Bifurcation of a Class of Discrete-time Neural Networks

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***Abstract*— In this paper, a class of discrete-time system modeling a network with two neurons is considered. Its flip bifurcations (also called period-doubling bifurcations for map) are demonstrated by deriving the equation describing the flow on the center manifold. In particular, the explicit formula for determining the direction and the stability of flip bifurcations are obtained. The theoretical analysis is verified by numerical simulations.**

I. INTRODUCTION

In the past few decades, neural networks have received intensive interest due to their wide applications, such as, pattern recognition, associative memory and combinational optimization, and its dynamical behavior plays an important role. Many works [1-13] have been published to investigate the dynamics of neural networks since Hopfield [10] constructed a simplified neural network model. Neural networks with one or two neurons are prototypes to understand the dynamics of large-scale networks, many results have been made for such simplified networks [3,4,6-9,11,12].

In 2003, Yuan and Huang [13] studied the asymptotical behavior of the following difference system:

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + a_{11} f(x_1(n)) + a_{12} f(x_2(n)), \\ x_2(n+1) &= \beta x_2(n) + a_{21} f(x_1(n)) + a_{22} f(x_2(n)), \\ & \qquad n = 0, 1, 2, ..., \end{aligned}$$

where $\beta \in (0,1)$ is a constant and $f : \mathbb{R} \to \mathbb{R}$ is the activation function given by the piecewise constant McCulloch-Pitts nonlinearity

$$f(u) = \begin{cases} -1, & u > \sigma, \\ +1, & u \le \sigma, \end{cases}$$

where $\sigma \in \mathbb{R}$ is a constant, and acts as the threshold. In 2004, Yuan et al. [6] considered the following system:

$$\begin{aligned} x_1(n+1) =& \beta x_1(n) + (1-\beta) f(\alpha x_1(n)) \\ &+ (1-\beta) f(\gamma_1 x_2(n)), \\ x_2(n+1) =& \beta x_2(n) - (1-\beta) f(\gamma_2 x_1(n)) \\ &+ (1-\beta) f(\alpha x_2(n)), \qquad n = 0, 1, 2, ..., \end{aligned}$$

where $\beta \in (0,1)$ is internal decay of the neurons, the constant $\alpha > 0$ and $\gamma_i (i = 1, 2)$ denote the gain parameters, $f : \mathbb{R} \to \mathbb{R}$ is a continuous transfer function and $f(0) = 0$. They discussed the global and local stability of the equilibrium, which gave some sufficient conditions to guarantee the existence of bifurcation, meanwhile got a formula to determine the direction and stability of bifurcation. In 2005, Yuan et al. [7] introduced a more general model based on [6]. They studied the following discrete-time neural network model with self-connection in the following form

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + a_{11} f_1(x_1(n)) + a_{12} f_2(x_2(n)), \\ x_2(n+1) &= \beta x_2(n) + a_{21} f_1(x_1(n)) + a_{22} f_2(x_2(n)), \\ & \qquad n = 0, 1, 2, ..., \end{aligned}$$

where $\beta \in (0,1)$ is internal decays of the neurons. Some sufficient conditions were given to guarantee the stability of the equilibrium and the existence of Neimark-Sacker bifurcation. The direction and the stability of Neimark-Sacker bifurcation were discussed. In 2007, He et al. [14] further studied the following neural networks with different internal decay of the neurons

$$\begin{aligned} x_1(n+1) &= \alpha x_1(n) + a_{11} f_1(x_1(n)) + a_{12} f_2(x_2(n)), \\ x_2(n+1) &= \beta x_2(n) + a_{21} f_1(x_1(n)) + a_{22} f_2(x_2(n)), \\ & \qquad n = 0, 1, 2, ..., \end{aligned} \tag{1}$$

where $x_i (i = 1, 2)$ denotes the state of the i-th neuron, $\alpha \in (0,1)$, $\beta \in (0,1)$ are internal decays of the neurons, the constants $a_{ij} (i, j = 1, 2)$ denotes the connection weights, $f_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous transfer functions and $f_i(0) = 0 (i = 1, 2)$. Not only the stability of equilibrium and the existence of Neimark-Sacker bifurcation but also the direction of the Neimark-Sacker bifurcation and the stability of the bifurcating periodic solution of system (1) are investigated.

However, in addition to Neimark-Sacker bifurcation, system (1) may exhibit more plentiful behaviors of dynamics such as saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation, flip bifurcation, and even chaos([14,15]). In this paper, all of our effort will be concentrated on the flip bifurcations at equilibrium $(0,0)$ of system (1). In fact, the direction and stability of the flip bifurcations at equilibrium $(0,0)$ are determined by approximately computing a center manifold.

The organization of this paper is as follows. In next section, we devote to the direction and stability of the flip bifurcations. In section 3, some simulations are made to demonstrate our results.

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II. Direction and Stability of the Flip Bifurcation

In this section, the formulas for determining the direction and the stability of flip bifurcations of system (1) at the equilibrium (0,0) will be presented by employing the center manifold theory. we assume that the transfer functions in (1) satisfy

(H_1) $f_i \in C^3(\mathbb{R},\mathbb{R}),\ f_i(0)=0,\ f_i'(0)\neq 0,\ i=1,2.$

For the sake of simplicity and the need of discussion, the following parameters are defined:

$$T_1=\frac{1}{2}(\alpha+a_{11}f_1'(0)),\quad T_2=\frac{1}{2}(\beta+a_{22}f_2'(0)),$$
$$D=-a_{12}a_{21}f_1'(0)f_2'(0).$$

Further, we assume that

(H_2) $a_{12}a_{21}\neq 0,\ T_1+T_2\neq 0,\ (T_1-T_2)^2>D.$

Now system (1) can be rewritten as a mapping $F:\mathbb{R}^2\to\mathbb{R}^2$:

$$\begin{bmatrix} x_1(n)\\ x_2(n)\end{bmatrix}\mapsto \begin{bmatrix}\alpha+a_{11}f_1'(0) & a_{12}f_2'(0)\\ a_{21}f_1'(0) & \beta+a_{22}f_2'(0)\end{bmatrix}\times\begin{bmatrix} x_1(n)\\ x_2(n)\end{bmatrix}+\begin{bmatrix}F_1(x(n))\\ F_2(x(n))\end{bmatrix}, \tag{2}$$

where $x(n)=(x_1(n),x_2(n))^T\in\mathbb{R}^2$. From assumption (H_1), we know that $F_i(i=1,2)$ in (2) can be expanded as

$$\begin{aligned}F_1(x)=&\frac{a_{11}}{2}f_1''(0)x_1^2+\frac{a_{12}}{2}f_2''(0)x_2^2+\frac{a_{11}}{6}f_1'''(0)x_1^3\\&+\frac{a_{12}}{6}f_2'''(0)x_2^3+\mathcal{O}(4),\\F_2(x)=&\frac{a_{21}}{2}f_1''(0)x_1^2+\frac{a_{22}}{2}f_2''(0)x_2^2+\frac{a_{21}}{6}f_1'''(0)x_1^3\\&+\frac{a_{22}}{6}f_2'''(0)x_2^3+\mathcal{O}(4),\end{aligned}$$

where $\mathcal{O}(4)$ means terms of order≥ 4. Then the Jacobian matrix of (2) at $(0,0)$ is

$$DF(0,0)=\begin{bmatrix}\alpha+a_{11}f_1'(0) & a_{12}f_2'(0)\\ a_{21}f_1'(0) & \beta+a_{22}f_2'(0)\end{bmatrix}$$

and its eigenvalues are

$$\lambda_{1,2}=T_1+T_2\pm\sqrt{(T_1-T_2)^2-D}.$$

Lemma 2.1: Suppose that (H_1) and (H_2) hold and $(2T_1+1)(2T_2+1)=-D$. If $T_1+T_2<-1$, then $\lambda_1=-1$ and $\lambda_2<-1$; if $T_1+T_2>-1$, then $\lambda_2=-1$, $\lambda_1>-1$ and $\lambda_1\neq 1$.

The proof of Lemma 2.1 is trivial and will be omitted.

We first consider the case $T_1+T_2<-1$. For convenience, we let

$$\begin{aligned}A\ :=&\ \frac{1}{(\lambda_1-\lambda_2)a_{12}f_2'(0)},\\
A_1:=&\ \frac{A}{2(\lambda_1^2-\lambda_2)}\big[(a_{12}f_2'(0))^2f_1''(0)(a_{21}a_{12}f_2'(0)\\&-\lambda_1a_{11})+\lambda_1^2f_2''(0)(a_{22}a_{12}f_2'(0)-\lambda_1a_{12})\big],\\
A_2:=&-\frac{A}{2}\big[(a_{11}(2T_1-\lambda_2)+a_{21}a_{12}f_2'(0))f_1''(0)\\&\times(a_{12}f_2'(0))^2+(a_{12}(2T_1-\lambda_2)\\&+a_{22}a_{12}f_2'(0))f_2''(0)(2T_1-\lambda_1)^2\big],\\
A_3:=&-AA_1\big[(a_{11}(2T_1-\lambda_2)+a_{21}a_{12}f_2'(0))f_1''(0)\\&\times(a_{12}f_2'(0))^2+(a_{12}(2T_1-\lambda_2)\\&+a_{22}a_{12}f_2'(0))f_2''(0)(2T_1-\lambda_1)(2T_1-\lambda_2)\big]\\&+\frac{A}{6}\big[\big(a_{11}(2T_1-\lambda_2)+a_{21}a_{12}f_2'(0)\big)\\&\times(a_{12}f_2'(0))^3f_1'''(0)-\big(a_{12}(2T_1-\lambda_2)\\&+a_{22}a_{12}f_2'(0)\big)(2T_1-\lambda_1)^3f_2'''(0)\big],\\
A^*:=&\ A_2^2+A_3.\end{aligned}$$

Then we have the following theorem.

Theorem 2.1: Suppose that (H_1) and (H_2) hold. If $(2T_1+1)(2T_2+1)=-D$, $T_1+T_2<-1$ and $A^*\neq 0$, then a flip bifurcation occurs at the equilibrium $(0,0)$. More concretely, for $A^*>0$, a 2-periodic orbit of system (1) emerges near the $(0,0)$ when $T_1<-\frac{1}{2}-\frac{D}{2(2T_2+1)}$, which is actually attractive, but the 2-periodic orbit does not exist when $T_1\geq -\frac{1}{2}-\frac{D}{2(2T_2+1)}$; for $A^*<0$, a repellent 2-periodic orbit of system (1) emerges near $(0,0)$ when $T_1>-\frac{1}{2}-\frac{D}{2(2T_2+1)}$, but the 2-periodic orbit does not exist when $T_1\leq -\frac{1}{2}-\frac{D}{2(2T_2+1)}$.

Proof: We can easily see that the matrix $DF(0,0)$ has eigenvectors $(-a_{12}f_2'(0),2T_1-\lambda_1)^T$ and $(-a_{12}f_2'(0),2T_1-\lambda_2)^T$ corresponding to λ_1 and λ_2 respectively. Hence the matrix $DF(0,0)$ can be diagonalized by the change of variables $(x_1,x_2)^T=H(u,v)^T$, where

$$H=\begin{bmatrix}-a_{12}f_2'(0) & -a_{12}f_2'(0)\\ 2T_1-\lambda_1 & 2T_1-\lambda_2\end{bmatrix},$$

and therefore the mapping F can be changed into $\Phi_{T_1}\colon\mathbb{R}^2\to\mathbb{R}^2$,

$$\begin{aligned}\begin{bmatrix}u\\ v\end{bmatrix}\ \mapsto\ &\begin{bmatrix}\lambda_1u\\ \lambda_2v\end{bmatrix}+\frac{1}{(\lambda_1-\lambda_2)a_{12}f_2'(0)}\\&\times\begin{bmatrix}-(2T_1-\lambda_2)\tilde{F}_1(u,v)-a_{12}f_2'(0)\tilde{F}_2(u,v)\\ (2T_1-\lambda_1)\tilde{F}_1(u,v)+a_{12}f_2'(0)\tilde{F}_2(u,v)\end{bmatrix}\\&+\mathcal{O}(4)\end{aligned}\tag{3}$$

where

$$\begin{aligned}\tilde{F}_1(u,v) =&\frac{a_{11}}{2}f_1''(0)(a_{12}f_2'(0))^2(u+v)^2\\&+\frac{a_{12}}{2}f_2''(0)[(2T_1(u+v)-(\lambda_1 u+\lambda_2 v)]^2\\&+\frac{a_{11}}{6}f_1'''(0)(a_{12}f_2'(0))^3(u+v)^3\\&+\frac{a_{12}}{6}f_2'''(0)[(2T_1(u+v)-(\lambda_1 u+\lambda_2 v)]^3,\\ \tilde{F}_2(u,v) =&\frac{a_{21}}{2}f_1''(0)(a_{12}f_2'(0))^2(u+v)^2\\&+\frac{a_{22}}{2}f_2''(0)[(2T_1(u+v)-(\lambda_1 u+\lambda_2 v)]^2\\&+\frac{a_{21}}{6}f_1'''(0)(a_{12}f_2'(0))^3(u+v)^3\\&+\frac{a_{22}}{6}f_2'''(0)[(2T_1(u+v)-(\lambda_1 u+\lambda_2 v)]^3.\end{aligned}$$

We now want to compute the center manifold and derive the mapping on the center manifold. We assume

$$v = h(u,T_1) = au^2 + buT_1 + cT_1^2 + \mathcal{O}(3) \tag{4}$$

near the origin. By Theorem 2.1.4 in [15], those coefficients a, b and c can be determined by the equation

$$\begin{aligned}&\mathcal{N}\,(h(u,T_1))\\&:= h\big(\lambda_1 u - A(2T_1-\lambda_2)\tilde{F}_1(u,h(u,T_1))\\&\quad -Aa_{12}f_2'(0)\tilde{F}_2(u,h(u,T_1)),\ T_1\big)\\&\quad -\lambda_2 h(u,T_1) - A(2T_1-\lambda_1)\tilde{F}_1(u,h(u,T_1))\\&\quad -Aa_{12}f_2'(0)\tilde{F}_2(u,h(u,T_1))\\&= 0.\end{aligned} \tag{5}$$

where

$$\begin{aligned}\tilde{F}_1(u,h) =&\frac{a_{11}}{2}f_1''(0)(a_{12}f_2'(0))^2\Big(u+au^2+buT_1+cT_1^2\Big)^2\\&+\frac{a_{12}}{2}f_2''(0)\Big((2T_1(u+au^2+buT_1+cT_1^2)\\&-\lambda_1 u-\lambda_2(au^2+buT_1+cT_1^2)\Big)^2+\mathcal{O}(4),\\ \tilde{F}_2(u,h) =&\frac{a_{21}}{2}f_1''(0)(a_{12}f_2'(0))^2\Big(u+au^2+buT_1+cT_1^2\Big)^2\\&+\frac{a_{22}}{2}f_2''(0)\Big((2T_1(u+au^2+buT_1+cT_1^2)\\&-\lambda_1 u-\lambda_2(au^2+buT_1+cT_1^2)\Big)^2+\mathcal{O}(4).\end{aligned}$$

Comparing coefficients of u^2, uT_1 and T_1^2 in (5), we get

$$\begin{aligned}a\lambda_1^2-a\lambda_2 =& -\frac{A}{2}\Big[a_{11}f_1''(0)(a_{12}f_2'(0))^2\lambda_1\\&+a_{12}f_2''(0)\lambda_1^3\\&+a_{21}a_{12}f_2'(0)f_1''(0)(a_{12}f_2'(0))^2\\&-a_{22}a_{12}f_2'(0)f_2''(0)\lambda_1^2\Big],\\ b\lambda_1-b\lambda_2 =& 0,\\ c\ -c\lambda_2 =& 0,\end{aligned}$$

from which we solve

$$\begin{aligned}a &= \frac{A}{2(\lambda_1^2-\lambda_2)}\big[(a_{12}f_2'(0))^2f_1''(0)(a_{21}a_{12}f_2'(0)\\&\quad-\lambda_1 a_{11})+\lambda_1^2 f_2''(0)(a_{22}a_{12}f_2'(0)-\lambda_1 a_{12})\big]\\&= A_1,\\ b &= 0,\\ c &= 0.\end{aligned}$$

Thus the expression of (4) is determined and

$$v = h(u,s) = A_1u^2 + \mathcal{O}(3). \tag{6}$$

Substituting (6) into the first equation in (3), we obtain a one-dimensional mapping $u \mapsto \phi_{T_1}(u)$, where

$$\begin{aligned}&\phi_{T_1}(u)\\&= \lambda_1 u - \frac{1}{(\lambda_1-\lambda_2)a_{12}f_2'(0)}\Big[(2T_1-\lambda_2)\tilde{F}_1(u,v)\\&\quad+a_{12}f_2'(0)\tilde{F}_2(u,v)\Big]+\mathcal{O}(\|u\|^4)\\&= \lambda_1 u-\frac{A}{2}\Big[\Big(a_{11}(2T_1-\lambda_2)+a_{21}a_{12}f_2'(0)\Big)f_1''(0)\\&\quad\times\big(a_{12}f_2'(0)\big)^2+\big(a_{12}(2T_1-\lambda_2)+a_{22}a_{12}f_2'(0)\big)\\&\quad\times f_2''(0)\lambda_1^2(2T_1-\lambda_1)^2\Big]u^2-Aa\Big[(a_{11}(2T_1-\lambda_2)\\&\quad+a_{21}a_{12}f_2'(0))f_1''(0)(a_{12}f_2'(0))^2+\Big(a_{12}(2T_1-\lambda_2)\\&\quad+a_{22}a_{12}f_2'(0)\Big)f_2''(0)(2T_1-\lambda_1)(2T_1-\lambda_2)\Big]u^3\\&\quad+\frac{A}{6}\Big[\big(a_{11}(2T_1-\lambda_2)+a_{21}a_{12}f_2'(0)\big)\\&\quad\times(a_{12}f_2'(0))^3f_1'''(0)-\Big(a_{12}(2T_1-\lambda_2)\\&\quad+a_{22}a_{12}f_2'(0)\Big)(2T_1-\lambda_1)^3f_2'''(0)\Big]u^3+\mathcal{O}(\|u\|^4)\\&=\lambda_1 u+A_2u^2+A_3u^3+\mathcal{O}(\|u\|^4).\end{aligned}$$

Here we note the dependence of λ_1, λ_2 on T_1. Further, by directly computing and in view of the assumptions, we get that

$$\begin{aligned}\Big[\frac{\partial\phi_{T_1}}{\partial T_1}\frac{\partial^2\phi_{T_1}}{\partial u^2} &+ 2\frac{\partial^2\phi_{T_1}}{\partial u\partial T_1}\Big]\Big|_{(u,T_1)=(0,T_1)}\\ &= \frac{2T_2+1}{T_1+T_2+1}\neq 0,\end{aligned} \tag{7}$$

$$\begin{aligned}\Big[\frac{1}{2}\Big(\frac{\partial^2\phi_{T_1}}{\partial u^2}\Big)^2 &+ \frac{1}{3}\frac{\partial^3\phi_{T_1}}{\partial u^3}\Big]\Big|_{(u,T_1)=(0,T_1)}\\ &= 2(A_2^2+A_3)\ \neq 0.\end{aligned} \tag{8}$$

Thus, the conditions (F_1) and (F_2) of Theorem 3.5.1 in [14] are checked by (7) and (8) respectively, implying that a flip bifurcation occurs at $(u,T_1)=(0,-\frac{1}{2}-\frac{D}{2(2T_2+1)})$ and a 2-periodic orbit arises as stated in the theorem. □

Next, we consider the case $T_1 + T_2 > -1$. Let

$$
\begin{aligned}
B \; &:= \frac{1}{(\lambda_2 - \lambda_1) a_{12} f_2'(0)}, \\
B_1 &:= \frac{B}{2(\lambda_2^2 - \lambda_1)} \big[(a_{12} f_2'(0))^2 f_1''(0) (a_{21} a_{12} f_2'(0) \\
&\quad - \lambda_2 a_{11}) + \lambda_2^2 f_2''(0) (a_{22} a_{12} f_2'(0) - \lambda_2 a_{12}) \big], \\
B_2 &:= -\frac{B}{2} \big[(a_{11}(2T_1 - \lambda_1) + a_{21} a_{12} f_2'(0)) f_1''(0) \\
&\quad \times (a_{12} f_2'(0))^2 + (a_{12}(2T_1 - \lambda_1) \\
&\quad + a_{22} a_{12} f_2'(0)) f_2''(0) (2T_1 - \lambda_2)^2 \big], \\
B_3 &:= -B B_1 \big[(a_{11}(2T_1 - \lambda_1) + a_{21} a_{12} f_2'(0)) f_1''(0) \\
&\quad \times (a_{12} f_2'(0))^2 + (a_{12}(2T_1 - \lambda_1) \\
&\quad + a_{22} a_{12} f_2'(0)) f_2''(0) (2T_1 - \lambda_1)(2T_1 - \lambda_2) \big] \\
&\quad + \frac{B}{6} \big[\big(a_{11}(2T_1 - \lambda_1) + a_{21} a_{12} f_2'(0) \big) \\
&\quad \times (a_{12} f_2'(0))^3 f_1'''(0) - \big(a_{12}(2T_1 - \lambda_1) \\
&\quad + a_{22} a_{12} f_2'(0) \big) (2T_1 - \lambda_2)^3 f_2'''(0) \big], \\
B^* &:= B_2^2 + B_3.
\end{aligned}
$$

Then we have the following theorem.

Theorem 2.2: Suppose that (H_1) and (H_2) hold. If $(2T_1 + 1)(2T_2 + 1) = -D$, $T_1 + T_2 > -1$ and $B^* \neq 0$, then a flip bifurcation occurs at the equilibrium $(0, 0)$. More concretely, for $B^* > 0$, a 2-periodic orbit of system (1) emerges near the $(0, 0)$ when $T_1 < -\frac{1}{2} - \frac{D}{2(2T_2+1)}$, which is actually attractive, but the 2-periodic orbit does not exist when $T_1 \geq -\frac{1}{2} - \frac{D}{2(2T_2+1)}$; for $B^* < 0$, a repellent 2-periodic orbit of system (1) emerges near $(0, 0)$ when $T_1 > -\frac{1}{2} - \frac{D}{2(2T_2+1)}$, but the 2-periodic orbit does not exist when $T_1 \leq -\frac{1}{2} - \frac{D}{2(2T_2+1)}$.

The proof of Theorem 2.2 is similar to that of Theorem 2.1 and because of limited space it will be omitted.

III. Numerical simulations

In this section, we give numerical simulations to support our theoretical analysis.

Example: Let $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$, $f_1(u) = sin(u)$, $f_2(u) = arctan(u/2)$ in the system (1). By the simple calculation, we obtain

$$
\begin{aligned}
f_1'(0) \; &= 1, \quad f_1''(0) = f_2''(0) = 0 \quad f_2'(0) = 0.5, \\
f_1'''(0) &= -1, \quad f_2'''(0) = -0.25.
\end{aligned}
$$

(I) If $a_{12} = -2$, $a_{21} = 2$, $a_{22} = -1$, then $T_2 = 0.125$, $D = 2$. When $a_{11} = -2.8$, we get $T_1 + T_2 = -1.175 < -1$, $(T_1 - T_2)^2 = 2.031 > D$ and $A^* = A_3 = 1.985 > 0$. By Theorem 2.1, we know that a flip bifurcation occurs at the equilibrium $(0, 0)$. A 2-periodic orbit of system (1) emerges near the $(0, 0)$ when $T_1 < -\frac{1}{2} - \frac{D}{2(2T_2+1)} = -1.3 (i.e., a_{11} < -2.8)$, but the 2-periodic orbit does not exist when $T_1 \geq -1.3 (i.e., a_{11} > -2.8)$. The corresponding bifurcation plot is shown in Fig.1.

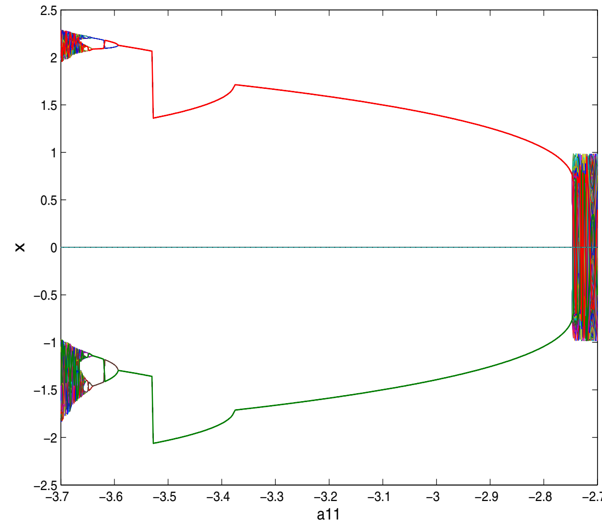

Fig.1. Period-two orbit bifurcates from (0,0) for $a_{11} < -2.8$.

(II) If $a_{12} = -1$, $a_{21} = 2$, $a_{22} = -1$, then $T_2 = 0.125$, $D = 1$. When $a_{11} = -2$, we get $T_1 + T_2 = -0.775 > -1$, $(T_1 - T_2)^2 = 1.051 > D$ and $B^* = B_3 = 0.085 > 0$. By Theorem 2.2, we know that a flip bifurcation occurs at the equilibrium $(0, 0)$. A 2-periodic orbit of system (1) emerges near the $(0, 0)$ when $T_1 < -\frac{1}{2} - \frac{D}{2(2T_2+1)} = -0.9 (i.e., a_{11} < -2)$, but the 2-periodic orbit does not exist when $T_1 \geq -0.9 (i.e., a_{11} > -2)$. The corresponding bifurcation plot is shown in Fig.2.

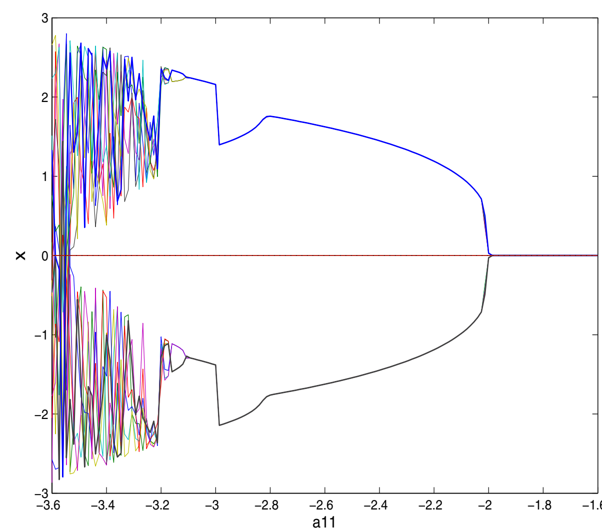

Fig. 2. Period-two orbit bifurcates from (0,0) for $a_{11} < -2$.

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