Knowledge Reduction of Dominance-based Fuzzy Rough Set in Fuzzy Decision System

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***Abstract*—Dominance-based rough set approach is an useful extension of the classical rough set approach and it has been successfully applied into multi-criteria decision analysis problems. This paper present an explorative research focusing on knowledge reduction of fuzzy rough set model in fuzzy decision system. The investigated fuzzy rough set model is different from the classical fuzzy rough set model because it is based on the dominance principle of memberships of objects on the attributes. We introduce the concept of reducts of fuzzy lower and upper approximations. They are minimal subsets of attributes which preserve the fuzzy lower and upper approximate memberships for each object belongs to the universe. The judgment theorems and discernibility matrixes associated with these two reducts are also obtained. An numerical examples is employed to substantiate the conceptual arguments.**

I. INTRODUCTION

As one of the new mathematical tools to deal with uncertain and vague information, rough set theory[1], [2], [3], [4], [5], [6] was firstly proposed by Pawlak. In classical rough set model, the lower and upper approximations are defined based on the two extreme cases regarding the relationships between an equivalence class and a set. The lower approximation requires that the equivalence class is a subset of the set, while the upper approximation requires that the equivalence class has a intersectant part with the set. However, a lack of consideration for the preference-ordered domains of attributes limits the applications of Pawlak's rough set and has motivated many researchers to investigate dominance-based generalizations of rough set model. The Dominance-based Rough Set Approach (DRSA)[7], [8], [9], [10] was firstly proposed by Greco, this approach is different from the classical rough set approach because it takes into account the preference orders in the domains of attributes and in the set of decision classes. This innovation is mainly based on substitution of the indiscernibility relation (equivalence relation) by a dominance relation[7].

By generalizing the information (decision) system from crisp to fuzzy case, Greco et al. have also introduced the DRSA into fuzzy systems and proposed a new fuzzy rough set approach. This fuzzy rough approach is different from most known fuzzy set extensions of rough set theory, does not use any fuzzy logical connectives (t-norm, t-conorm, fuzzy implication) because it is based on the ordinal properties of fuzzy membership degrees only[11].

One of the most important problems which can be solved using the rough set concept is reducing attributes, i.e., knowledge reduction. In recent years, many authors have proposed different concepts of reducts[12], [13], [14], [15], [16] in classical information systems in rough set research, each of which aimed at some basic requirements. By eliminating the unnecessary attributes, one can generate different kinds of simplified or optimal decision rules from the decision system. However, only a limited number of researches on the knowledge reductions of DRSA have been presented presently. They are summarized as follows. In Ref.[17], we have proposed four types of approximate distribution reducts based on the DRSA. In Ref.[18], Shao and Zhang have presented the approach to knowledge reduction in consistent decision system by considering the ordinal properties of values of attributes.

The purpose of this paper is to present explorative research focusing on knowledge reduction of the fuzzy rough set model proposed by Greco which is based on the ordinal properties of fuzzy membership degrees. To generate simplified decision rules from fuzzy decision system, we introduce the concept of reducts of fuzzy lower and upper approximations, from which can obtain minimal subsets of attributes which preserve the lower and upper approximate memberships for each object.

To facilitate our discussion, we first present basic notions of DRSA and the relevant fuzzy rough models in fuzzy decision system. The approach to compute reducts of fuzzy rough approximations are then introduced in Section 3. An illustrative example is analyzed in Section 4. Results are summarized in Section 5.

II. PRELIMINARIES

A. Dominance-based rough set model

A decision system is a 4-tuple $\Omega =< U, AT \cup d, V, f >$. U is a non-empty finite set of objects called universe and AT is a non-empty finite set of condition attributes, such that $\forall a \in AT : U \rightarrow V_a$ where V_a is the domain of attribute a, d is

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the decision attribute where $AT \cap d = \emptyset$, V is regarded as the domain of all attributes and then $V = V_{AT} \cup V_d$, for $\forall x \in U$, $f(x, a)$ is the value that x holds on $a (a \in AT \cup d)$.

By considering the preference-ordered domains of attributes, Greco et al. have proposed an extension of the rough set that is able to deal with inconsistencies typical to exemplary decisions in Multi-Criteria Decision Making (MCDM) problems, which is called the Dominance-based Rough Set Approach (DRSA)[7], [8]. Let $\succeq_a$ be a weak preference relation on U (often called outranking) representing a preference on the set of objects with respect to criterion a; $x \succeq_a y$ means "x is at least as good as y with respect to criterion a". We say that x dominates y with respect to AT, (or, x AT-dominates y), denoted by $xD_{AT}y$, if $x \succeq_a y$ for all $a \in AT$. Assuming, without loss of generality, that domains of all criterions are ordered such that preference increases with the value, $xD_{AT}y$ is equivalent to: $f(x, a) \geq f(y, a)$ for $\forall a \in AT$. Therefore, the "granules of knowledge" used in DRSA are[7]:

- A set of objects dominating x, called A-dominating set, $D^+_{AT}(x) = \{y \in U : yD_{AT}x\}$;
- A set of objects dominated by x, called A-dominated set, $D^-_{AT}(x) = \{y \in U : xD_{AT}y\}$.

Moreover, assume that the decision attribute d makes a partition of U into a finite number of classes; let $\mathbf{CL} = \{CL_n, n \in N\}$, $N = \{1, 2, \cdots, m\}$, be a set of these classes that are ordered, that is, for $\forall r_1, r_2 \in N$ such that $r_1 > r_2$, the objects from CL_{r_1} are preferred to the objects from CL_{r_2}. The sets to be approximated are an upward union and a downward union of decision classes, which are defined respectively as $CL^{\geq}_n = \bigcup_{n' \geq n} CL_{n'}$, $CL^{\leq}_n = \bigcup_{n' \leq n} CL_{n'}$, $n, n' \in N$. The statement $x \in CL^{\geq}_n$ means "x belongs to at least class CL_n", where $x \in CL^{\leq}_n$ means "x belongs to at most class CL_n".

Suppose that we want to approximate the upward and downward unions of decision classes by using dominance principle, the lower and upper approximations of $CL^{\geq}_n$ are defined as[7], [8]:

$$\begin{aligned} \underline{AT}(CL^{\geq}_n) &= \{x \in U : D^+_{AT}(x) \subseteq CL^{\geq}_n\}, && (1) \\ \overline{AT}(CL^{\geq}_n) &= \{x \in U : D^-_{AT}(x) \cap CL^{\geq}_n \neq \emptyset\}; && (2) \end{aligned}$$

the lower and upper approximations of $CL^{\leq}_n$ are defined as[7], [8]:

$$\begin{aligned} \underline{AT}(CL^{\leq}_n) &= \{x \in U : D^-_{AT}(x) \subseteq CL^{\leq}_n\}, && (3) \\ \overline{AT}(CL^{\leq}_n) &= \{x \in U : D^+_{AT}(x) \cap CL^{\leq}_n \neq \emptyset\}. && (4) \end{aligned}$$

B. DRSA in fuzzy decision system

The fuzzy decision system represents the formulation of a problem with fuzzy samples (samples containing fuzzy representations). Thus, an decision system is called a fuzzy information system if $V = [0, 1]$, i.e., each object is described by a fuzzy membership value on each attribute. To distinguish with the classical information system, the fuzzy decision system is denoted by $\Omega^F =< U, AT \cup d, V, f >$.

In fuzzy decision table Ω^F, if $AT = \{a_1, \cdots, a_m\}$ is the set of condition attributes, d is the decision attribute, then we consider a universe of discourse U and $m + 1$ fuzzy sets, denoted by $\widetilde{a_1}, \cdots, \widetilde{a_m}$ and $\widetilde{d}$, defined on U by means of membership functions $\mu_{\widetilde{a_i}} : U \to [0, 1]$, $i \in \{1, \cdots, m\}$ and $\mu_{\widetilde{d}} : U \to [0, 1]$. $\mu_{\widetilde{a_i}}$ and $\mu_{\widetilde{d}}$ represent the values of the object x with respect to the condition attribute a_i and decision attribute d respectively. Suppose that we want to approximate knowledge contained in decision d using knowledge about $\{\widetilde{a_1}, \cdots, \widetilde{a_m}\}$. Then, the lower approximation of fuzzy set $\widetilde{d}$ given the information on $\widetilde{a_1}, \cdots, \widetilde{a_m}$ is a fuzzy set $\underline{App}(AT, \widetilde{d})$, whose membership function for each $x \in U$, denoted by $\mu[\underline{App}(AT, \widetilde{d}), x]$, is defined as follows[11]:

$$\mu[\underline{App}(AT, \widetilde{d}), x] = \min_{z \in D^{\uparrow}_{AT}(x)} \{\mu_{\widetilde{d}}(z)\}; \quad (5)$$

where for each $x \in U$, $D^{\uparrow}(x)$ is a non-empty set defined by

$$D^{\uparrow}_{AT}(x) = \{y \in U : \mu_{\widetilde{a_i}}(y) \geq \mu_{\widetilde{a_i}}(x) \text{ for each } a_i \in AT\}. \quad (6)$$

$D^{\uparrow}_{AT}(x)$ is the set of objects dominating x in terms of set of condition attributes.

The lower approximation $\mu[\underline{App}(AT, \widetilde{d}), x]$ can be interpreted as follows: in the universe U the following implication holds[11]:

If $\mu_{\widetilde{a_1}}(y) \geq \mu_{\widetilde{a_1}}(x)$ and $\mu_{\widetilde{a_2}}(y) \geq \mu_{\widetilde{a_2}}(x)$ and $\cdots$ and $\mu_{\widetilde{a_m}}(y) \geq \mu_{\widetilde{a_m}}(x)$, then $\mu_{\widetilde{d}}(y) \geq \mu[\underline{App}(AT, \widetilde{d}), x]$.

Similarity, the upper approximation of $\widetilde{d}$ given the information on $\widetilde{a_1}, \cdots, \widetilde{a_m}$ is a fuzzy set $\overline{App}(AT, \widetilde{d})$, whose membership function for each $x \in U$, denoted by $\mu[\overline{App}(AT, \widetilde{d}), x]$, is defined as follows[11]:

$$\mu[\overline{App}(AT, \widetilde{d}), x] = \max_{z \in D^{\downarrow}_{AT}(x)} \{\mu_{\widetilde{d}}(z)\}; \quad (7)$$

where for each $x \in U$, $D^{\downarrow}_{AT}(x)$ is a non-empty set defined by

$$D^{\downarrow}_{AT}(x) = \{y \in U : \mu_{\widetilde{a_i}}(y) \leq \mu_{\widetilde{a_i}}(x) \text{ for each } a_i \in AT\}. \quad (8)$$

$D^{\downarrow}_{AT}(x)$ is the set of objects dominated by x in terms of set of condition attributes.

Upper approximation $\mu[\overline{App}(AT, \widetilde{d}), x]$ can be interpreted as follows: in the universe U the following implication holds[11]:

If $\mu_{\widetilde{a_1}}(y) \leq \mu_{\widetilde{a_1}}(x)$ and $\mu_{\widetilde{a_2}}(y) \leq \mu_{\widetilde{a_2}}(x)$ and $\cdots$ and $\mu_{\widetilde{a_m}}(y) \leq \mu_{\widetilde{a_m}}(x)$, then $\mu_{\widetilde{d}}(y) \leq \mu[\overline{App}(AT, \widetilde{d}), x]$.

$[\underline{App}(AT, \widetilde{d}), \overline{App}(AT, \widetilde{d})]$ is referred to as a pair of rough set of fuzzy set $\widetilde{d}$ by using knowledge about $\{\widetilde{a_1}, \cdots, \widetilde{a_m}\}$. For more details about properties of $\left[\underline{App}(AT, \widetilde{d}), \overline{App}(AT, \widetilde{d})\right]$, we refer the readers to Ref.[11].

III. Knowledge Reduction

Definition 1: Given a fuzzy decision system Ω^F, $A \subseteq AT$,

1) A is referred to as a reduct of lower approximation $\underline{App}(AT, \widetilde{d})$ if and only if

a) for each $x \in U$, $\mu[\underline{App}(AT,\widetilde{d}),x] = \mu[\underline{App}(A,\widetilde{d}),x]$,
b) $\exists x \in U$ such that $\mu[\underline{App}(AT,\widetilde{d}),x] \neq \mu[\underline{App}(B,\widetilde{d}),x]$, for $\forall B \subset A$;

2) A is referred to as a reduct of upper approximation $\overline{App}(AT,\widetilde{d})$ if and only if
a) for each $x \in U$, $\mu[\overline{App}(AT,\widetilde{d}),x] = \mu[\overline{App}(A,\widetilde{d}),x]$,
b) $\exists x \in U$ such that $\mu[\overline{App}(AT,\widetilde{d}),x] \neq \mu[\overline{App}(B,\widetilde{d}),x]$ for $\forall B \subset A$.

By Definition 1, we can see that the reducts of lower and upper approximations are minimal subsets of attributes which preserve the lower and upper approximation memberships for each $x \in U$. In the following, we will present practical approach to compute the above two types of reducts.

Lemma 1: Given a fuzzy decision system Ω^F, $A \subseteq AT$, for $\forall x \in U$, we have

$$y \in D_A^{\uparrow}(x) \Leftrightarrow D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x), \tag{9}$$

$$y \in D_A^{\downarrow}(x) \Leftrightarrow D_A^{\downarrow}(y) \subseteq D_A^{\downarrow}(x), \tag{10}$$

$$D_A^{\uparrow}(x) = \bigcup\{D_{AT}^{\uparrow}(y) : y \in D_A^{\uparrow}(x)\}, \tag{11}$$

$$D_A^{\downarrow}(x) = \bigcup\{D_{AT}^{\downarrow}(y) : y \in D_A^{\downarrow}(x)\}. \tag{12}$$

Proof: Obviously, $D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x) \Rightarrow y \in D_A^{\uparrow}(x)$, thus, it must be proved that $y \in D_A^{\uparrow}(x) \Rightarrow D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x)$. For $\forall z \in D_A^{\uparrow}(y)$, since $y \in D_A^{\uparrow}(x)$, we have $\mu_{\widetilde{a_i}}(y) \geq \mu_{\widetilde{a_i}}(x)$ and $\mu_{\widetilde{a_i}}(z) \geq \mu_{\widetilde{a_i}}(y)$ for each $a_i \in A$, from which we can conclude that $\mu_{\widetilde{a_i}}(z) \geq \mu_{\widetilde{a_i}}(x)$ for each $a_i \in A$, thus $z \in D_A^{\uparrow}(x)$.

Similarity, it is not difficult to prove formula (10).

For $\forall y \in D_A^{\uparrow}(x)$, since $y \in D_{AT}^{\uparrow}(y)$, we have $D_A^{\uparrow}(x) \subseteq \bigcup\{D_{AT}^{\uparrow}(y) : y \in D_A^{\uparrow}(x)\}$. Consequently, it must be proved that $D_A^{\uparrow}(x) \supseteq \bigcup\{D_{AT}^{\uparrow}(y) : y \in D_A^{\uparrow}(x)\}$.

For $\forall z \in \bigcup\{D_{AT}^{\uparrow}(y) : y \in D_A^{\uparrow}(x)\}$, there must be $y \in D_A^{\uparrow}(x)$ such that $z \in D_{AT}^{\uparrow}(y)$. Since $A \subseteq AT$, we have $z \in D_A^{\uparrow}(y)$. Thus, $z \in D_A^{\uparrow}(x)$.

Similarity, it is not difficult to prove formula (12).

Theorem 2: Given a fuzzy decision system Ω^F, $A \subseteq AT$, we have

1) $\mu[\underline{App}(AT,\widetilde{d}),x] = \mu[\underline{App}(A,\widetilde{d}),x]$ for $\forall x \in U \Leftrightarrow \forall x,y \in U$, if $\mu[\underline{App}(AT,\widetilde{d}),x] > \mu[\underline{App}(AT,\widetilde{d}),y]$, then $D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x)$ does not hold;
2) $\mu[\overline{App}(AT,\widetilde{d}),x] = \mu[\overline{App}(A,\widetilde{d}),x]$ for $\forall x \in U \Leftrightarrow \forall x,y \in U$, if $\mu[\overline{App}(AT,\widetilde{d}),y] > \mu[\overline{App}(AT,\widetilde{d}),x]$, then $D_A^{\downarrow}(y) \subseteq D_A^{\downarrow}(x)$ does not hold.

Proof: 1) "$\Rightarrow$": For $\forall x,y \in U$, if $D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x)$, then by formula (5), we have $\mu[\underline{App}(A,\widetilde{d}),x] \leq \mu[\underline{App}(A,\widetilde{d}),y]$. By assumption we have $\mu[\underline{App}(AT,\widetilde{d}),x] = \mu[\underline{App}(A,\widetilde{d}),x]$ and $\mu[\underline{App}(AT,\widetilde{d}),y] = \mu[\underline{App}(A,\widetilde{d}),y]$, it follows that $\mu[\underline{App}(AT,\widetilde{d}),x] \leq \mu[\underline{App}(AT,\widetilde{d}),y]$.

"$\Leftarrow$": Since $A \subseteq AT$, we have $D_A^{\uparrow}(x) \supseteq D_{AT}^{\uparrow}(x)$, from which we can conclude that $\mu[\underline{App}(A,\widetilde{d}),x] \leq \mu[\underline{App}(AT,\widetilde{d}),x]$. Thus, it must be proved that $\mu[\underline{App}(A,\widetilde{d}),x] \geq \mu[\underline{App}(AT,\widetilde{d}),x]$. By formulas (9) and (11) we have $D_A^{\uparrow}(x) = \bigcup\{D_{AT}^{\uparrow}(y) : y \in D_A^{\uparrow}(x)\} = \bigcup\{D_{AT}^{\uparrow}(y) : D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x)\}$. Thus, $\mu[\underline{App}(A,\widetilde{d}),x] = \max\{\mu[\underline{App}(AT,\widetilde{d}),y] : D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x)\}$. By assumption we have $D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x) \Rightarrow \mu[\underline{App}(AT,\widetilde{d}),x] \leq \mu[\underline{App}(AT,\widetilde{d}),y]$. It follows that $\mu[\underline{App}(AT,\widetilde{d}),x] \leq \mu[\underline{App}(A,\widetilde{d}),x]$.

2) Similarity, it is not difficult to prove 2).

Definition 2: Given a fuzzy decision system Ω^F, denote by

$$\theta_{AT}^L = \{(y,x) : \mu[\underline{App}(AT,\widetilde{d}),x] > \mu[\underline{App}(AT,\widetilde{d}),y]\};$$

$$\theta_{AT}^U = \{(y,x) : \mu[\overline{App}(AT,\widetilde{d}),y] > \mu[\overline{App}(AT,\widetilde{d}),x]\};$$

where

$$\theta_{AT}^L(y,x) = \begin{cases} \{a_i \in AT : \mu_{\widetilde{a_i}}(y) < \mu_{\widetilde{a_i}}(x)\} : (y,x) \in \theta_{AT}^L \\ AT : (y,x) \notin \theta_{AT}^L \end{cases}$$

$$\theta_{AT}^U(y,x) = \begin{cases} \{a_i \in AT : \mu_{\widetilde{a_i}}(x) < \mu_{\widetilde{a_i}}(y)\} : (y,x) \in \theta_{AT}^U \\ AT : (y,x) \notin \theta_{AT}^U \end{cases}$$

$\theta_{AT}^L(y,x)$ and $\theta_{AT}^U(y,x)$ are referred to as lower and upper approximate discernibility attributes sets of y and x, respectively, $\boldsymbol{D}^L = \{\theta_{AT}^L(y,x)\}$ and $\boldsymbol{D}^U = \{\theta_{AT}^U(y,x)\}$ are referred to as lower and upper approximate discernibility matrixes of fuzzy decision system, respectively.

Theorem 3: Given a fuzzy decision system Ω^F, $A \subseteq AT$,

1) $\mu[\underline{App}(AT,\widetilde{d}),x] = \mu[\underline{App}(A,\widetilde{d}),x]$ for $\forall x \in U \Leftrightarrow$ for each $(y,x) \in \theta_{AT}^L$, $A \cap \theta_{AT}^L(y,x) \neq \emptyset$;
2) $\mu[\overline{App}(AT,\widetilde{d}),x] = \mu[\overline{App}(A,\widetilde{d}),x]$ for $\forall x \in U \Leftrightarrow$ for each $(y,x) \in \theta_{AT}^U$, $A \cap \theta_{AT}^U(y,x) \neq \emptyset$.

Proof: 1) "$\Rightarrow$": For $\forall (y,x) \in \theta_{AT}^L$, we have $\mu[\underline{App}(AT,\widetilde{d}),x] > \mu[\underline{App}(AT,\widetilde{d}),y]$. By Theorem 1 we can see that $D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x)$ does not hold, which implies that there must be $z \in U$ such that $z \in D_A^{\uparrow}(y)$ and $z \notin D_A^{\uparrow}(x)$. Suppose that $y \in D_A^{\uparrow}(x)$, then we have $z \in D_A^{\uparrow}(x)$ because $z \in D_A^{\uparrow}(y)$. This is contradictive to $z \notin D_A^{\uparrow}(x)$, from which we can conclude that there must be $a_i \in A \subseteq AT$ such that $\mu_{\widetilde{a_i}}(y) < \mu_{\widetilde{a_i}}(x)$, i.e., $a_i \in \theta_{AT}^L(y,x)$, $A \cap \theta_{AT}^L(y,x) \neq \emptyset$.

"$\Leftarrow$": $(y,x) \in \theta_{AT}^L \Rightarrow \mu[\underline{App}(AT,\widetilde{d}),x] > \mu[\underline{App}(AT,\widetilde{d}),y]$. If for $\forall (y,x) \in \theta_{AT}^L$, $A \cap \theta_{AT}^L(y,x) \neq \emptyset$, then there must be $a_i \in A$ such that $\mu_{\widetilde{a_i}}(y) < \mu_{\widetilde{a_i}}(x)$, i.e., $y \notin D_A^{\uparrow}(x)$. Since $y \in D_A^{\uparrow}(y)$, then we have $D_A^{\uparrow}(y) \subseteq D_A^{\uparrow}(x)$ does not hold. By Theorem 1 we can conclude that $\mu[\underline{App}(AT,\widetilde{d}),x] = \mu[\underline{App}(A,\widetilde{d}),x]$ for $\forall x \in U$.

2) The proof of 2) is similar to the proof of 1).

Definition 3: Given a fuzzy decision system Ω^F, denote by

$$\Delta^L = \bigwedge_{(y,x)\in\theta^L_{AT}} \bigvee \theta^L_{AT}(y,x);$$

$$\Delta^U = \bigwedge_{(y,x)\in\theta^U_{AT}} \bigvee \theta^U_{AT}(y,x);$$

Δ^L and Δ^U are referred to as lower and upper approximation discernibility functions respectively.

Theorem 4: Given a fuzzy decision system Ω^F, $A \subseteq AT$, then A is reduct of $\overline{App}(AT,\widetilde{d})$ or $\underline{App}(AT,\widetilde{d})$, if and only if $\wedge_{a\in A} a$ is a prime implicant of the discernibility function Δ^L or Δ^U.

Proof: It follows directly from Theorem 3 and the definition of minimal disjunctive normal forms of the discernibility functions.

IV. An Illustrative Example

To demonstrate the above concepts, we consider data in Table 1, which describes a small training set with fuzzy samples.

Table 1 is a summary of cars' evaluations. This table details 6 cars evaluated by means of five attributes: a_1: Mileage; a_2: Power; a_3: Compression-ratio; a_4: Max-speed; d: Global evaluation.

The universe of discourse is $U = \{x_1, \cdots, x_6\}$, $AT = \{a_1, a_2, a_3, a_4\}$ is the set of condition attributes and d is the decision attribute. The global evaluation indicates that the higher value a car holds on decision d, the better this car should be.

By using the fuzzy rough set approach presented in Section 2, we obtain the following:

$$\underline{App}(AT,\widetilde{d}) = \frac{0.8}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.65}{x_4} + \frac{0.8}{x_5} + \frac{0.75}{x_6},$$

$$\overline{App}(AT,\widetilde{d}) = \frac{0.85}{x_1} + \frac{0.85}{x_2} + \frac{0.5}{x_3} + \frac{0.65}{x_4} + \frac{0.8}{x_5} + \frac{0.75}{x_6}.$$

In the following, we can calculate the reducts of $\underline{App}(AT,\widetilde{d})$ and $\overline{App}(AT,\widetilde{d})$ by using the approach studied in Section 3.

Step 1: By Definition 2, we have

$$\theta^L_{AT} = \{(x_3,x_1),(x_4,x_1),(x_6,x_1),(x_3,x_2),(x_4,x_2),(x_6,x_2),(x_3,x_4),(x_3,x_5),(x_4,x_5),(x_6,x_5),(x_3,x_6),(x_4,x_6)\}.$$

$$\theta^U_{AT} = \{(x_1,x_3),(x_1,x_4),(x_1,x_5),(x_1,x_6),(x_2,x_3),(x_2,x_4),(x_2,x_5),(x_2,x_6),(x_4,x_3),(x_5,x_3),(x_5,x_4),(x_5,x_6),(x_6,x_3),(x_6,x_4)\}.$$

Step 2: Compute the lower and upper approximate discernibility matrixes of Table 1.

Here, we only present the lower approximate discernibility matrix of Table 1 as Table 2 shows.

Step 3: By Definition 3, we have $\Delta^L = \Delta^U = a_2 \wedge a_3$. According to Theorem 4, $\{a_2, a_3\}$ is the reduct of lower and upper approximations $\underline{App}(AT,\widetilde{d})$ and $\overline{App}(AT,\widetilde{d})$.

From discussion above, we can generate simplified decision rules from Table 1 as following show.

TABLE I
Cars' Evaluations

U	a_1	a_2	a_3	a_4	d
x_1	0.6	0.9	0.8	0.7	0.8
x_2	0.5	0.85	0.7	0.6	0.85
x_3	0.3	0.5	0.7	0.6	0.5
x_4	0.5	0.7	0.7	0.6	0.65
x_5	0.9	0.7	0.8	0.5	0.8
x_6	0.7	0.9	0.6	0.7	0.75

TABLE II
Lower Approximate Discernibility Matrix

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	AT	AT	AT	AT	AT	AT
x_2	AT	AT	AT	AT	AT	AT
x_3	AT	a,b	AT	a,b	a,b,c	a,b,d
x_4	AT	b	AT	AT	a,c	a,b,d
x_5	AT	AT	AT	AT	AT	AT
x_6	c	b,c	AT	AT	a,c	AT

- Decision rules w.r.t. the lower approximation:
  r_1: $\mu_{\widetilde{a_2}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.8 \Rightarrow \mu_{\widetilde{d}}(y) \geq 0.8$;
  r_2: $\mu_{\widetilde{a_2}}(y) \geq 0.85 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.7 \Rightarrow \mu_{\widetilde{d}}(y) \geq 0.8$;
  r_3: $\mu_{\widetilde{a_2}}(y) \geq 0.5 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.7 \Rightarrow \mu_{\widetilde{d}}(y) \geq 0.5$;
  r_4: $\mu_{\widetilde{a_2}}(y) \geq 0.7 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.7 \Rightarrow \mu_{\widetilde{d}}(y) \geq 0.65$;
  r_5: $\mu_{\widetilde{a_2}}(y) \geq 0.7 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.8 \Rightarrow \mu_{\widetilde{d}}(y) \geq 0.8$;
  r_6: $\mu_{\widetilde{a_2}}(y) \geq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \geq 0.6 \Rightarrow \mu_{\widetilde{d}}(y) \geq 0.75$.
- Decision rules w.r.t. the upper approximation:
  r_1: $\mu_{\widetilde{a_2}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.8 \Rightarrow \mu_{\widetilde{d}}(y) \leq 0.85$;
  r_2: $\mu_{\widetilde{a_2}}(y) \leq 0.85 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.7 \Rightarrow \mu_{\widetilde{d}}(y) \leq 0.85$;
  r_3: $\mu_{\widetilde{a_2}}(y) \leq 0.5 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.7 \Rightarrow \mu_{\widetilde{d}}(y) \leq 0.5$;
  r_4: $\mu_{\widetilde{a_2}}(y) \leq 0.7 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.7 \Rightarrow \mu_{\widetilde{d}}(y) \leq 0.65$;
  r_5: $\mu_{\widetilde{a_2}}(y) \leq 0.7 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.8 \Rightarrow \mu_{\widetilde{d}}(y) \leq 0.8$;
  r_6: $\mu_{\widetilde{a_2}}(y) \leq 0.9 \wedge \mu_{\widetilde{a_3}}(y) \leq 0.6 \Rightarrow \mu_{\widetilde{d}}(y) \leq 0.75$.

V. Conclusions

We have developed a general framework for approach to knowledge reduction of fuzzy rough set in fuzzy decision system. It must be noticed that the fuzzy rough model we used here is different from the classical fuzzy rough approach because it is based on the ordinal properties of fuzzy membership degrees, i.e., dominance principle. To preserve the lower and upper memberships for each object with minimal number of attributes, we introduced the concept of reducts of fuzzy lower and upper approximations. The judgement theorems and discernibility functions we discussed in this paper present a practical approach to compute the above two types of reducts.

In further research, we will develop the proposed approaches to more complicated fuzzy systems such as interval-valued fuzzy system and incomplete fuzzy system.

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