Stochastic State Estimation and Control for Stochastic Descriptor Systems

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***Abstract*—In this paper, a stochastic observer is proposed, which can make the estimation error dynamics to be stochastically asymptotical stable and impulse-free. Moreover, an integrated estimation and feedback control mechanism is presented, which can make both the state and error dynamics stochastically asymptotical stable and impulse-free. Simulated results demonstrate the efficiency of the proposed approach. Finally, this technique is extended to fuzzy stochastic systems.**

***Keywords*—stochastic systems, decriptor systems, signal estimation, fuzzy systems.**

I. INTRODUCTION

State and signal estimation and reconstruction are always hot in signal processing and control community. There have been huge results reported [1-3].

Stochastic descriptor system is a more complex system compared with the conventional deterministic or stochastic models. This is because those stochastic descriptor systems possess singular nature and stochastic behaviors. Therefore, investigation on signal estimation and control for stochastic systems is of significance, but challenging. However, there are few results for stochastic descriptor systems, except for some limited work in Kalman filtering [4-5].

In this study, we consider a descriptor system model with Itô formula. A descriptor stochastic estimator is proposed. Moreover, an integrated estimation and control mechanism is developed.

II. PROBLEM STATEMENT

Consider the following stochastic system

$$\begin{aligned} Edx(t) &= Ax(t)dt + Jx(t)dw + Bu(t)dt \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input, $y(t) \in R^p$ is the output vector, and w is the one-dimension Brownian motion.

Now we give the following definitions:

Definition 1. Stochastic system (1) is called stochastic stable, if for any $\varepsilon \in (0,1)$ and $r > 0$, there is always a positive number $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\{|x(t; t_0, x_0)| < r \quad for\ all\ t \ge t_0\} \ge 1 - \varepsilon$$

where $|x_0| < \delta$.

Definition 2. Stochastic system (1) is stochastic asymptotically stable, if system (1) is stochastic stable, and for any $\varepsilon \in (0,1)$, there is $\delta_0 = \delta_0(\varepsilon,\ t_0) > 0$ such that

$$P\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\} \ge 1 - \varepsilon$$

where $|x_0| < \delta_0$.

Definition 3. Stochastic system (1) is admissible if the system is stochastic asymptotically stable and impulse-free.

Lemma 1 [6]**.** Stochastic system (1) is admissible if there is matrix X such that the following inequalities hold:

$$\begin{aligned} &E^T X = X^T E \ge 0 \\ &A^T X + X^T A + J^T (E^+)^T E^T X E^+ J < 0 \end{aligned}$$

where E^+ is the Moore-Penrose inverse of E.

The goal of the study is to design an estimator such that the estimation error is admissible.

III. Estimator Design for Stochastic Systems

A. Estimator Design

Design the following estimator:

$$\begin{aligned} Ed\hat{x}(t) &= A\hat{x}(t)dt + Bu(t)dt + G(\hat{y}(t) - y(t))dt \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \quad (2)$$

Subtracting (2) from (1), and letting $e(t) = x(t) - \hat{x}(t)$, the error dynamics are governed by the following equation:

$$Ede(t) = (A + GC)e(t)dt + Jx(t)dw \quad (3)$$

Let $\xi(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$, and then construct the following augmented system

$$\tilde{E}d\xi(t) = \tilde{A}e(t)dt + \tilde{J}\xi(t)dw \quad (4)$$

where

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A + GC \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} J & 0 \\ J & 0 \end{bmatrix}. \quad (5)$$

It is obvious that $rank\tilde{E} = rank(\tilde{E} \quad \tilde{J})$, which means that the noise term does not change the system structure.

Theorem 1. There is a stochastic observer in the form (2) for system (1), if there is matrix X such that

$$\tilde{E}^T X = X^T \tilde{E} \geq 0 \quad (6a)$$

$$\tilde{A}^T X + X^T \tilde{A} + \tilde{J}^T (\tilde{E}^+)^T \tilde{E}^T X \tilde{E}^+ \tilde{J} < 0 \quad (6b)$$

where $\tilde{E}$, $\tilde{A}$ and $\tilde{J}$ are defined as (5).

Proof: This result can be obtained directly by using Lemma 1.

Remark 1. Theorem 1 indicates that $\xi(t) \to 0$, and then means $e(t) \to 0$ and $x(t) \to 0$ equivalently. This means that the proposed observer above requires system (1) is admissible. However, this condition does not hold always in practical cases. Therefore, this motivates us to make some improving for the estimator design.

B. Estimator-Based Controller

Next, we will design a mechanism to make both the observed system and estimation error to be admissible.

Applying the following feedback

$$u(t) = K\hat{x}(t) \quad (7)$$

to system (1), one has

$$\begin{aligned} Edx(t) &= Ax(t)dt + Jx(t)dw + BK\hat{x}(t)dt \\ &= (A + BK)x(t)dt + Jx(t)dw - BKe(t)dt \end{aligned} \quad (8)$$

Denote by $\xi(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$. Using (3) and (8), one has

$$\bar{E}d\xi(t) = \bar{A}e(t)dt + \bar{J}\xi(t)dw \quad (9)$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \quad \bar{A} = \begin{bmatrix} A + BK & -BK \\ 0 & A + GC \end{bmatrix} \quad \bar{J} = \begin{bmatrix} J & 0 \\ J & 0 \end{bmatrix}. \quad (10)$$

The following theorem is given.

Theorem 2. System (9) is admissible if there is matrix Z such that

$$\bar{E}^T Z = Z^T \bar{E} \geq 0 \quad (11a)$$

$$\bar{A}^T Z + Z^T \bar{A} + \bar{J}^T (\bar{E}^+)^T \bar{E}^T Z \bar{E}^+ \bar{J} < 0 \quad (11b)$$

where $\bar{E}$, $\bar{A}$ and $\bar{J}$ are defined as (10).

Proof: The result can be obtained directly in terms of Lemma 1.

Theorem 3. Consider system (1), state estimator (2), and state feedback (7). If there are matrices X and Y such that

$$E^T X = X^T E \geq 0 \quad (12a)$$

$$\begin{aligned} &(A + BK)^T X + X^T (A + BK) \\ &+ J^T (E^+)^T E^T X E^+ J < 0 \end{aligned} \quad (12b)$$

$$E^T Y = Y^T E \geq 0 \quad (12c)$$

$$(A + GC)^T Y + Y^T (A + GC) < 0 \quad (12d)$$

then the closed-loop system is admissible.

Proof: For X and Y satisfying (12a)-(12d), let $Z = \begin{bmatrix} X & 0 \\ 0 & \alpha Y \end{bmatrix}$, where α is a positive number. Then, one has

$$\bar{E}^T Z = \begin{bmatrix} E^T X & 0 \\ 0 & \alpha E^T Y \end{bmatrix} = \begin{bmatrix} X^T E & 0 \\ 0 & \alpha Y^T E \end{bmatrix} = Z^T \bar{E} \geq 0$$

In addition, according to $\bar{E}^+ = \begin{bmatrix} E^+ & 0 \\ 0 & E^+ \end{bmatrix}$, one has

$$\bar{A}^T Z + Z^T \bar{A} + \bar{J}^T (\bar{E}^+)^T \bar{E}^T Z \bar{E}^+ \bar{J}$$

$$= \begin{bmatrix} A + BK & -BK \\ 0 & A + GC \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & \alpha Y \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & \alpha Y \end{bmatrix}^T \begin{bmatrix} A + BK & -BK \\ 0 & A + GC \end{bmatrix}$$

$$+\begin{bmatrix} J & 0 \\ J & 0 \end{bmatrix}^T \left(\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}^+ \right)^T \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & \alpha Y \end{bmatrix}$$

$$\times \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}^+ \begin{bmatrix} J & 0 \\ J & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (A+BK)^T X + X^T(A+BK) & -X^T(BK) \\ -(BK)^T X & \alpha\Pi_2 \end{bmatrix}^T$$

$$+\begin{bmatrix} J & 0 \\ J & 0 \end{bmatrix}^T \begin{bmatrix} (E^+)^T E^T X E^+ & 0 \\ 0 & \alpha(E^+)^T E^T Y E^+ \end{bmatrix} \begin{bmatrix} J & 0 \\ J & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Pi_1 & -X^T(BK) \\ -(BK)^T X & \alpha\Pi_2 \end{bmatrix}$$

where

$$\Pi_1 = (A+BK)^T X + X^T(A+BK) + J^T(E^+)^T E^T X E^+ J$$

$$\Pi_2 = (A+GC)^T Y + Y^T(A+GC)$$

Using the Schur complement, one has

$$\begin{bmatrix} \Pi_1 & -X^T(BK) \\ -(BK)^T X & \alpha\Pi_2 \end{bmatrix} < 0$$

if and only if

$$\alpha\Pi_2 < 0$$

$$\Pi_1 - \alpha^{-1} X^T(BK)\Pi_2^{-1}(BK)^T X < 0$$

For $\Pi_1 < 0$ and $\Pi_2 < 0$, one can find a sufficient large positive number α (α^{-1} sufficient small) such that

$$\Pi_1 - \alpha^{-1} X^T(BK)\Pi_2^{-1}(BK)^T X < 0$$

That is, for X and Y satisfying (12b) and (12d), one can choose a sufficient large α such that $Z = \begin{bmatrix} X & 0 \\ 0 & \alpha Y \end{bmatrix}$ meets

$$\overline{A}^T Z + Z^T \overline{A} + \overline{J}^T(\overline{E}^+)^T \overline{E}^T Z \overline{E}^+ \overline{J} < 0$$

In the meanwhile, when X and Y satisfy (12a) and (12c), the matrix $Z = \begin{bmatrix} X & 0 \\ 0 & \alpha Y \end{bmatrix}$ make the following hold

$$\overline{E}^T Z = Z^T \overline{E} \geq 0$$

This completes the proof.

Remark 2. Theorem 3 has shown that the separation principle of estimator and state feedback controller. That is, one can design the estimator gain G and state feedback gain K, respectively. In this case, we do not need the system dynamics to be admissible.

IV. Estimator-based Controller for Fuzzy Stochastic Systems

Fuzzy model is an effective tool for handling nonlinear systems [7-9]. In this section, we will discuss the estimator-based design problem for stochastic fuzzy systems.

Consider the following system described by IF-THEN rules:

Rules i : IF z_1 is M_{1i} and … and z_p is M_{pi}, THEN

$$\begin{aligned} Edx(t) &= A_i x(t)dt + J_i x(t)dw + B_i u(t)dt \\ y(t) &= C_i x(t) \end{aligned} \tag{13}$$

wher $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the input, $z = \lfloor z_1 \ \ \dots \ \ z_p \rfloor$ are premise variables, $M_{1i} \dots M_{pi}$ are fuzzy sets. $i = 1, 2, \dots, l$, l is the number of the fuzzy rules.

The whole systems is obtained by taking the weighted average of all subsystems :

$$\begin{aligned} Edx(t) &= \sum_{i=1}^{l} h_i(z)(A_i x(t)dt + J_i x(t)dw + B_i u(t)dt) \\ y(t) &= \sum_{i=1}^{l} h_i(z) C_i x(t) \end{aligned} \tag{14}$$

A state estimator is given as:

Rules i : IF z_1 is M_{1i} and … and z_p is M_{pi}, THEN

$$\begin{aligned} Ed\hat{x}(t) &= A_i \hat{x}(t)dt + B_i u(t)dt + G_i(\hat{y}(t) - y(t)) \\ y(t) &= C_i \hat{x}(t) \end{aligned} \tag{15}$$

The whole estimator is express as:

$$\begin{aligned} Ed\hat{x}(t) &= \sum_{i=1}^{l} h_i(z)(A_i \hat{x}(t)dt + B_i u(t)dt + G_i(\hat{y}(t) - y(t))) \\ \hat{y}(t) &= \sum_{i=1}^{l} h_i(z) C_i \hat{x}(t) \end{aligned} \tag{16}$$

The estimator-based controller can be given as:

Rules i : IF z_1 is M_{1i} and … and z_p is M_{pi}, THEN

$$u(t) = K\hat{x}(t) \tag{17}$$

The control for the whole system is the following:

$$u(t) = \sum_{i=1}^{l} h_i(z)\hat{x}(t) \tag{18}$$

Subtracting (15) from (14), and using (18), and letting $e(t) = x(t) - \hat{x}(t)$, one can get the following error dynamic equation:

$$Ede(t) = \sum_{i=1}^{l}\sum_{i=1}^{l} h_i(z)h_j(z)\big((A_i + G_i C_j)e(t)dt + J_i x(t)dw\big) \tag{19}$$

Let $\xi(t)=\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}$, and construct the following augmented plant:

$$\overline{E}d\xi(t)=\sum_{i=1}^{l}\sum_{i=1}^{l}h_i(z)h_j(z)(\overline{A}_{ij}\xi(t)dt+\overline{J}_i\xi(t)dw) \quad (20)$$

where

$$\overline{E}=\begin{bmatrix}E & 0\\ 0 & E\end{bmatrix}\quad \overline{A}=\begin{bmatrix}A_i+B_iK_j & -B_iK_j\\ 0 & A_i+G_iC_j\end{bmatrix}\quad \overline{J}=\begin{bmatrix}J_i & 0\\ J_i & 0\end{bmatrix} \quad (21)$$

Lemma 2. A fuzzy stochastic system (14) can be admissible by using the estimator-based controller (16)-(17), if there exists a matrix Z such that

$$\overline{E}^T Z = Z^T\overline{E}\geq 0 \quad (22)$$

$$\overline{A}_{ij}^T Z+Z\overline{A}_{ij}+\overline{J}_i^T(\overline{E}^+)^T\overline{E}^T Z\overline{E}^+\overline{J}_i<0 \quad (23)$$

where $\overline{E}$, $\overline{A}_{ij}$ and $\overline{J}_i$ are defined by (21).

Proof. The result can be obtained directly by using Lemma1.

Now we discuss the separation property of the design for the state-feedback gain and the observer gain.

Theorem 4. A fuzzy stochastic system (14) can be admissible by using the estimator-based controller (16)-(17), if there exist matrices X and Y such that

$$E^T X = X^T E\geq 0 \quad (24a)$$

$$(A+B_iK_j)^T X+X^T(A+B_iK_j)+J_i^T(E^+)^T E^T XE^+J_i<0 \quad (24b)$$

$$\overline{E}^T Y = Y^T\overline{E}\geq 0 \quad (24c)$$

$$(A+G_iC_j)^T Y+Y^T(A+G_iC_j)+J_i^T(E^+)^T E^T YE^+J_i<0 \quad (24d)$$

Proof. Suppose there are matrices X and Y to satisfy (24a)-(24d). Let $Z=\begin{bmatrix}X & 0\\ 0 & \alpha Y\end{bmatrix}$, and α be any positive number. Therefore, one has

$$\overline{E}^T Z=\begin{bmatrix}E^T X & 0\\ 0 & \alpha E^T Y\end{bmatrix}=\begin{bmatrix}X^T E & 0\\ 0 & \alpha Y^T E\end{bmatrix}=Z^T\overline{E}\geq 0$$

Denote by $\overline{E}^+=\begin{bmatrix}E^+ & 0\\ 0 & E^+\end{bmatrix}$. One can derive that

$$\overline{A}_{ij}^T Z+Z\overline{A}_{ij}+\overline{J}_i^T(\overline{E}^+)^T\overline{E}^T Z\overline{E}^+\overline{J}_i$$

$$=\begin{bmatrix}A_i+B_iK_j & -B_iK_j\\ 0 & A_i+G_iC_j\end{bmatrix}^T\begin{bmatrix}X & 0\\ 0 & \alpha Y\end{bmatrix}+\begin{bmatrix}X & 0\\ 0 & \alpha Y\end{bmatrix}^T\begin{bmatrix}A_i+B_iK_j & -B_iK_j\\ 0 & A_i+G_iC_j\end{bmatrix}$$

$$+\begin{bmatrix}J_i & 0\\ J_i & 0\end{bmatrix}^T\left(\begin{bmatrix}E & 0\\ 0 & E\end{bmatrix}^+\right)^T\begin{bmatrix}E & 0\\ 0 & E\end{bmatrix}^T\begin{bmatrix}X & 0\\ 0 & \alpha Y\end{bmatrix}\times\begin{bmatrix}E & 0\\ 0 & E\end{bmatrix}^+\begin{bmatrix}J_i & 0\\ J_i & 0\end{bmatrix}$$

$$=\begin{bmatrix}(A_i+B_iK_j)^T X+X^T(A_i+B_iK_j) & -X^T(B_iK_j)\\ -(B_iK_j)^T X & \alpha\Pi_{ij}\end{bmatrix}^T$$

$$+\begin{bmatrix}J_i & 0\\ J_i & 0\end{bmatrix}^T\begin{bmatrix}(E^+)^T E^T XE^+ & 0\\ 0 & \alpha(E^+)^T E^T YE^+\end{bmatrix}\begin{bmatrix}J_i & 0\\ J_i & 0\end{bmatrix}$$

$$=\begin{bmatrix}\Theta_{ij} & -X^T(B_iK_j)\\ -(B_iK_j)^T X & \alpha\Pi_{ij}\end{bmatrix}$$

where

$$\Theta_{ij}=(A_i+B_iK_j)^T X+X^T(A_i+B_iK_j)+J_i^T(E^+)^T E^T XE^+J_i$$

$$\Pi_{ij}=(A_i+G_iC_j)^T Y+Y^T(A_i+G_iC_j)$$

According to the Schur complement, it is obvious that

$$\begin{bmatrix}\Theta_{ij} & -X^T(B_iK_j)\\ -(B_iK_j)^T X & \alpha\Pi_{ij}\end{bmatrix}<0$$

if and only if

$$\Pi_{ij}<0$$

$$\Theta_{ij}-\alpha^{-1}X^T(B_iK_j)\Pi_{ij}^{-1}(B_iK_j)^T X<0$$

From the inequalities, one can conclude that for $\Theta_{ij}<0$ and $\Pi_{ij}<0$, one can find a sufficiently large positive number α such that

$$\Theta_{ij}-\alpha^{-1}X^T(B_iK_j)\Pi_{ij}^{-1}(B_iK_j)^T X<0$$

In other words, for X and Y satisfying (24b) and (24d), one can always find a sufficiently large positive number α such that

$$\overline{A}_{ij}^T Z+Z\overline{A}_{ij}+\overline{J}_i^T(\overline{E}^+)^T\overline{E}^T Z\overline{E}^+\overline{J}_i<0$$

where $Z=\begin{bmatrix}X & 0\\ 0 & \alpha Y\end{bmatrix}$.

Moreover, for X and Y satisfying (24a) and (24c), one can obtain:

$$\overline{E}^T Z = Z^T \overline{E} \geq 0$$

where $Z = \begin{bmatrix} X & 0 \\ 0 & \alpha Y \end{bmatrix}$.

As a result, the fuzzy stochastic system (14) can be stabilized by the estimator-based controller (16)-(17) by Lemma 2.

Remark 3. Theorem 4 has illustrated the separation principle of fuzzy estimator and fuzzy state feedback controller. Therefore, we can design the fuzzy estimator-based controller conveniently.

V. EXAMPLE AND SIMULATION

Consider a stochastic system in the form of (1), where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix} \quad J = \begin{bmatrix} 0 & 0.2 & 0.1 \\ 0.1 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Design the following state estimator-based controller

$$Ed\hat{x}(t) = A\hat{x}(t)dt + Bu(t)dt + G(\hat{y}(t) - y(t))dt$$
$$\hat{y}(t) = C\hat{x}(t)$$
$$u(t) = K\hat{x}(t)$$

From Theorem 3, one can obtain the gains

$$K = \begin{bmatrix} -1.8283 & -6.4148 & -3.2162 \\ 1.8240 & -2.6031 & -0.0378 \end{bmatrix}$$

$$G = \begin{bmatrix} -3.0241 & -1.1514 \\ -4.3028 & -8.7416 \\ -5.2353 & -81177 \end{bmatrix}$$

Let $x(0) = (0.5 \quad 1 \quad -1)^T$. From the simulated curves, the estimation and control performance are desired.

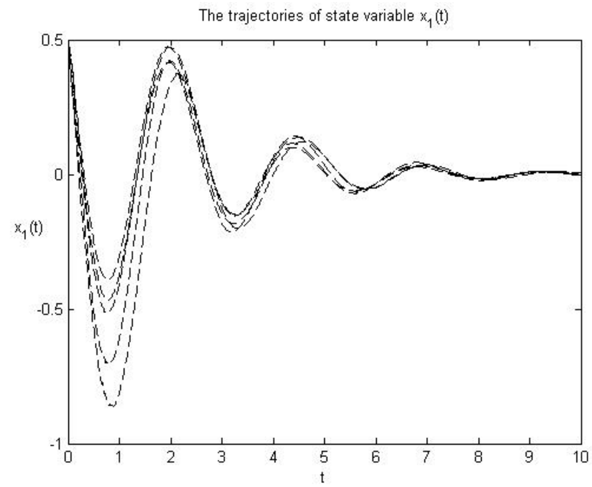

Figure 1. State $x_1(t)$: estimator-based control

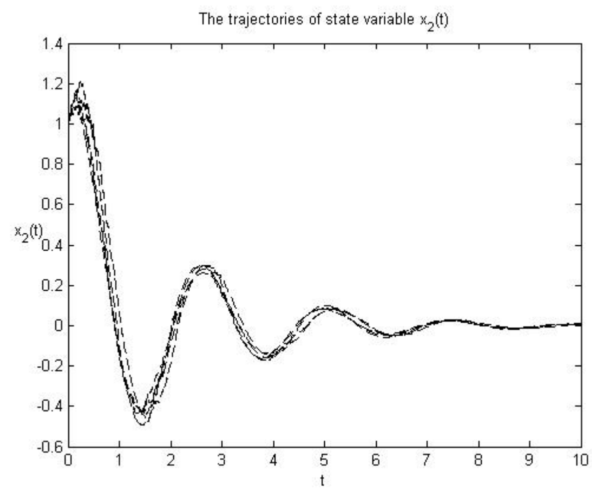

Figure 2. State $x_2(t)$: estimator-based control

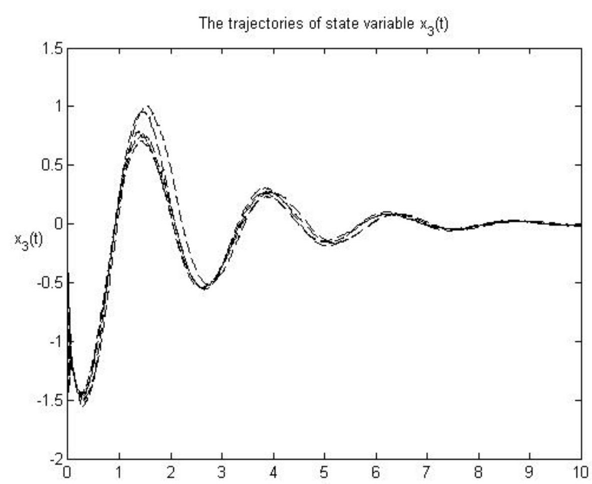

Figure 3. State $x_3(t)$: estimator-based control.

VI. CONCLUSION

In this study, a descriptor stochastic estimator has been proposed. Moreover, an integrated estimation and control mechanism has also been developed. The fuzzy stochastic case has been also investigated. The proposed estimation and control techniques will find many applications in signal processing and control issues, such as robust signal estimation, signal change detection and signal compensations.

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