# What is a Camera?

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**Abstract:** *This paper addresses the problem of characterizing a general class of cameras under reasonable, "linear" assumptions. Concretely, we use the formalism and terminology of classical projective geometry to model cameras by two-parameter linear families of straight lines—that is, degenerate* reguli *(rank-3 families) and non-degenerate* linear congruences *(rank-4 families). This model captures both the* general linear cameras *of Yu and McMillan [16] and the* linear oblique cameras *of Pajdla [8]. From a geometric perspective, it affords a simple classification of all possible camera configurations. From an analytical viewpoint, it also provides a simple and unified methodology for deriving general formulas for projection and inverse projection, triangulation, and binocular and trinocular geometry.*

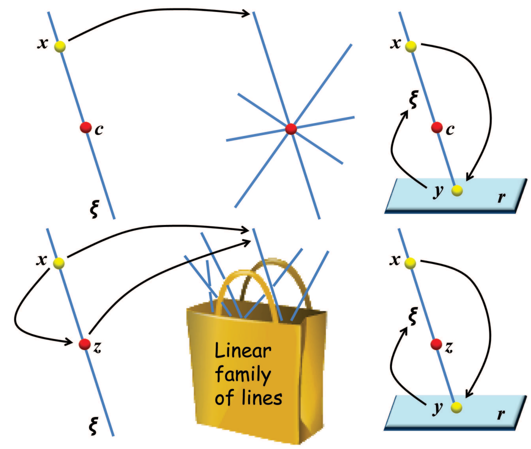


Figure 1. What a camera is. See text for details.

## 1. Introduction

Several formal models for "general" cameras have been proposed in the past few years [5, 7, 8, 11, 12, 13, 14, 16, 17]. We propose in this paper a simple framework that subsumes these models by representing cameras as two-dimensional linear families of lines. It is well known that classical pinhole (or *central*) cameras correspond to rank-3 *line bundles* (Figure 1, top): As argued in [9], the perspective projection process can be decomposed into an *essential* part associating with any point $\boldsymbol{x}$ the line (or *ray*) $\boldsymbol{\xi}$ passing through the pinhole $\boldsymbol{c}$, and a *secondary* one that associates with $\boldsymbol{\xi}$ its intersection $\boldsymbol{y}$ with the retinal plane $\boldsymbol{r}$—that is, its projection (the projective structure of the image is indeed independent of the choice of $\boldsymbol{r}$ as long as it does not pass through $\boldsymbol{c}$). In the classical projective geometry terminology of Veblen and Young [15], line bundles are linear families of lines of rank 3. More generally, we will show in this paper that many other *non-central* cameras can be represented by other rank-3 or rank-4 families, including *elliptic*, *hyperbolic*, and *parabolic linear congruences*, corresponding respectively to *linear oblique (or bilinear) cameras* [8, 16] and *stereo panoramas or cyclographs* [12], *two-slit* [4, 17] and *linear pushbroom cameras* [5], and *pencil cameras* [16]. Subsets of these have previously been modeled under the formalisms of the *general linear cameras* of Yu and McMillan [16] and the *linear oblique cameras* of Pajdla [8], but never, to the best of our knowledge, in a fully unified manner. From an analytical perspective, our model also provides a simple and unified methodology for deriving general formulas for direct and inverse projection, triangulation, and binocular and trinocular geometry.

### 1.1. Contributions

The main contributions of this paper are theoretical. They can be summarized as follows:

• **A unified framework for modeling many types of central and non-central imaging devices.** A camera as defined in this presentation is (roughly) a device for associating straight lines with points (Figure 1, bottom). In essence, a camera has at its disposal a "linear bag" of lines from which it picks, for any point $\boldsymbol{x}$, the corresponding line $\boldsymbol{\xi}$. As will be shown later in this paper, this process can be decomposed into first finding a point $\boldsymbol{z} = \mathcal{A}\boldsymbol{x}$ on the line $\boldsymbol{\xi}$, which in turn defines this line. This is the essential part of imaging. As before, there is a secondary part that associates with $\boldsymbol{x}$ its image $\boldsymbol{y}$ in some retinal plane $\boldsymbol{r}$. Conversely, an *inverse projection* process associates with each image point $\boldsymbol{y}$ its preimage $\boldsymbol{\xi}$.

• **A geometric classification of the possible camera configurations.** As shown in Section 2, classical (line) projective geometry [10, 15] provides a complete characteriza-

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tion of all physical cameras captured by our model, including *linear oblique (or bilinear) cameras* [8, 16] and *stereo panoramas or cyclographs* [12], *two-slit* [4, 17] and *linear pushbroom cameras* [5], and *pencil cameras* [16].

• **A unified analytical framework for describing the mapping between points and the corresponding lines.** We show in the appendix that this mapping can always be represented in terms of a $4 \times 4$ matrix $\mathcal{A}$. In turn, this affords simple and general formulas for direct and inverse projection as well as triangulation (Section 3).

• **A general approach to multi-view geometry.** We show in Sections 2 and 3 that the fundamental matrix and trifocal tensor characterizing admissible binocular or trinocular correspondences for pinhole cameras are readily generalized to *arbitrary* pairs or triples of cameras in our model.

As stated earlier, these contributions are (so far) mainly of theoretical interest. Experiments will follow, and we hope that the formulas afforded by our framework will prove of interest for practitioners interested in stereopsis with rigs that include some non-traditional cameras. In the mean time, we also hope they will be of interest to people who want to know "what is under the hood", so to speak, of the cameras that are the bread and butter of our profession.

## 1.2. Related work

Several authors have proposed using a *two-plane parameterization* of lines to model non-central cameras [12, 16]. In this model, two parallel planes $\boldsymbol{p}_{st}$ and $\boldsymbol{p}_{uv}$ equipped with $(s,t)$ and $(u,v)$ coordinate systems are used to parameterize all lines by the vectors $\boldsymbol{r} = (s,t,u,v)^T$ in $\mathbb{R}^4$ associated with their intersections with the two planes. In particular, Yu and McMillan propose in [16] a *general linear camera* (or *GLC*) model where different types of cameras are defined in terms of affine combinations of the vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$ representing three arbitrary lines $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, and $\boldsymbol{\delta}_3$. They go on to show that pinhole, pushbroom, two-slit and bilinear cameras can be represented in a linear manner using this model, and that they can be classified in terms of the number of fixed lines that all rays in a camera pass through (see [13] for a similar idea). Although this model is attractive, and actually captures many classical camera models, we believe that it overemphasizes the role of camera parameterization relative to their geometry. In particular, a given GLC model only captures a subset of the corresponding camera instances. For example, as shown in [16, Lemma 4], the two-slit cameras associated with a GLC model must have slits parallel to the corresponding two planes.

The general linear camera models proposed in [16] are mostly concerned with monocular imaging geometry. Multi-view geometry is well understood for pinhole cameras, and it can be represented analytically in the projective case in terms of the fundamental matrix and trifocal tensor associated with two or three cameras. The past few years have witnessed initial attempts at modeling the multi-view geometry of specific types of non-central cameras (e.g., [4, 5, 8, 12, 13, 14]). To go further, Seitz and Kim present in [12] a classification of all stereo image pairs captured by two central or non-central cameras that can be fused in the sense that epipolar correspondences always correspond (possibly after rectification) to image rows. This constraint naturally imposes that the lines associated with the two cameras sweep a doubly ruled surface—that is, as shown in [6], a (possibly degenerate) regulus. This setup imposes a symmetry between the epipolar curves associated with two matching points: They must be the projection of the same surface in space. Although this constraint is necessary for a person to visually fuse two pictures, it is not clear that it is of great importance for automated approaches to stereopsis, where the epipolar constraint is mostly used to restrict the search for correspondences to a one-dimensional locus. Instead, we derive in this paper *general* formulas for the epipolar curves associated with *any pair* of cameras captured by our model, and these curves will, as often as not, have different preimages.

Pajdla examines in [8] *oblique cameras* such that no two rays in one camera ever intersect (these are a superset of the *bilinear cameras* discussed in [16]), and uses a criterion similar to that of Seitz and Kim to characterize the families of oblique camera rays that can be fused in stereopsis. He goes further and shows that a linear class (in the projective geometry sense used in this presentation) of oblique cameras can be characterized by associating with any point $\boldsymbol{x}$ in space the ray joining it to the point $\mathcal{A}\boldsymbol{x}$, where $\mathcal{A}$ is a particular type of nonsingular projective transformation (*elliptic involution*). As will be shown later in this paper, all our camera models can also be characterized in this form, except for the fact that the matrix $\mathcal{A}$ may be singular. Pajdla does not give explicit formulas for the epipolar geometry of oblique cameras. Sturm takes a major step in this direction in [13], where he gives another classification of camera models in terms of the number of lines intersected by any line in the corresponding family. Because he does not impose any constraint on non-central cameras in this hierarchy, the multi-view constraints he derives are expressed directly in terms of the coordinates of matching camera rays in some global coordinate system instead of image coordinates. In a sense, this camera model is "too general": In particular, unlike the model presented in the rest of this presentation, it does not afford direct or inverse projection formulas given some image plane and coordinate basis. Instead, the association between image points and rays is assumed to be known through some external calibration process. At the other end of the spectrum, let us note that Sturm and Barreto propose in [14] a general model of a specific class of cameras —namely, central catadioptric ones— in terms of non-linear families of lines (quadratic complexes).

## 2. The geometric picture

### 2.1. Camera model

From an abstract viewpoint, a camera is a device for recording scene radiance along a two-dimensional set of (oriented) straight lines (or, equivalently, for recording a two-dimensional subset of the *light field*): For example, a pinhole camera maps all the rays passing through the pinhole onto their intersections with some image plane. The geometric picture is clear in this idealized case, and the configuration of the projections is (projectively) independent of the choice of the plane. In this paper, we will ignore photometry and focus on geometric camera properties for more general families of lines, modeling them using the tools of (unoriented, projective) line geometry, and ignoring "nuisances" such as distortion, limited field of view, etc. Our presentation is rooted in the notion of *linear dependence* among lines in $\mathbb{P}^3$, which is defined in the classical text of Veblen and Young [15] in purely axiomatic terms (see Section 3 for an analytical characterization in terms of Plücker coordinates): In this formalism, when two lines are coplanar, the lines of the flat pencil that contains them—that is the set of all lines lying in the same plane and passing through the point where they intersect—are said to be linearly dependent on them. When two lines are skew, the only lines linearly dependent on them are the lines themselves. The lines linearly dependent on three skew lines are the lines of the regulus of which they are rulers (Figure 2, top right). In general, we say that a line $\boldsymbol{\delta}$ is linearly dependent on $n \geq 3$ lines $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_n$ if there exists some sequence of lines $\boldsymbol{\delta}_{n+1}, \ldots, \boldsymbol{\delta}_{n+k}$, with $\boldsymbol{\delta} = \boldsymbol{\delta}_{n+k}$, such that each $\boldsymbol{\delta}_{n+i}$, for $i = 1, \ldots, k$, is linearly dependent on two or three lines among $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{n+i-1}$. A set of lines, none of which is linearly dependent on the others, is said to be linearly independent. A subset $A$ of any set of lines $B$ is said to *span* $B$ when all the lines in $B$ are linearly dependent on the lines in $A$, and the *rank* of $B$ is the minimum number of lines spanning it.

**Definition:** *A* camera *is a two-parameter linear family of lines—that is, a degenerate* regulus *(rank-3 family), or a non-degenerate* linear congruence *(rank-4 family).*

This model encompasses pinhole perspective cameras since, as will be shown next, the corresponding rays form a bundle of lines passing through the optical center and thus a degenerate regulus. More generally, it is mathematically reasonable to restrict the set of cameras to those with such a linear structure. We will see in the rest of this paper that this is sufficient to capture most common camera types.

### 2.2. A classification of camera configurations

According to our definition, any camera corresponds to a two-dimensional linear family of lines. It turns out that all of the corresponding line configurations are known, and

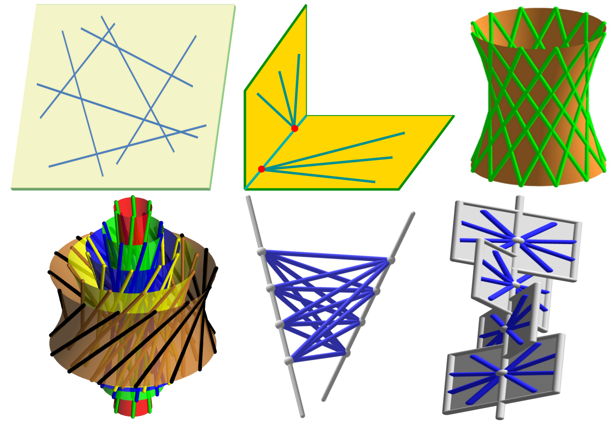

Figure 2. Top: reguli. From left to right: a line field formed by all the lines lying in a plane; two flat pencils lying in different planes but sharing a line; and a non-degenerate regulus formed by one of the two rulings of a hyperboloid of one sheet or of a hyperbolic paraboloid. Bottom: linear congruences. From left to right: an elliptic congruence, visualized as a one-parameter family of non-intersecting reguli; a hyperbolic one; and a parabolic one. Copyright Hans Havlicek, Vienna University of Technology.

classified according to their (linear) rank [2, 10, 15].

**The rank-3 families** are known as *reguli* [15, Ch. XI, Th. 13] and they are (Figure 2, top): (3a) the *line fields* formed by all the lines in a plane; (3b) all lines passing through some point (line bundle, see Figure 1, top); (3c) the union of all lines belonging to two flat pencils lying in different planes but sharing one line; and (3d) all lines belonging to a non-degenerate regulus.

**The rank-4 families** are known as *linear congruences* and they are [15, Ch. XI, Th. 14] (Figure 2, bottom): (4a) the congruences generated by four skew lines (elliptical congruence); (4b) all lines incident to two skew lines (hyperbolic congruence); (4c) a one-parameter family of flat pencils, having one line in common (parabolic congruence); and (4d) all lines lying in a plane or passing through a point in this plane (degenerate congruence, not shown). As shown in [10] for example, they can be classified in terms of the number of *focal lines* all rays in the congruence intersect: two real lines for hyperbolic congruences, one (double) real line for parabolic ones, and two (conjugate) complex lines for elliptic ones (see [13, 16] for a classification of camera models similar in spirit, if slightly less general). We will come back to this characterization in Section 3.

Our definition of cameras as rank-3 or rank-4 linear families of lines leads to the following classification:

**Line fields ≡ epipolar-plane cameras.** The epipolar-plane image [1] formed by a 1D camera moving along an arbitrary path in a plane that contains all its instances generates all lines in that plane. The corresponding rays span a rank-3 line field (Figure 2, top left). We will not discuss these "cameras" any further since they can only be used to im-

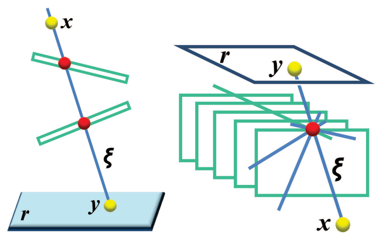

Figure 3. A two-slit camera and a linear pushbroom camera. Projectively, the two models are equivalent, the "parallel" planes of the pushbroom camera meeting along a line "at infinity".

age a single plane. Note that non-degenerate reguli and the degenerate reguli formed by two flat pencils sharing a line only intersect any retinal plane along a conic or a pair of lines instead of a two-dimensional image region. For this reason, they do not qualify as bona fide cameras either.

**Line bundles $\equiv$ pinhole cameras.** Line bundles are dual to line fields and, as discussed earlier, they correspond to pinhole cameras (Figure 1, top).

**Elliptic congruences $\equiv$ linear oblique cameras.** An elliptic congruence is generated by four skew lines such that none of them intersects the regulus generated by the other three. Its elements form a one-parameter family of "concentric" reguli [10] (Figure 2, bottom left), with exactly one line going through each point of space. This is a *linear oblique camera* as defined in [8] (or a *bilinear camera* as defined in [16]). Physical instances of this camera type include *stereo panoramas or cyclographs* [12], as well as the catadioptric systems described in [8].

**Hyperbolic congruences $\equiv$ two-slit cameras.** Hyperbolic congruences correspond to two-slit cameras [4, 17] (Figure 3, left). Linear pushbroom cameras [5] are formed by sweeping one-dimensional pinhole cameras along some line orthogonal to their retinal planes (Figure 3, right). Although this is not always recognized, they are just an instance of two-slit cameras, the second slit being the line at infinity common to the parallel planes.

**Parabolic congruences $\equiv$ pencil cameras.** A parabolic congruence is a one-parameter continuous family of flat pencils sharing a common line. This appears to correspond to a *pencil camera* as defined by Yu and McMillan [16]. Although we are not aware of any physical realization of pencil cameras, one can certainly imagine a one-dimensional camera whose optical center translates along a line as it rotates about it (Figure 2, bottom right).

**Degenerate congruences $\equiv$ two-slit cameras when the slits are coplanar.** A degenerate congruence consists of the union of all lines in a plane with the bundle of lines through a point in that plane. This is exactly the set of all lines intersecting two coplanar lines, the center of the bundle being the point where the two slits intersect [4]. We will not consider these any further.

### 2.3. Multi-view geometry

Consider two arbitrary cameras in our model. By construction, a line $\boldsymbol{\delta}_1$ associated with the first camera is already linearly dependent on three or four lines depending on the rank $k$ of the corresponding family. For the point where $\boldsymbol{\delta}_1$ pierces the image plane to match a point in the second image, $\boldsymbol{\delta}_1$ must intersect the corresponding line $\boldsymbol{\delta}_2$, and thus satisfy one additional linear constraint. In particular, the lines from the first camera intersecting $\boldsymbol{\delta}_2$ must form a rank $k-1$ family—that is, depending on $k$, a regulus or a flat pencil of lines. The lines from the second family intersecting $\boldsymbol{\delta}_1$ also form a flat pencil or a regulus, but the two line sets may not sweep a common surface, and the epipolar locus will consist of lines or conics. We will give analytical formulas characterizing these in the next section.

## 3. The analytical picture

We have so far restricted our discussion of cameras to a purely geometric one. It is now time to take a more analytical viewpoint which, in turn, requires choosing parameterizations for our cameras. For this, it will be convenient to introduce the *Plücker* parameterization of straight lines, defined in the next section, along with the fundamental *join* and *meet operators*. We will assume throughout this section that the projective space $\mathbb{P}^3$ has been equipped with a fixed coordinate system, and identify points and planes with their (homogeneous) coordinate vectors in this basis, "$=$" denoting equality up to scale among these vectors.

### 3.1. Plücker coordinates

The *join* operator "$\vee$" associates with two points $\boldsymbol{x}$ and $\boldsymbol{y}$ the line $\boldsymbol{x} \vee \boldsymbol{y}$ joining them. This *geometric* operator has an *analytical* counterpart, and we will use the same notation for both: we define the *Plücker coordinate vector* of the line joining $\boldsymbol{x}$ and $\boldsymbol{y}$ as $\boldsymbol{x} \vee \boldsymbol{y} = (\boldsymbol{u}; \boldsymbol{v})$, where

$$\boldsymbol{u} = \begin{bmatrix} x_4 y_1 - x_1 y_4 \\ x_4 y_2 - x_2 y_4 \\ x_4 y_3 - x_3 y_4 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix},$$

and, in "Matlab fashion", we use the ";" symbol to indicate that the coordinates of $\boldsymbol{u}$ and $\boldsymbol{v}$ have been stacked onto each other to form a vector of $\mathbb{R}^6$. We will from now on identify lines with their Plücker coordinate vectors in some fixed but otherwise arbitrary coordinate system. Plücker coordinates are homogeneous, and lines form a quadratic hypersurface $\mathbb{L}^4$ of dimension 4—the *Klein quadric*—in the projective space $\mathbb{P}^5$: Indeed, it follows immediately from the definition of the join that the Plücker coordinate vector $\boldsymbol{\delta} = (\boldsymbol{u}; \boldsymbol{v})$ of a line satisfies the quadratic constraint $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. It is in fact possible to define an inner product in $\mathbb{L}^4$ by the formula $(\boldsymbol{\delta}|\boldsymbol{\eta}) \stackrel{\text{def}}{=} \boldsymbol{u} \cdot \boldsymbol{t} + \boldsymbol{v} \cdot \boldsymbol{s}$, where $\boldsymbol{\delta} = (\boldsymbol{u}; \boldsymbol{v})$ and

$\boldsymbol{\eta} = (\boldsymbol{s};\boldsymbol{t})$. A vector $\boldsymbol{\delta}$ in $\mathbb{R}^6$ represents a line if and only if $(\boldsymbol{\delta}|\boldsymbol{\delta}) = 0$, and it can also be shown that a necessary and sufficient condition for two lines $\boldsymbol{\delta}$ and $\boldsymbol{\eta}$ to be coplanar is that $(\boldsymbol{\delta}|\boldsymbol{\eta}) = 0$. When $\boldsymbol{\delta} = (\boldsymbol{u};\boldsymbol{v})$, it is convenient to define the vector $\boldsymbol{\delta}^* = (\boldsymbol{v};\boldsymbol{u})$ so that $(\boldsymbol{\delta}|\boldsymbol{\eta}) = \boldsymbol{\delta}^* \cdot \boldsymbol{\eta} = \boldsymbol{\delta} \cdot \boldsymbol{\eta}^*$. The *meet* operator associates with two planes $\boldsymbol{p}$ and $\boldsymbol{q}$ the line $\boldsymbol{p} \wedge \boldsymbol{q}$. Analytically, the corresponding Plücker vector is $\boldsymbol{p} \wedge \boldsymbol{q} = (\boldsymbol{p} \vee \boldsymbol{q})^*$. The join operator can be extended to lines and points: Given a line $\boldsymbol{\delta}$ and a point $\boldsymbol{x}$ not lying on $\boldsymbol{\delta}$, we define the join $\boldsymbol{\delta} \vee \boldsymbol{x}$ of $\boldsymbol{\delta}$ and $\boldsymbol{x}$ as the plane spanned by the line and the point. Likewise, we define the meet $\boldsymbol{\delta} \wedge \boldsymbol{p}$ of a line $\boldsymbol{\delta}$ and a plane $\boldsymbol{p}$ that does not contain $\boldsymbol{\delta}$ as the point where the intersect. Analytically, it is easy to show that if $\boldsymbol{\lambda} = \boldsymbol{p} \wedge \boldsymbol{q}$ and $\boldsymbol{\mu} = \boldsymbol{x} \vee \boldsymbol{y}$, then

$$\begin{cases} \boldsymbol{\lambda} \vee \boldsymbol{x} = [\boldsymbol{\lambda}_\vee]\boldsymbol{x} \quad \text{where} \quad [\boldsymbol{\lambda}_\vee] \stackrel{\text{def}}{=} \boldsymbol{p}\boldsymbol{q}^T - \boldsymbol{q}\boldsymbol{p}^T, \\ \boldsymbol{\mu} \wedge \boldsymbol{p} = [\boldsymbol{\mu}_\wedge]\boldsymbol{p} \quad \text{where} \quad [\boldsymbol{\mu}_\wedge] \stackrel{\text{def}}{=} \boldsymbol{x}\boldsymbol{y}^T - \boldsymbol{y}\boldsymbol{x}^T. \end{cases}$$

A necessary and sufficient condition for a point $\boldsymbol{x}$ to lie on a line $\boldsymbol{\delta}$ is that $\boldsymbol{\delta} \vee \boldsymbol{x} = \boldsymbol{0}$. Likewise, a necessary and sufficient condition for a plane $\boldsymbol{p}$ to contain a line $\boldsymbol{\delta}$ is that $\boldsymbol{\delta} \wedge \boldsymbol{p} = \boldsymbol{0}$. Finally, the join of three points $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ is the plane spanned by these points, defined analytically by $\boldsymbol{x} \vee \boldsymbol{y} \vee \boldsymbol{z} \stackrel{\text{def}}{=} (\boldsymbol{x} \vee \boldsymbol{y}) \vee \boldsymbol{z}$. The meet of three planes $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ is the point where these planes intersect, defined analytically by $\boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} \stackrel{\text{def}}{=} (\boldsymbol{p} \wedge \boldsymbol{q}) \wedge \boldsymbol{r}$.

### 3.2. Screws and linear complexes

Straight lines can be identified with points on the Klein quadric via Plücker coordinates. Following the mechanics literature, we call *screws* the elements $\boldsymbol{s}$ of $\mathbb{P}^5$ not necessarily lying on this quadric—that is, they may not verify $(\boldsymbol{s}|\boldsymbol{s}) = 0$, and say that two screws $\boldsymbol{s}$ and $\boldsymbol{t}$ are *reciprocal* when $(\boldsymbol{s}|\boldsymbol{t}) = 0$. A screw $\boldsymbol{s}$ can be identified with the *linear complex* formed by all lines $\boldsymbol{\delta}$ such that $(\boldsymbol{s}|\boldsymbol{\delta}) = 0$. Two screws $\boldsymbol{s}$ and $\boldsymbol{t}$ define a straight line $\lambda\boldsymbol{s} + \mu\boldsymbol{t}$ in $\mathbb{P}^5$, or equivalently, a *pencil of linear complexes* whose *carrier*, defined as the set of all lines reciprocal to all screws in the pencil, is a linear congruence.

There are three types of pencils of complexes and, accordingly, three types of linear congruences (Figure 4, left): a hyperbolic pencil intersects the Klein quadric in two real points, an elliptic pencil intersects it in two conjugate complex points, and a parabolic pencil intersects the Klein quadric in a double real point (tangency). The lines in the corresponding congruences respectively intersect two real lines, two complex conjugate lines, and a double real line. In the latter case, the pencil and the associated congruence are defined by a line $\boldsymbol{\delta}$ and a screw $\boldsymbol{s}$ in the hyperplane $\boldsymbol{T}$ tangent to the Klein quadric in $\boldsymbol{\delta}$ (Figure 4, right). It is easy to show that the two are reciprocal. The intersection $\boldsymbol{\gamma}$ of the Klein quadric with $\boldsymbol{T}$ consists of all the lines intersecting $\boldsymbol{\delta}$.

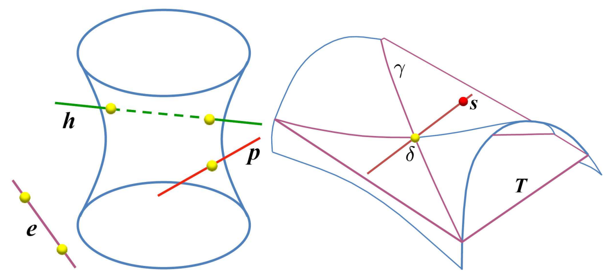

Figure 4. Left: The straight lines $\boldsymbol{e}$, $\boldsymbol{h}$, and $\boldsymbol{p}$ respectively depict elliptic, hyperbolic, and parabolic pencils of complexes intersecting the Klein quadric in two complex conjugate points (and no real point), two real points, and a double real point. Right: A line $\boldsymbol{\delta}$ and a screw $\boldsymbol{s}$ in the tangent hyperplane $\boldsymbol{T}$ to the Klein quadric in $\boldsymbol{\delta}$ define a parabolic pencil of complexes and the corresponding congruence. These drawings are for illustration only since the corresponding geometric elements "live" in $\mathbb{P}^5$.

### 3.3. The essential map of a camera

It is easy—if a bit tedious (see appendix)—to derive for every camera captured by the formalism proposed in this paper an explicit formula for the corresponding *essential* map—that is, the map $\boldsymbol{x} \rightarrow \boldsymbol{\xi}$ associating with every point $\boldsymbol{x}$ the ray $\boldsymbol{\xi}$ of the camera passing through this point.

For a pinhole camera, $\boldsymbol{\xi} = \boldsymbol{c} \vee \boldsymbol{x}$. As shown in the appendix, the essential map for the camera associated with any linear congruence can always be written as $\boldsymbol{\xi} = \boldsymbol{x} \vee \mathcal{A}\boldsymbol{x}$ for some $4 \times 4$ matrix $\mathcal{A}$. This was originally shown by Pajdla [4] for linear oblique cameras—that is, elliptic linear congruences—for which $\mathcal{A}$ is a nonsingular *elliptic involution*. We show in the appendix that this is also the case for hyperbolic and parabolic congruences, although the matrices $\mathcal{A}$ are singular in these cases. We also show in the appendix that, in the case of hyperbolic and elliptic cameras, we can write the essential map in a different form—that is, as $\boldsymbol{\xi} = \mathcal{X}\hat{\boldsymbol{x}}$, where $\mathcal{X}$ is a $6 \times 4$ matrix whose columns are screws, and $\hat{\boldsymbol{x}}$ is a vector in $\mathbb{R}^4$ whose components are quadratic forms in $\boldsymbol{x}$. In the rest of this section, we use these analytical forms of the essential map to derive explicit formulas for several fundamental geometric problems.

**Note:** We assume from now on that the "internal projective parameters" of our cameras are known—that is, the coordinates of all points, planes, and lines defining them are known in some fixed but arbitrary projective coordinate system. We will discuss briefly totally uncalibrated and fully (metrically) calibrated cameras in Section 4.

### 3.4. Direct projection

Let us consider an image plane $\boldsymbol{r}$ equipped with a (projective) basis $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$,[1] and denote by $\boldsymbol{u}$ =

[1] We assume here that the *relative* scales of the vectors $\boldsymbol{y}_i$ are fixed. This can be done (for example) by picking a fourth *unit* point in $\boldsymbol{r}$ to complete the specification of a projective coordinate system for that plane.

$(u_1, u_2, u_3)^T$ the coordinate vector of any point $\boldsymbol{y}$ in the plane $\boldsymbol{r}$. In particular, we can write $\boldsymbol{y} = \mathcal{Y}\boldsymbol{u}$, where $\mathcal{Y}$ is the $4 \times 3$ matrix $\mathcal{Y} = [\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3]$. This matrix is rectangular and non invertible. However, it admits a three-parameter family of *pseudo inverses* $\mathcal{Y}^+_{\boldsymbol{z}}$ such that $\mathcal{Y}^+_{\boldsymbol{z}}\mathcal{Y} = \mathrm{Id}$, and we have

$$\boldsymbol{u} = \mathcal{Y}^+_{\boldsymbol{z}}\boldsymbol{y}, \quad \text{where} \quad \mathcal{Y}^+_{\boldsymbol{z}} \stackrel{\text{def}}{=} \begin{bmatrix} (\boldsymbol{y}_2 \vee \boldsymbol{y}_3 \vee \boldsymbol{z})^T \\ (\boldsymbol{y}_3 \vee \boldsymbol{y}_1 \vee \boldsymbol{z})^T \\ (\boldsymbol{y}_1 \vee \boldsymbol{y}_2 \vee \boldsymbol{z})^T \end{bmatrix} \tag{1}$$

for some point $\boldsymbol{z}$ arbitrarily chosen outside the image plane. Note that the $3 \times 4$ matrix $\mathcal{Y}^+_{\boldsymbol{z}}$ is the dual of an *inverse projection matrix* as defined in [3] for example.

Let us use this notation to derive the projection equations for a pinhole camera in a purely projective manner. We consider a pinhole camera with optical center $\boldsymbol{x}$, the optical ray $\boldsymbol{\xi}$ associated with a point $\boldsymbol{x}$ is $\boldsymbol{\xi} = \boldsymbol{x} \vee \boldsymbol{c}$ and the corresponding image point is $\boldsymbol{y} = \boldsymbol{\xi} \wedge \boldsymbol{r} = (\boldsymbol{c} \cdot \boldsymbol{r})\boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{r})\boldsymbol{c}$. Now, using Eq. (1) with $\boldsymbol{z} = \boldsymbol{c}$, we obtain $\boldsymbol{u} = (\boldsymbol{c} \cdot \boldsymbol{r})\mathcal{Y}^+_{\boldsymbol{c}}\boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{r})\mathcal{Y}^+_{\boldsymbol{c}}\boldsymbol{c}$. Thus, since $\boldsymbol{c}$ belongs to the three planes associated with the rows of $\mathcal{Y}^+_{\boldsymbol{c}}$, we have $\boldsymbol{u} = \mathcal{P}\boldsymbol{x}$, where $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{Y}^+_{\boldsymbol{c}}$.

Let us now come back to the case of a linear congruence for which the ray associated with some point $\boldsymbol{x}$ is $\boldsymbol{\xi} = \boldsymbol{x} \vee \mathcal{A}\boldsymbol{x}$. The image of $\boldsymbol{x}$ is

$$\boldsymbol{y} = \boldsymbol{\xi} \wedge \boldsymbol{r} = ((\mathcal{A}\boldsymbol{x}) \cdot \boldsymbol{r})\boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{r})\mathcal{A}\boldsymbol{x},$$

and the coordinate vector $\boldsymbol{u}$ of the image point $\boldsymbol{y}$ can thus be written as

$$\boldsymbol{u} = ((\mathcal{A}\boldsymbol{x}) \cdot \boldsymbol{r})\mathcal{Y}^+_{\boldsymbol{z}}\boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{r})\mathcal{Y}^+_{\boldsymbol{z}}\mathcal{A}\boldsymbol{x},$$

or, since $\boldsymbol{r} = \boldsymbol{y}_1 \vee \boldsymbol{y}_2 \vee \boldsymbol{y}_3$,

$$\boldsymbol{u} = |\mathcal{Y}, \mathcal{A}\boldsymbol{x}|\mathcal{Y}^+_{\boldsymbol{z}}\boldsymbol{x} - |\mathcal{Y}, \boldsymbol{x}|\mathcal{Y}^+_{\boldsymbol{z}}\mathcal{A}\boldsymbol{x},$$

where $|\mathcal{U}, \boldsymbol{v}|$ denotes the determinant of the $4 \times 4$ matrix whose first three columns are given by the $3 \times 4$ matrix $\mathcal{U}$, and whose last column is the vector $\boldsymbol{v}$ of $\mathbb{R}^4$. The choice of $\boldsymbol{z}$ (outside of the plane $\boldsymbol{r}$) is once again arbitrary.

### 3.5. Inverse projection

Let us now consider an image point $\boldsymbol{y}$ with coordinate vector $\boldsymbol{u}$ in some coordinate system $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$ of the retina. Its preimage is

$$\boldsymbol{\xi} = \boldsymbol{y} \vee \mathcal{A}\boldsymbol{y} = (\sum_{i=1}^{3} u_i \boldsymbol{y}_i) \vee (\sum_{i=1}^{3} u_i \mathcal{A}\boldsymbol{y}_i),$$

which can be rewritten as

$$\boldsymbol{\xi} = \sum_{i=1}^{3} u_i^2 \boldsymbol{\xi}_{ii} + \sum_{i<j\leq 3} u_i u_j (\boldsymbol{\xi}_{ij} + \boldsymbol{\xi}_{ji}),$$

where $\boldsymbol{\xi}_{ij} = \boldsymbol{y}_i \vee \mathcal{A}\boldsymbol{y}_j$. In other words $\boldsymbol{\xi}$ can be written as a linear combination of six fixed screws (when $\boldsymbol{\xi}_{ij}$ and $\boldsymbol{\xi}_{ji}$ are not coplanar, their sum is a screw but not a line). In fact, it is shown in the appendix that it is always possible to write $\boldsymbol{\xi}$ as a linear combination of $k$ fixed screws $\boldsymbol{\xi}_i$ $(= 1, \dots, k)$, with $k = 4$ for hyperbolic and elliptic congruences, and $k = 5$ for parabolic ones, and with coefficients that are quadratic functions in $\boldsymbol{u}$. In other words, we can always write

$$\boldsymbol{\xi} = \hat{\mathcal{P}}^T \hat{\boldsymbol{u}}, \quad \text{where} \quad \hat{\mathcal{P}}^T = [\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k], \tag{2}$$

and $\hat{\boldsymbol{u}}$ is a vector of $\mathbb{R}^k$ whose components are quadratic forms in $\boldsymbol{u}$ (with known coefficients given our assumption of known internal projective parameters). Note the similarity with the familiar $3 \times 6$ *line projection matrix* associated with pinhole cameras [3].

### 3.6. Triangulation

Let us consider $n \geq 2$ image points $\boldsymbol{y}_i$ $(i = 1, \dots, n)$ with coordinates $\boldsymbol{u}_i$. The triangulation problem is to find the point $\boldsymbol{x}$ that projects onto these cameras. We write that the corresponding rays $\boldsymbol{\xi}_i = \hat{\mathcal{P}}_i^T \hat{\boldsymbol{u}}_i$ must pass through $\boldsymbol{x}$, or $[\boldsymbol{\xi}_{i\vee}]\boldsymbol{x} = 0$ for $i = 1, \dots, n$. This is a $4n \times 4$ system of homogeneous linear equations in $\boldsymbol{x}$, of which $2n$ are independent since each matrix $[\boldsymbol{\xi}_{i\vee}]$ has only rank 2, and the corresponding homogeneous linear least-squares problem can be solved as an eigenvalue problem.

### 3.7. Epipolar geometry

As noted in Section 3.5, for any non-degenerate linear congruence, the line $\boldsymbol{\xi}$ associated with a point $\boldsymbol{y}$ with coordinates $\boldsymbol{u}$ in some image plane $\boldsymbol{r}$ can be written in the form of Eq. (2) for some $k \times 6$ matrix $\hat{\mathcal{P}}$ with $k$ equal to 4 or 5, and some vector $\hat{\boldsymbol{u}}$ of $\mathbb{R}^k$ whose components are quadratic forms in $\boldsymbol{u}$. This equation also holds for $k = 3$ when $\hat{\mathcal{P}}$ is a $3 \times 6$ line projection matrix and $\hat{\boldsymbol{u}} = \boldsymbol{u}$ [3]. It follows that, given any two cameras with $k_1 \times 6$ and $k_2 \times 6$ matrices $\hat{\mathcal{P}}_1$ and $\hat{\mathcal{P}}_2$, and two image points with coordinates $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and associated projection rays $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, the epipolar constraint can be written as $0 = (\boldsymbol{\xi}_1|\boldsymbol{\xi}_2) = \boldsymbol{\xi}_1^T \boldsymbol{\xi}_2^*$, or

$$\hat{\boldsymbol{u}}_1^T \mathcal{F} \hat{\boldsymbol{u}}_2 = 0, \quad \text{where} \quad \mathcal{F} = \hat{\mathcal{P}}_1 \hat{\mathcal{P}}_2^{*T}.$$

where $\hat{\mathcal{P}}^*$ is the matrix obtained from $\hat{\mathcal{P}}$ by swapping its first three columns with its last three. The $k_1 \times k_2$ matrix $\mathcal{F}$ is the generalization of the usual *fundamental matrix* associated with pinhole cameras. It should be noted that since the matrix $\hat{\mathcal{P}}$ associated with a two-slit camera (hyperbolic congruence) has only 4 rows ($k = 4$), the fundamental matrix associated with two two-slit cameras is only a $4 \times 4$ matrix, a more “economical” parameterization of epipolar geometry than the rank-4 $6 \times 6$ matrix constructed in [4] (see [13] for a different construction of a $4 \times 4$ fundamental matrix for metrically calibrated two-slit cameras).

### 3.8. Three-view geometry

Formulas for the trinocular case are easily derived as well using the general trilinear constraints obeyed by three lines intersecting in a point [9], i.e.,

$$\mathcal{T}_i(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) = 0 \quad \text{for} \quad i = 1, \ldots, 4,$$

where $\mathcal{T}_i$ is a $6 \times 6 \times 6$ tensor. This translates immediately into the following three-view constraints verified by three matching image points with coordinates $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$:

$$\hat{\mathcal{T}}_i(\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \hat{\boldsymbol{u}}_3) \stackrel{\text{def}}{=} \mathcal{T}_i(\hat{\mathcal{P}}_1^T \hat{\boldsymbol{u}}_1, \hat{\mathcal{P}}_2^T \hat{\boldsymbol{u}}_2, \hat{\mathcal{P}}_3^T \hat{\boldsymbol{u}}_3) = 0$$

for $i = 1, \ldots, 4$. These polynomial constraints are the generalization of the trifocal constraints associated with pinhole cameras. The maximum size of the corresponding tensor is $5 \times 5 \times 5$, with at most triquadratic constraints.

## 4. Discussion

We have presented a unified framework for representing many of the cameras discussed in the literature as linear families of lines of rank 3 or 4. In turn, this has allowed us to derive simple and general formulas for monocular and multi-view geometry. Besides practical applications—e.g., an implementation of the multi-view tensors presented in this paper—many open theoretical problems remain. For example, is it possible to find a simple parameterization of linear parabolic congruences in terms of four fixed screws, as was done for elliptic and hyperbolic ones? This would bring down the size of all fundamental matrices to $4 \times 4$. Our approach has been purely projective. What about Euclidean cameras, their "internal" parameters, and self calibration? Conversely, what about totally uncalibrated cameras, for which the (inverse) projection matrices are unknown? In this case, the parameters of the quadratic forms associated with inverse projection and multi-view geometry are also unknown, leading back to $6 \times 6$ fundamental matrices. Does knowing these determine the cameras' projective parameters? Another interesting issue is to go back to photometry and study how our cameras sample the *light field*, as proposed in [7]. Much remains to be done.

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## 5. Appendix

We derive in this appendix the equation relating, for each non-degenerate line congruence, any point $\boldsymbol{x}$ to the corresponding line. In each case, we show that it can be written in terms of some operator $\mathcal{A}$ that maps $\boldsymbol{x}$ onto some other point $\mathcal{A}\boldsymbol{x}$ on $\boldsymbol{\xi}$, which in turn allows us to write the action of a camera in the form $\boldsymbol{x} \to \boldsymbol{x} \vee \mathcal{A}\boldsymbol{x}$.

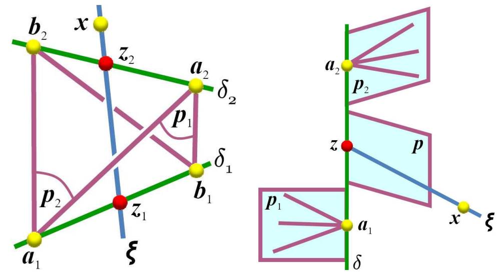


Figure 5. Parameterizing a hyperbolic linear congruence (left) and a parabolic one (right, the plane $\boldsymbol{p}_3$ and the corresponding flat pencil are not shown in the figure).

### 5.1. Hyperbolic linear congruences

Let us consider a line $\boldsymbol{\xi}$ passing through some point $\boldsymbol{x}$ in the hyperbolic congruence formed by all lines intersecting the two real *focal lines* $\boldsymbol{\delta}_1 = \boldsymbol{p}_1 \wedge \boldsymbol{q}_1$ and $\boldsymbol{\delta}_2 = \boldsymbol{p}_2 \wedge \boldsymbol{q}_2$ (Figure 5, left). We can write

$$\begin{aligned}\boldsymbol{\xi} &= (\boldsymbol{\delta}_1 \vee \boldsymbol{x}) \wedge (\boldsymbol{\delta}_2 \vee \boldsymbol{x}) \\ &= [(\boldsymbol{q}_1 \cdot \boldsymbol{x})\boldsymbol{p}_1 - (\boldsymbol{p}_1 \cdot \boldsymbol{x})\boldsymbol{q}_1] \wedge [(\boldsymbol{q}_2 \cdot \boldsymbol{x})\boldsymbol{p}_2 - (\boldsymbol{p}_2 \cdot \boldsymbol{x})\boldsymbol{q}_2],\end{aligned}$$

which is easily rewritten as $\boldsymbol{\xi} = \mathcal{X}\hat{\boldsymbol{x}}$, where

$$\mathcal{X} = [\boldsymbol{q}_1 \wedge \boldsymbol{q}_2, \boldsymbol{p}_2 \wedge \boldsymbol{q}_1, \boldsymbol{q}_2 \wedge \boldsymbol{p}_1, \boldsymbol{p}_1 \wedge \boldsymbol{p}_2],$$

and

$$\hat{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x}^T(\boldsymbol{p}_1\boldsymbol{p}_2^T)\boldsymbol{x} \\ \boldsymbol{x}^T(\boldsymbol{p}_1\boldsymbol{q}_2^T)\boldsymbol{x} \\ \boldsymbol{x}^T(\boldsymbol{q}_1\boldsymbol{p}_2^T)\boldsymbol{x} \\ \boldsymbol{x}^T(\boldsymbol{q}_1\boldsymbol{q}_2^T)\boldsymbol{x} \end{bmatrix}.$$

Note that the columns of $\mathcal{X}$ are lines in the congruence. Given an image point $\boldsymbol{y}$ with coordinates $\boldsymbol{u}$ in some basis $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$ of the image plane, its preimage is thus $\boldsymbol{\xi} = \hat{\mathcal{P}}^T\hat{\boldsymbol{u}}$, where $\hat{\mathcal{P}} = \mathcal{X}^T$, and $\hat{\boldsymbol{u}}$ is a vector of $\mathbb{R}^4$ with components that are quadratic forms in $\boldsymbol{u}$.[2] This provides a more "compact" alternative to using the general machinery of Section 3.5 to construct a $6 \times 6$ matrix $\hat{\mathcal{P}}$.

Let us now show that, as mentioned in Section 3.5, hyperbolic congruences can in fact be represented by a linear mapping $\mathcal{A}$. We define the four points: $\boldsymbol{a}_1 = \boldsymbol{p}_1 \wedge \boldsymbol{q}_1 \wedge \boldsymbol{p}_2$, $\boldsymbol{b}_1 = \boldsymbol{p}_1 \wedge \boldsymbol{q}_1 \wedge \boldsymbol{q}_2$, $\boldsymbol{a}_2 = \boldsymbol{p}_2 \wedge \boldsymbol{q}_2 \wedge \boldsymbol{p}_1$, and $\boldsymbol{b}_2 = \boldsymbol{p}_2 \wedge \boldsymbol{q}_2 \wedge \boldsymbol{q}_1$, so that we have $\boldsymbol{\delta}_i = \boldsymbol{a}_i \vee \boldsymbol{b}_i$ for $i = 1, 2$. It is easy to show that the points where $\boldsymbol{\xi}$ intersects $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are respectively[3]

$$\begin{cases} \boldsymbol{z}_1 = (\boldsymbol{q}_2 \cdot \boldsymbol{x})\boldsymbol{a}_1 - (\boldsymbol{p}_2 \cdot \boldsymbol{x})\boldsymbol{b}_1, \\ \boldsymbol{z}_2 = (\boldsymbol{q}_1 \cdot \boldsymbol{x})\boldsymbol{a}_2 - (\boldsymbol{p}_1 \cdot \boldsymbol{x})\boldsymbol{b}_2. \end{cases}$$

[2] The coefficients of these quadratic forms are projective invariants, but depend of course on the choice of the parameterization of the camera by the planes $\boldsymbol{p}_1$, $\boldsymbol{q}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{q}_2$, and by the points $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{y}_3$.

[3] This formula is correct for this particular choice of points on $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$. It would be false for arbitrary points not lying in the proper combination of planes $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$.

In particular, we have $\boldsymbol{\xi} = \boldsymbol{x} \vee (\mathcal{A}_1 \boldsymbol{x}) = \boldsymbol{x} \vee (\mathcal{A}_2 \boldsymbol{x})$, with $\mathcal{A}_1 = \boldsymbol{a}_1 \boldsymbol{q}_2^T - \boldsymbol{b}_1 \boldsymbol{p}_2^T$ and $\mathcal{A}_2 = \boldsymbol{a}_2 \boldsymbol{q}_1^T - \boldsymbol{b}_2 \boldsymbol{p}_1^T$.

### 5.2. Parabolic linear congruences

A parabolic congruence is a one-parameter continuous family of flat pencils sharing a common line $\boldsymbol{\delta}$, such that the successive planes $\boldsymbol{p}_t$ in the pencil of planes $\boldsymbol{\delta}^*$ passing through $\boldsymbol{\delta}$ and the centers $\boldsymbol{a}_t$ of the corresponding flat pencils are in projective correspondence [10, Th. 3.2.9]. In particular, let us consider three planes $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ in $\boldsymbol{\delta}^*$ and the corresponding points $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$. A line $\boldsymbol{\xi}$ in the congruence passing through the point $\boldsymbol{x}$ can be characterized as follows (Figure 5, right): It lies in a plane $\boldsymbol{p}$ in the pencil $\boldsymbol{\delta}^*$ and passes through a point $\boldsymbol{z}$ of $\boldsymbol{\delta}$ such that

$$\begin{cases} \boldsymbol{z} = \alpha \boldsymbol{a}_1 + \beta \boldsymbol{a}_2 \\ \boldsymbol{p} = \alpha \boldsymbol{p}_1 + \beta \boldsymbol{p}_2 \end{cases} \quad \text{with} \quad \begin{cases} \boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2 \\ \boldsymbol{p}_3 = \boldsymbol{p}_1 + \boldsymbol{p}_2 \end{cases},$$

where the rightmost equalities fix the relative scales of all vectors. Since $\boldsymbol{\delta} = \boldsymbol{p}_1 \wedge \boldsymbol{p}_2$, we have $\boldsymbol{p} = \boldsymbol{\delta} \vee \boldsymbol{x} = (\boldsymbol{p}_1 \boldsymbol{p}_2^T - \boldsymbol{p}_2 \boldsymbol{p}_1^T)\boldsymbol{x}$, and it follows that $\alpha = \boldsymbol{x} \cdot \boldsymbol{p}_2$ and $\beta = -\boldsymbol{x} \cdot \boldsymbol{p}_1$. In turn, we have

$$\boldsymbol{\xi} = \boldsymbol{z} \vee \boldsymbol{x} = [(\boldsymbol{x} \cdot \boldsymbol{p}_2)\boldsymbol{a}_1 - (\boldsymbol{x} \cdot \boldsymbol{p}_1)\boldsymbol{a}_2)] \vee \boldsymbol{x}. \tag{3}$$

Note that Eq. (3) can be rewritten as $\boldsymbol{\xi} = \boldsymbol{x} \vee \mathcal{A}\boldsymbol{x}$, similar to the formula for *linear oblique cameras* given in [4], but the projective transformation $\mathcal{A} = \boldsymbol{a}_1 \boldsymbol{p}_2^T - \boldsymbol{a}_2 \boldsymbol{p}_1^T$ is singular with rank 2 in our case.

Given an image point $\boldsymbol{y}$ with coordinate vector $\boldsymbol{u}$ in some basis $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$ of the image plane, we can now use the general method of Section 3 to write its preimage $\boldsymbol{\xi}$ as a linear combination of six of the lines $\boldsymbol{y}_i \vee \mathcal{A}\boldsymbol{y}_j$ $(i, j = 1, 2, 3)$. Since any of these lines intersects the line $\boldsymbol{\delta}$, only five of them are in fact linearly independent, which finally allows us to write $\boldsymbol{\xi} = \hat{\mathcal{P}}^T \hat{\boldsymbol{u}}$ for some $5 \times 6$ matrix $\hat{\mathcal{P}}$ easily computed from the given points $\boldsymbol{y}_i$.

### 5.3. Elliptic linear congruences

Pajdla [4] has shown that elliptic congruences can be characterized by an elliptic involution $\mathcal{A}$ such that the line $\boldsymbol{\xi}$ associated with a point $\boldsymbol{x}$ is $\boldsymbol{\xi} = \boldsymbol{x} \vee \mathcal{A}\boldsymbol{x}$. Remember that an elliptic linear congruence is formed of all lines intersecting two conjugate complex *focal lines* $\boldsymbol{\delta}$ and $\bar{\boldsymbol{\delta}}$, where $\bar{\boldsymbol{u}}$ denotes the conjugate of any vector $\boldsymbol{u}$ in $\mathbb{C}^n$ (Figure 4). Let us take $\boldsymbol{\delta} = \boldsymbol{p} \wedge \boldsymbol{q}$, then the conjugate of $\boldsymbol{\delta}$ can be written as $\bar{\boldsymbol{\delta}} = \bar{\boldsymbol{p}} \wedge \bar{\boldsymbol{q}}$. Similar to Section 5.1, we obtain

$$\begin{aligned} \boldsymbol{\xi} &= [(\boldsymbol{q} \cdot \boldsymbol{x})\boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{x})\boldsymbol{q}] \wedge [(\bar{\boldsymbol{q}} \cdot \boldsymbol{x})\bar{\boldsymbol{p}} - (\bar{\boldsymbol{p}} \cdot \boldsymbol{x})\bar{\boldsymbol{q}}], \\ &= (\boldsymbol{q} \cdot \boldsymbol{x})(\bar{\boldsymbol{q}} \cdot \boldsymbol{x})\boldsymbol{p} \wedge \bar{\boldsymbol{p}} - (\boldsymbol{q} \cdot \boldsymbol{x})(\bar{\boldsymbol{p}} \cdot \boldsymbol{x})\boldsymbol{p} \wedge \bar{\boldsymbol{q}} \\ &- (\boldsymbol{p} \cdot \boldsymbol{x})(\bar{\boldsymbol{q}} \cdot \boldsymbol{x})\boldsymbol{q} \wedge \bar{\boldsymbol{p}} + (\boldsymbol{p} \cdot \boldsymbol{x})(\bar{\boldsymbol{p}} \cdot \boldsymbol{x})\boldsymbol{q} \wedge \bar{\boldsymbol{q}}. \end{aligned}$$

Now, let $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ denote the real and imaginary parts of $\boldsymbol{p}$, and $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ denote the real and imaginary parts of $\boldsymbol{q}$. Using the properties of complex conjugates and the antisymmetry of the meet operator, it is then easy (if a bit tedious) to show that $\boldsymbol{\xi} = \mathcal{X}\hat{\boldsymbol{x}}$, where

$$\mathcal{X} = [\boldsymbol{p}_1 \wedge \boldsymbol{p}_2, \boldsymbol{p}_2 \wedge \boldsymbol{q}_1 + \boldsymbol{q}_2 \wedge \boldsymbol{p}_1, \boldsymbol{p}_1 \wedge \boldsymbol{q}_1 + \boldsymbol{p}_2 \wedge \boldsymbol{q}_2, \boldsymbol{q}_1 \wedge \boldsymbol{q}_2]$$

and

$$\hat{\boldsymbol{x}} = \begin{bmatrix} (\boldsymbol{q}_1 \cdot \boldsymbol{x})^2 + (\boldsymbol{q}_2 \cdot \boldsymbol{x})^2 \\ (\boldsymbol{q}_1 \cdot \boldsymbol{x})(\boldsymbol{p}_1 \cdot \boldsymbol{x}) + (\boldsymbol{q}_2 \cdot \boldsymbol{x})(\boldsymbol{p}_2 \cdot \boldsymbol{x}) \\ (\boldsymbol{q}_2 \cdot \boldsymbol{x})(\boldsymbol{p}_1 \cdot \boldsymbol{x}) - (\boldsymbol{q}_1 \cdot \boldsymbol{x})(\boldsymbol{p}_2 \cdot \boldsymbol{x}) \\ (\boldsymbol{p}_1 \cdot \boldsymbol{x})^2 + (\boldsymbol{p}_2 \cdot \boldsymbol{x})^2 \end{bmatrix}.$$

The second and third column of $\mathcal{X}$ are *not* lines but only screws since the lines $\boldsymbol{p}_2 \wedge \boldsymbol{q}_1$ and $\boldsymbol{q}_2 \wedge \boldsymbol{p}_1$ are mutually skew, and so are $\boldsymbol{p}_1 \wedge \boldsymbol{q}_1$ and $\boldsymbol{p}_2 \wedge \boldsymbol{q}_2$. It follows that the preimage of an image point $\boldsymbol{y}$, with coordinates $\boldsymbol{u}$ in some basis $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$ of the image plane, can be written in the usual form $\boldsymbol{\xi} = \hat{\mathcal{P}}^T \hat{\boldsymbol{u}}$, with $\hat{\mathcal{P}} = \mathcal{X}^T$.

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