A Novel algorithm of Constructing LDPC Codes with Graph Theory

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***Abstract*—There are many algorithms to construct good Low-Density Parity-Check (LDPC) Code, the most typical algorithms are bit-filling algorithm, randomly construction algorithm, and PEG (Progressive Edge Growth) algorithm. As shown in [1], the error floor of LDPC Code is decreased by using PEG algorithm, but the error correction performance in waterfall region is compromised since a stopping set with small size will form the codeword with small Hamming weight over AWGN [2]. In this paper, we propose a novel algorithm to construct LDPC Codes. In our algorithm, we construct LDPC Code to avoid small girth and small stopping set by detecting the complete associated matrix of check node (defined in this paper) that converted from bipartite graph of LDPC Code based on the graph theory. Simulation shows that the LDPC code constructed by our algorithm has lower error floor than randomly constructed LDPC Code. The performance improvement of our algorithm is 0.1dB at BER of 10^{-3} compared with PEG algorithm.**

***Keywords*—LDPC, Graph, Parity check matrix, stopping set**

I. INTRODUCTION

Low-Density Parity-Check (LDPC) Codes were introduced by Gallager [3]. It's shown that LDPC Codes can achieve performance close to the channel capacity at low complexity when iterative decoding is used. Recently LDPC Codes have increasingly been drawn attention due to its superior error correction capability and low complexity [4]. Galllager considered only regular LDPC codes whose parity check matrix have a fixed number of "1" in each row and also have a fixed number of "1" in each column. It has been shown that the performance of LDPC Code can be improved using irregular scheme [2].

As we know, the smaller length girth and smaller stopping set would reduce the performance of LDPC Code, especially in waterfall region [5]. To keep longer length girth and larger stopping sets are two main objectives to construct LDPC Code presented in this paper.

Based on the knowledge of graph theory, the check nodes in a girth in bipartite graph would form an Euler loop, and this sub-graph can be described by a matrix. So we can determine whether a girth will be formed by some check nodes by detecting the matrix correspond to the sub-graph generated from original bipartite graph. Similarly, based on the definition of stopping set in graph, we can get another description of it with matrix. So during the construction of LDPC Code as following two steps: 1. Check whether there will be a stopping set if we set "1" at current position. If it is, fill "0" in the current position. Else: 2. Check whether the check node represented by current position of parity check matrix will form a girth less than pre-defined by designer. If it is, set "0" at the current position. Or "1" is set. The method mentioned above can avoid the appearance of small stopping set and small girth in the parity check matrix.

This paper is organized as follows. Section 2 introduces some definitions of LDPC Code. The basic idea of construction of LDPC Code with graph theory is described in section 3. In section 4, we give the algorithm of construction of LDPC Code with graph theory. The results of simulation are shown in section 5. The conclusion is given in section 6.

II. BASIC DEFINITION OF LDPC CODE

Assume the length of a LDPC Code is n. This LDPC Code can be described by specifying its binary parity check matrix H. We call this code low density parity check code because H is a sparse matrix. An LDPC Code and its decoding are represented by a bipartite graph which can be described as $G(H)=(V,E)$ with $m+n$ vertices $V=(1,2,\cdots,m,m+1,\cdots,m+n)$. The first m vertices correspond to m parity check equations also referred to m check nodes in bipartite graph. The rest vertices correspond to n variable nodes. An example of bipartite graph of regular LDPC Code is represented as in figure 1 (a). This bipartite graph is corresponded to the parity check matrix H of this LDPC Code that is shown in figure 1 (b). The parity check matrix and its bipartite graph of LDPC with length $n=10$ and number of parity check equations of $m=5$ are shown in figure 1.

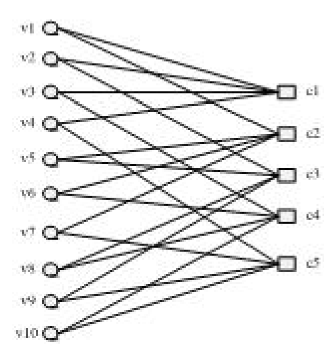

Figure. 1 (a) Bipartite graph of a regular LDPC Code

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Figure. 1 (b) Parity check matrix of a regular LDPC Code corresponded to bipartite graph shown in fig. 1(a)

As shown in the figure 1 (a), we call the number of edges that connected to a node is the degree of this node. So the degrees of variable (check) nodes are all same for regular LDPC Code as shown in figure 1 (a).

For irregular LDPC Code, the degrees of each variable (check) node are not same.

We assume λ_i and ρ_i are fraction of edges emanating from variable and check codes of degree i. Then the expression of $\lambda(x) = \sum_{d=1}^{d_v} \lambda_d x^{d-1}$ and $\rho(x) = \sum_{d=1}^{d_c} \rho_d x^{d-1}$ are the variable node degree distribution and check node degree distribution respectively where d_v and d_c are the maximum degree of variable nodes and check nodes.

III. ALGORITHM OF CONSTRUCTION WITH GRAPH THEORY

A. Basic idea of construction with graph theory

As we know, during the construction of LDPC Code, one of the main targets is to make sure there is not the small length girth in the bipartite graph correspond to parity check matrix. Considering the construction of LDPC Code with its bipartite graph directly, it will be too complex because of the large number of vertices and edges in the bipartite graph. So we try to generate a new graph or matrix that can depicts the girth in original bipartite graph. As shown in graph theory, the girth in bipartite graph is a loop. We list each vertices of a girth, eliminate the variable vertices. Then an Euler loop will be formed whose vertices are all check nodes in girth in bipartite graph. We can also write the matrix that can depicts the graph generated from original bipartite graph. So we can avoid the girth by controlling the matrix that we generated as above.

B. Abbreviations and Acronyms

Definition 1. We call the graph as Structure graph of Parity check node if this graph represents the relationship of each check node shown in bipartite graph.

For example, the parity check matrix is shown as follows:

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Assume that $c1, c2, \cdots, c6$ denotes the check nodes and $v1, v2, \cdots v12$ denotes the variable nodes in bipartite graph which is correspond to this parity check matrix. From the parity check matrix, we can get that there is a cycle in its bipartite graph as $c2 \to v1 \to c4 \to v4 \to c5 \to v5 \to c2$. Then we eliminate the information of variable node, we can get the structure graph of parity check nodes as

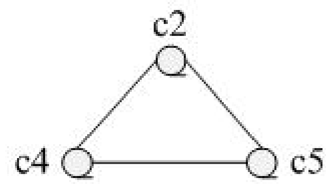

Figure. 2 Structure graph of parity check matrix with 6-cycle

Definition 2. We call a loop as Euler Loop if all vertices are passed when each edge in the graph is passed once and only once.

Definition 3. We call a graph as Euler graph if there is an Euler loop.

Definition 4. Assume graph $G = < V, E >$ and graph $G' = < V', E' >$, we call these two graphs are isomorphic if there exists a mapping one by one: g: $v_i \to v_i'$ and $e = (v_i, v_j)$ is an edge of G if and only if $e' = (g(v_i), g(v_j))$ is an edge of G'.

Theorem 1. The structure graphs of parity check matrix with cycles that have the same length are all Euler graph and isomorphic.

The proof of this theorem is shown in [4].

Definition 5 Assume G is a graph. Let $v_1, v_2, \cdots, v_p$ and $e_1, e_2, \cdots, e_q$ are vertices and edges of G respectively. Matrix $M(G) = (m_{ij})$ in which

$$m_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is associated with } e_j \\ 0, & \text{if } v_i \text{ is not associated with } e_j \end{cases}$$

Then we call matrix M as complete associated matrix of graph G.

Definition 6. We call a matrix without an all “0” sub-matrix (with or without column permutation) and all the numbers of “1” of rows are less or equal than 2 as sub-matrix that can not be divided.

Theorem 2. If there is a sub-matrix of complete associated matrix that can not be divided, and the numbers of “1” in each rows are not all 2, then the check nodes represented by this sub-matrix can not form a cycle in bipartite graph.

Proof:

a) Because the sub-matrix of complete associated matrix can not be divided as shown in the definition 6, we can get that this graph is a simple connected graph.

b) We assume that the check nodes mentioned in theorem 2 consist a cycle in bipartite graph. Based on the definition of complete associated matrix, these check nodes form an Euler loop. Based on the definition of Euler loop, we know that the degrees of vertices are all two. This result is opposite from the condition in the theorem 2. According to the above, we know the check nodes mentioned in theorem 2 represented by the sub-matrix given in the theorem can not form a cycle in bipartite graph. The theorem is proofed completely.

Based on the definitions and theorems above, we can choose a check node that avoid the appearance of d-cycle (d is pre-determined by user) bipartite graph by detecting the complete associated matrix of this check node with other $\frac{d}{2}-1$ check nodes during the construction of LDPC Code.

As mentioned in [3], the performance of LDPC Code is decreased since every codeword with small Hamming weight is caused by a stopping set with small size. So preventing small stopping sets also helps to increase the minimum distance and performance of LDPC. The stopping set can be described as follows:

Definition 7 A stopping set S of variable nodes is said to form a stopping set if all its neighboring check nodes are connected to S at least twice.

As described in definition 7, the stopping set can also be presented by parity check matrix as follows:

First we define a function:

$$f_{SC}(\alpha)=\sum_i I(\alpha[i]=1)$$

Where α indicates column vector and $I(x)$ indicates the indicator function. Thus f_{SC} counts the number of 1's in a column vector.

Consider a subset S of the columns of the parity check matrix of size $m\times n$. Let $\Delta=\sum_{i\in S} v_i$, where the sum is over the real field (not over GF (2)). The set S forms a stopping set if $f_{SC}(\Delta)=0$.

For example, assume the parity check matrix as

$$H=\begin{pmatrix}1&0&1&0&1\\1&1&1&1&0\\0&1&1&1&1\end{pmatrix}$$

The rows denote the check nodes $c1, c2, c3$ and the columns denote the variable nodes $v1, v2, \cdots, v5$. Then the set of variable nodes $v1, v2, v3$ is such that $v1+v2+v3=[2,3,2]^T$ which does not have a 1 in any component. So, it forms a stopping set. So we also should make sure that we did not form a small stopping set during the construction of LDPC Code.

According to the definitions and theorems shown above, we randomly choose some check nodes to determine whether these check nodes and other variables nodes could form a girth, we can firstly get the structure graph of these parity check nodes and its complete associated matrix as shown in definition 1 and 5. Then the girth can be found by detecting if this structure graph of these parity check nodes can form a Euler graph according to theorem 2.

IV. CONSTRUCTION OF LDPC CODE WITH GRAPH THEORY

According to all the above, we summarize the algorithm of construction of LDPC Code with graph theory. Consider we construct an irregular LDPC Code with the column weight as $D=(d_0, d_1, \cdots, d_{N-1})$, row weight as $Q=(q_0, q_1, \cdots, q_{M-1})$ and length of cycle greater than p. We set "1" at proper position for each column one by one and others are "0". The criterion of construction is shown as follows:

Step 1. Initialize the parity check matrix and the complete associated matrix of structure graph of parity check matrix of all the check nodes.
Step 2. Set "1" or "0" for each element by column. This step will divided into the following three cases:

1). If the number of "1" in this column is zero, then choose the row whose weight is the least one and less than q_i to set "1".
2). If the number of "1" in this column is less than d_j and number of "1" in this row is less than q_i:
   - ✓ Determine whether there will be a cycle with length less than p according to theorem 1 and 2.
   - ✓ Determine whether there will be a small stopping set based on the definition described with matrix.
3). If all the cases are satisfied, then set "1" at this row otherwise choose another check node and go to b.
4). If the number of "1" in this column is equal to d_j, then go to a for the next column.

Figure. 3 Criterion of construction of LDPC code

V. SIMULATION AND RESULT

We did the simulation of irregular LDPC Code with length 10000 and rate 1/2. In this LDPC code, we avoided the appearance the cycle of length less than 6. The variable node degree distribution is $\lambda(x)=0.4999x^2+0.3294x^3+0.1707x^8$ and the check node degree distribution is $\rho(x)=0.9514x^7+0.0486x^8$. The BP algorithm is used in the decoding. The simulation used Adding White Gaussian Noise (AWGN) channel and BPSK modulation technology.

We compared the performance of LDPC Code constructed by graph theorem proposed in this paper and that of LDPC Code whose edges of bipartite graph are selected randomly by avoiding cycles of length four and also the LDPC Code constructed by PEG algorithm. The performances of the figures are shown in the figure 4. The horizontal axis denotes the signal-to-noise (SNR) and the vertical axis denotes the frame error rate (FER).

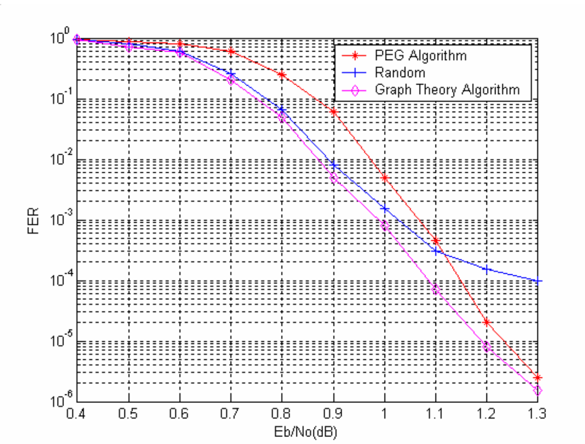

Figure. 4 Performances of different methods for construction of LDPC Code

As shown in the figure 4, we can get the best performance of LDPC Code constructed with graph theory method proposed in this paper. The randomly constructed LDPC Code shows a relatively higher error floor. It also shown from the figure that the construction algorithm proposed in this paper outperforms the PEG algorithm by more than 0.1 dB.

VI. CONCLUSION

In this paper, we proposed a new algorithm of construction of LDPC Code with graph theory. This algorithm can avoid the appearance of d -cycle in the bipartite graph (d is pre-determined by designer of LDPC Code). Furthermore, the small stopping set is avoided in our LDPC Code constructing algorithm with graph theory. So the error floor constructed by our algorithm proposed in this paper is lower than that of randomly construction algorithm and the performance in waterfall region is also be improved than original PEG algorithm.

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