Toward Automatic 3D Modeling of Scenes using a Generic Camera Model

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Abstract

This document contains proofs referenced in the CVPR'08 paper. They are useful for readers which would like to check several assertions.

1. Point Reconstruction by Ray Intersection

We show that function $\alpha_i(\tilde{\mathbf{P}}) = \arccos(\mathbf{d}_i^\top \frac{\tilde{\mathbf{P}}-\mathbf{o}_i}{||\tilde{\mathbf{P}}-\mathbf{o}_i||})$ is not C^1 continuous at point $\mathbf{P}$ such that $\alpha_i(\mathbf{P}) = 0$.

Without loss of generality, we change the coordinate system such that $\mathbf{d}_i = [0\ 0\ 1]^\top$ and write

$$\alpha_i(t) = \arccos(\mathbf{d}_i^\top \mathbf{D}(t)) \text{ with } \mathbf{D}(t) = \frac{[x(t)\ y(t)\ z(t)]^\top}{||[x(t)\ y(t)\ z(t)]||}$$

and $x(t), y(t), z(t)$ three real C^1 continuous functions with parameter t such that $[x(0)\ y(0)\ z(0)] = [0\ 0\ 1]$.

Apply the Chain Rule to $\alpha_i = \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}})$ with $\arccos'(u) = \frac{-1}{\sqrt{1-u^2}}$ if $|u| < 1$:

$$\begin{aligned}
\alpha_i' &= \arccos'(\frac{z}{\sqrt{x^2+y^2+z^2}})(\frac{z}{\sqrt{x^2+y^2+z^2}})' \\
&= -\sqrt{\frac{x^2+y^2+z^2}{x^2+y^2}}(\frac{z'}{\sqrt{x^2+y^2+z^2}} - z\frac{xx'+yy'+zz'}{(x^2+y^2+z^2)^{\frac{3}{2}}}) \\
&= -\frac{z'(x^2+y^2+z^2) - z(xx'+yy'+zz')}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} \\
&= \frac{-1}{x^2+y^2+z^2}(z'\sqrt{x^2+y^2} - \frac{z}{\sqrt{x^2+y^2}}(xx'+yy'))
\end{aligned}$$

Since $[x(t)\ y(t)\ z(t)] \approx [x'(0)t\ y'(0)t\ 1]$, we obtain

$$\alpha_i'(t) \approx \frac{t}{|t|}\sqrt{(x'(0))^2 + (y'(0))^2}.$$

We see that two α_i' limits are obtained if t converges to 0: one for each possible t sign. Thus, $\tilde{\mathbf{P}} \mapsto \alpha_i(\tilde{\mathbf{P}})$ is not C^1 continuous at point $\mathbf{P}$ such that $\alpha_i(\mathbf{P}) = 0$.

2. Virtual Covariance and Uncertainty

This is the proof for the C^- expression given by Eq. 3 in the paper.

The Jacobian of $\pi([x\ y\ z]^\top) = [x/z\ y/z]^\top$ is

$$J_\pi([x\ y\ z]^\top) = \begin{pmatrix} \frac{1}{z} & 0 & -\frac{x}{z^2} \\ 0 & \frac{1}{z} & -\frac{y}{z^2} \end{pmatrix}.$$

Let $d_i = ||\mathbf{P} - \mathbf{o}_i||$. The Jacobian J_{α_i} at point $\mathbf{P}$ of function

$$\alpha_i(\tilde{\mathbf{P}}) = \pi(\mathtt{R}_i(\tilde{\mathbf{P}} - \mathbf{o}_i)) \text{ with } \mathtt{R}_i\mathbf{d}_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{d}_i = \frac{\mathbf{P} - \mathbf{o}_i}{||\mathbf{P} - \mathbf{o}_i||}$$

is equal to

$$J_{\alpha_i}(\mathbf{P}) = J_\pi([0\,0\,d_i]^\top)\mathtt{R}_i = \mathtt{A}\mathtt{R}_i \text{ with } \mathtt{A} = \frac{1}{d_i}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let J be the Jacobian of the function $\tilde{\mathbf{P}} \mapsto [\alpha_1^\top\ \cdots\ \alpha_I^\top]^\top$. The inverse of $C(\mathbf{P}) = \sigma_\alpha^2(J(\mathbf{P})^\top J(\mathbf{P}))^{-1}$ is equal to

$$\begin{aligned}
C^- &= \frac{1}{\sigma_\alpha^2}J(\mathbf{P})^\top J(\mathbf{P}) = \frac{1}{\sigma_\alpha^2}\sum_{i=1}^{I}\mathtt{R}_i^\top\mathtt{A}^\top\mathtt{A}\mathtt{R}_i \\
&= \frac{1}{\sigma_\alpha^2}\sum_{i=1}^{I}\frac{1}{d_i^2}\mathtt{R}_i^\top(\mathtt{I}_{3\times3} - \mathbf{k}\mathbf{k}^\top)\mathtt{R}_i \text{ with } \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Since $\mathtt{R}_i$ is a rotation such that $\mathtt{R}_i\mathbf{d}_i = \mathbf{k}$, we obtain

$$C^- = \frac{1}{\sigma_\alpha^2}\sum_{i=1}^{I}\frac{\mathtt{I}_{3\times3} - \mathbf{d}_i\mathbf{d}_i^\top}{||\mathbf{P} - \mathbf{o}_i||^2}.$$

3. Reliability for 3D Modeling of a Scene

We show that $R(\mathbf{P})$ is arbitrarily large in two cases: (1) nearly parallel $\mathbf{d}_i$ and (2) large values of $||\mathbf{P} - \mathbf{o}_i||$.

Let $d = \min_i ||\mathbf{P} - \mathbf{o}_i||$. The smallest eigenvalue e of C^- is such that

$$e \leq \mathbf{d}_1^\top C^- \mathbf{d}_1 = \sum_{i=1}^{I}\frac{1 - (\mathbf{d}_i^\top\mathbf{d}_1)^2}{\sigma_\alpha^2||\mathbf{P} - \mathbf{o}_i||^2} \leq \sum_{i=1}^{I}\frac{1 - (\mathbf{d}_i^\top\mathbf{d}_1)^2}{\sigma_\alpha^2 d^2}.$$

This inequality and definitions (Eq. 4 and 5 in the paper)

$$U(\mathbf{P}) = \sqrt{\frac{\mathcal{X}_3^2(p)}{e}}, \quad R(\mathbf{P}) = \frac{U(\mathbf{P})}{\min_i ||\mathbf{P} - \mathbf{o}_i||}$$

imply

$$R(\mathbf{P}) = \sqrt{\frac{\mathcal{X}_3^2(p)}{ed^2}} \geq \sigma_\alpha \sqrt{\frac{\mathcal{X}_3^2(p)}{\sum_{i=1}^I 1 - (\mathbf{d}_i^\top \mathbf{d}_1)^2}}.$$

Thus, $R(\mathbf{P})$ is arbitrarily large in case (1). Since case (2) implies case (1), $R(\mathbf{P})$ is also arbitrarily large in case (2).

4. Geometric Tests

This is the proof for the $d^2(\mathbf{P}_1, \Pi)$ expression in Eq. 6 of the paper (more details in [1]).

Since $C^{-1}(\mathbf{P}_1)$ is definite positive, there are the Choleski factorization $C^{-1}(\mathbf{P}_1) = \mathtt{K}^\top \mathtt{K}$ and $\tilde{\mathbf{P}}$ such that $\mathbf{P}_2 = \mathbf{P}_1 + \mathtt{K}^{-1}\tilde{\mathbf{P}}$. Thus,

$$d^2(\mathbf{P}_1, \mathbf{P}_2) = (\mathbf{P}_1 - \mathbf{P}_2)^\top C^{-1}(\mathbf{P}_1)(\mathbf{P}_1 - \mathbf{P}_2) = ||\tilde{\mathbf{P}}||^2.$$

Let Π be the plane $\mathbf{n}^\top \mathbf{X} + d = 0$. We have

$$\mathbf{P}_2 \in \Pi \iff \mathbf{n}^\top \mathbf{P}_2 + d = 0 \iff \tilde{\mathbf{n}}^\top \tilde{\mathbf{P}} + \tilde{d} = 0 \iff \tilde{\mathbf{P}} \in \tilde{\Pi}$$

with $\tilde{\mathbf{n}} = \mathtt{K}^{-\top}\mathbf{n}$ and $\tilde{d} = d + \mathbf{n}^\top \mathbf{P}_1$. Now, we see that

$$d^2(\mathbf{P}_1, \Pi) = \min_{\mathbf{P}_2 \in \Pi} d^2(\mathbf{P}_1, \mathbf{P}_2) = \min_{\tilde{\mathbf{P}} \in \tilde{\Pi}} ||\tilde{\mathbf{P}}||^2.$$

This is the Euclidean distance between $\mathbf{0}_{3\times 1}$ and the plane $\tilde{\Pi} : \tilde{\mathbf{n}}^\top \tilde{\mathbf{P}} + \tilde{d} = 0$. Thus,

$$d^2(\mathbf{P}_1, \Pi) = \frac{(\tilde{\mathbf{n}}^\top \mathbf{0}_{3\times 1} + \tilde{d})^2}{||\tilde{\mathbf{n}}||^2} = \frac{(\mathbf{n}^\top \mathbf{P}_1 + d)^2}{\mathbf{n}^\top C(\mathbf{P}_1)\mathbf{n}}.$$

5. Comparing Specific and Generic Cameras

We show that virtual uncertainties of generic and specific camera models are the same iff Eq. 8 in the paper is true. The proof requires more notations than the paper.

5.1. Definitions of Angles and Coordinate Systems

Let $\mathbf{p}_i^0$ be points in the i-th generic image. The calibration function applied to $\mathbf{p}_i^0$ gives a ray with origin $\mathbf{o}_i^c$ and direction $\mathbf{d}_i^c$ in the camera coordinate system. We define α_i^c in this coordinate system by $\alpha_i^c(\mathbf{X}^c) = \pi(\mathtt{R}_{\mathbf{d}_i^c}(\mathbf{X}^c - \mathbf{o}_i^c))$ with a rotation $\mathtt{R}_{\mathbf{d}_i^c}$ such that $\mathtt{R}_{\mathbf{d}_i^c}\mathbf{d}_i^c = [0\,0\,1]^\top$ and $\pi([x\,y\,z]^\top) = [x/z\;y/z]^\top$. Function α_i^c is C^2 continuous and $||\alpha_i^c(\mathbf{X}^c)||$ is the tangent of the angle between $\mathbf{X}^c - \mathbf{o}_i^c$ and $\mathbf{d}_i^c$.

Let $(\mathtt{R}_i, \mathbf{t}_i)$ be the i-th pose of the camera in the world coordinate system ($\mathtt{R}_i$ is a rotation and $\mathbf{t}_i$ a translation). The world coordinates of $\mathbf{o}_i^c, \mathbf{d}_i^c$ are $\mathbf{o}_i^w = \mathtt{R}_i\mathbf{o}_i^c + \mathbf{t}_i$, $\mathbf{d}_i^w = \mathtt{R}_i\mathbf{d}_i^c$. There are similar notations $\mathbf{X}^w = \mathtt{R}_i\mathbf{X}_i^c + \mathbf{t}_i$ for any point $\mathbf{X}^w$ in the world coordinate system. We define α_i^w by $\alpha_i^w(\mathbf{X}^w) = \alpha_i^c(\mathtt{R}_i^\top(\mathbf{X}^w - \mathbf{t}_i))$ and obtain

$$\alpha_i^w(\mathbf{X}^w) = \pi(\mathtt{R}_{\mathbf{d}_i^w}(\mathbf{X}^w - \mathbf{o}_i^w)) \text{ with } \mathtt{R}_{\mathbf{d}_i^w} = \mathtt{R}_{\mathbf{d}_i^c}\mathtt{R}_i^\top.$$

Thus, α_i^w is C^2 continuous and $||\alpha_i^w(\mathbf{X}^w)||$ is the tangent of the angle between $\mathbf{X}^w - \mathbf{o}_i^w$ and $\mathbf{d}_i^w$. This function is the function α_i introduced by Eq. 1 in the paper.

5.2. Link Between Projection and Angle Error

Here we assume that the image projection $p(\mathbf{X}^c)$ of a point $\mathbf{X}^c$ in the camera coordinate system is a well defined and C^1 continuous function.

In this part, we would like to define a C^1 continuous function ϕ_i such that $\phi_i \circ p = \alpha_i^c$. Let $\mathbf{p}_i$ be a point with the corresponding observation ray $(\mathbf{o'}_i^c, \mathbf{d'}_i^c)$ by the calibration function. We assume that the camera is central as in Section 4 of the paper: $\mathbf{o'}_i^c = \mathbf{o}_i^c$. The image by α_i^c of a ray point $X^c(\lambda) = \mathbf{o}_i^c + \lambda\mathbf{d'}_i^c$ is equal to $\pi(\mathtt{R}_{\mathbf{d}_i^c}\mathbf{d'}_i^c)$ and does not depend on λ. Thus we define $\phi_i(\mathbf{p}_i)$ by $\pi(\mathtt{R}_{\mathbf{d}_i^c}\mathbf{d'}_i^c)$ with $\mathbf{d'}_i^c$ obtained by the calibration function applied to $\mathbf{p}_i$. Now, ϕ_i is well defined, $\phi_i \circ p = \alpha_i^c$ is true, and ϕ_i is C^1 continuous since α_i (and the calibration function) is (assumed to be) C^1 continuous.

5.3. Definitions of Virtual Covariances

Let $\mathbf{P}^w$ be a point in the world coordinate system. The virtual covariance matrix of the generic camera model requires $\mathbf{p}_i^0 = p(\mathtt{R}_i^\top(\mathbf{P}^w - \mathbf{t}_i))$ and is defined by

$$C_\alpha(\mathbf{P}^w) = \sigma_\alpha^2 (\sum_{i=1}^I J_{\alpha_i^w}^\top(\mathbf{P}^w) J_{\alpha_i^w}(\mathbf{P}^w))^{-1}.$$

Let p_i be the function $p_i(\mathbf{X}^w) = p(\mathtt{R}_i^\top(\mathbf{X}^w - \mathbf{t}_i))$. The virtual covariance matrix of the specific camera model has a similar definition:

$$C_p(\mathbf{P}^w) = \sigma_p^2 (\sum_{i=1}^I J_{p_i}^\top(\mathbf{P}^w) J_{p_i}(\mathbf{P}^w))^{-1}.$$

The Chain Rule provides equations

$$\begin{aligned} C_\alpha^{-1}(\mathbf{P}^w) &= \frac{1}{\sigma_\alpha^2} \sum_{i=1}^I \mathtt{R}_i J_{\alpha_i^c}^\top(\mathbf{P}_i^c) J_{\alpha_i^c}(\mathbf{P}_i^c) \mathtt{R}_i^\top \\ C_p^{-1}(\mathbf{P}^w) &= \frac{1}{\sigma_p^2} \sum_{i=1}^I \mathtt{R}_i J_p^\top(\mathbf{P}_i^c) J_p(\mathbf{P}_i^c) \mathtt{R}_i^\top \end{aligned}$$

which will be useful latter with $\mathbf{P}_i^c = \mathtt{R}_i^\top(\mathbf{P}^w - \mathbf{t}_i)$.

5.4. Comparing Virtual Covariances

We see that the condition $C_\alpha = C_p$ is equivalent to

$$0 = \sum_{i=1}^I \mathtt{R}_i \{ \frac{1}{\sigma_\alpha^2} J_{\alpha_i^c}^\top(\mathbf{P}_i^c) J_{\alpha_i^c}(\mathbf{P}_i^c) - \frac{1}{\sigma_p^2} J_p^\top(\mathbf{P}_i^c) J_p(\mathbf{P}_i^c) \} \mathtt{R}_i^\top$$

for all rotations $\mathtt{R}_i$ and points $\mathbf{P}_i^c$. We would like

$$\frac{1}{\sigma_\alpha^2} J_{\alpha_i^c}^\top J_{\alpha_i^c} - \frac{1}{\sigma_p^2} J_p^\top J_p = 0 \Longleftrightarrow C_\alpha = C_p.$$

The implication "$\Longrightarrow$" is obvious and we focus on the implication "$\Longleftarrow$". Let $\mathbf{P}^c$ and $\mathbf{P}^w$ be points in the camera and world coordinate systems, respectively. We set $\mathbf{P}_i^c = \mathbf{P}^c$ for all i. For all rotations $\mathtt{R}_i$, there is a $\mathbf{t}_i$ such that $\mathbf{P}_i^c = \mathtt{R}_i^\top(\mathbf{P}^w - \mathbf{t}_i)$. Under these conditions, $C_\alpha = C_p$ implies

$$0 = \sum_{i=1}^{I} \mathtt{R}_i f(\mathbf{P}^c)\mathtt{R}_i^\top \text{ with } f = \frac{1}{\sigma_\alpha^2} J_{\alpha_i^c}^\top J_{\alpha_i^c} - \frac{1}{\sigma_p^2} J_p^\top J_p$$

for all rotations $\mathtt{R}_i$. We see that $\mathtt{R}_i f(\mathbf{P}^c)\mathtt{R}_i^\top$ is constant for all rotations $\mathtt{R}_i$. Thus, there is $\lambda \in \mathbb{R}$ such that $f(\mathbf{P}^c) = \lambda \mathtt{I}_{3\times 3}$. We deduce that

$$0 = \sum_{i=1}^{I} \mathtt{R}_i f(\mathbf{P}^c)\mathtt{R}_i^\top = \lambda \sum_{i=1}^{I} \mathtt{R}_i \mathtt{R}_i^\top = \lambda I$$

and $\lambda = 0$. Now, implication "$\Longleftarrow$" is obvious.

The relation $\phi_i \circ p = \alpha_i^c$ implies $J_{\phi_i} J_p = J_{\alpha_i^c}$ and

$$J_p^\top \left(\frac{1}{\sigma_\alpha^2} J_{\phi_i}^\top J_{\phi_i} - \frac{1}{\sigma_p^2} \mathtt{I}_{2\times 2}\right) J_p = 0 \Longleftrightarrow C_\alpha = C_p$$

We assume that J_p is full rank and obtain

$$\frac{1}{\sigma_\alpha^2} J_{\phi_i}^\top J_{\phi_i} = \frac{1}{\sigma_p^2} \mathtt{I}_{2\times 2} \Longleftrightarrow C_\alpha = C_p.$$

5.5. Condition in the Paper

Let $\mathbf{P}^w, \mathbf{X}^w$ be 3D points. Functions α_i^c, α_i^w are defined by $\mathbf{p}_i^0 = p_i(\mathbf{P}^w)$. Eq. 8 (in the paper) is

$$||\alpha_i^w(\mathbf{X}^w)|| \approx \frac{\sigma_\alpha}{\sigma_p} ||p_i(\mathbf{X}^w) - p_i(\mathbf{P}^w)||.$$

Using notations $\mathbf{P}_i^c = \mathtt{R}_i^\top(\mathbf{P}^w - \mathbf{t}_i)$ and $\mathbf{X}_i^c = \mathtt{R}_i^\top(\mathbf{X}^w - \mathbf{t}_i)$, Eq. 8 is

$$||\alpha_i^c(\mathbf{X}_i^c)|| \approx \frac{\sigma_\alpha}{\sigma_p} ||p(\mathbf{X}_i^c) - p(\mathbf{P}_i^c)||$$

with $\mathbf{p}_i^0 = p(\mathbf{P}_i^c)$. We have $\mathbf{p}_i^0 = p(\mathbf{P}_i^c) \Longrightarrow \alpha_i^c(\mathbf{P}_i^c) = 0$ using the α_i^c definition. Thus, Eq. 8 is equivalent to

$$||\alpha_i^c(\mathbf{X}_i^c) - \alpha_i^c(\mathbf{P}_i^c)|| \approx \frac{\sigma_\alpha}{\sigma_p} ||p(\mathbf{X}_i^c) - p(\mathbf{P}_i^c)||.$$

Since ϕ_i is C^1 continuous and $\alpha_i^c = \phi_i \circ p$, we approximate ϕ_i by its linear Taylor expansion at point $p(\mathbf{P}_i^c)$ and obtain

$$||J_{\phi_i}(\mathbf{p}_i^0).(p(\mathbf{X}_i^c) - p(\mathbf{P}_i^c))||^2 \approx \frac{\sigma_\alpha^2}{\sigma_p^2} ||p(\mathbf{X}_i^c) - p(\mathbf{P}_i^c)||^2$$

for all $p(\mathbf{X}_i^c)$ in a neighborhood of $p(\mathbf{P}_i^c)$. The quadratic forms in both sides are equal and we see that Eq. 8 is equivalent to

$$J_{\phi_i}^\top J_{\phi_i} \approx \frac{\sigma_\alpha^2}{\sigma_p^2} \mathtt{I}_{2\times 2}.$$

This concludes the proof.

References

[1] K. Schindler and H. Bischof, On Robust Regression in Photogrammetric Point Clouds, DAGM'03 (also *LNCS 2781*, pp. 172-178, 2003).