\mathbf{J}_{\mathcal{N}}(\Phi|(u,v))|_{(u_2,v_2,t_1)} &= \mathbf{J}_{\mathcal{N}}(\Phi|(u,v))|_{(u_1,v_1,t_1)} + \frac{\partial\mathbf{J}_{\mathcal{N}}(\Phi|(u,v))}{\partial(u,v)}|_{(u_1,v_1,t_1)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix},
\end{aligned} \tag{15}$$

into (14), we have

$$\begin{aligned}
\frac{\partial \mathcal{N}}{\partial t}|_{u_2,v_2,t_1}\mathbf{\Delta}t &= -(\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v)) + \frac{\partial \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))}{\partial(u,v)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix}) \left(\mathbf{J}_{\mathcal{N}}(\Phi|(u,v)) + \frac{\partial \mathbf{J}_{\mathcal{N}}(\Phi|(u,v))}{\partial(u,v)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix}\right)^{\mathbf{T}} \mathbf{b} \\
&= -\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))\mathbf{J}_{\mathcal{N}}(\Phi|(u,v))^{\mathbf{T}}\mathbf{b}_d - \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v)) \left(\frac{\partial \mathbf{J}_{\mathcal{N}}(\Phi|(u,v))}{\partial(u,v)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix}\right)^{\mathbf{T}} \mathbf{b}_d \\
&\quad - \frac{\partial \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))}{\partial(u,v)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix} \mathbf{J}_{\mathcal{N}}(\Phi|(u,v))^{\mathbf{T}}\mathbf{b}_d \\
&\quad - \frac{\partial \mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))}{\partial(u,v)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix} \left(\frac{\partial \mathbf{J}_{\mathcal{N}}(\Phi|(u,v))}{\partial(u,v)} \times_3 \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix}\right)^{\mathbf{T}} \mathbf{b}_d. \qquad (16)
\end{aligned}$$

From (16) or (25) in the main paper, we know $\begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix} = O(\Delta t)$. In addition, as $\mathbf{b}_d = O(\Delta t)$, the first term in the right hand side of (16) is $O(\Delta t)$ while the other terms are $O(\Delta t^2)$. Using assumption $\mathbf{A2}$, we can neglect $O(\Delta t^2)$ with respect to $O(\Delta t)$, and (16) becomes,

$$\frac{\partial \mathcal{N}}{\partial t}|_{u_2,v_2,t_1}\mathbf{\Delta}t \approx -(\mathbf{J}_{\mathcal{N}}(\mathcal{C}|(u,v))\mathbf{J}_{\mathcal{N}}(\Phi|(u,v))^{\mathbf{T}}\mathbf{b}_d. \qquad (17)$$