The Discrete Fuzzy Numbers on a Fixed Set With Finite Support Set*

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***Abstract*—It is known that to deep systematically study the theory in relation to general discrete fuzzy numbers always will encounter some difficulties since the metric and the difference of two general discrete fuzzy numbers cannot reasonably be defined. In this paper, we define a special sort discrete fuzzy numbers—discrete fuzzy number on a fixed set with finite support set, on which we can study the problems of metric and difference (we will study the problems in relation to metric and difference in afterwards papers). And then we obtain a representation theorem of such discrete fuzzy numbers, study the operations of scalar product, addition and multiplication, and obtain some results.**

***Keywords*—discrete fuzzy number, operation, support set**

I. Introduction

In 1972, Chang and Zadeh [2] introduced the conception of fuzzy numbers with the consideration of the properties of probability functions. Since then a lot of mathematicians have been studying on fuzzy number, and have obtained many results (for example, see [1,3,4,5,8,10]).

In 2001, Voxman [6] introduced the conception of discrete fuzzy numbers, which is useful in some applications; for example, discrete fuzzy number can be used to represent the pixel value in the center point of a window (see [9]). And in [6], Voxman gave out the Canonical representations of discrete fuzzy numbers. In [9] we discussed the representation of cut-set form of discrete fuzzy numbers, and using the representation, we shown that the usual addition of two discrete fuzzy numbers does not keep the closeness of the operation (at this point, discrete fuzzy numbers are not like non-discrete fuzzy numbers since non-discrete fuzzy numbers keep the closeness of usual addition (see [10])), and define a new addition for two discrete fuzzy numbers, which keeps the closeness of the operation. And we point out that when the usual addition of two discrete fuzzy numbers is still a discrete fuzzy number, the two additions are identical. In [7] we also discuss the representations and operations of scalar product, addition and multiplication for a special kind of discrete fuzzy numbers----fuzzy integers.

In recently, basing [9], we went on discussing the operations of addition and multiplication for discrete fuzzy numbers in [11,12]. Using representations (obtained in [7]) of discrete fuzzy numbers, we defined two new kinds of operations of addition and multiplication for discrete fuzzy numbers, and show that the new addition and multiplication keep the closeness of the operation. In addition, we also pointed out that when the usual addition of two discrete fuzzy numbers is still a discrete fuzzy number, the usual addition and the two new additions are identical, and the usual multiplication and the two new multiplications are identical.

However, for general discrete fuzzy numbers, we cannot reasonably define metric and the operation of difference, thus, the application is confined in some ways. In this paper, we define a special sort discrete fuzzy numbers—discrete fuzzy number on a fixed set, on which we can conveniently study the problems of metric and difference (we will study the problems relating with metric and difference in afterwards papers). We give the representation of such discrete fuzzy numbers. And then we study the operations of scalar product, addition and multiplication, discuss the properties of these operations, and obtain some results.

II. Preliminaries

Let R be the real number field. For any $A,B\subset R$ and $k\in R$, we define the addition and the multiplication of A and B, and the scalar product of k and A, respectively, by: $A+B=\{a+b: a\in A, b\in B\}$, $AB=\{ab: a\in A, b\in B\}$ and $kB=\{ka : a\in A\}$.

A fuzzy subset (in short, a fuzzy set) of R is a function $u: R\to[0,1]$. For each fuzzy set u, we denote $[u]^r=\{x\in R: u(x)\geq r\}$ for any $r\in(0,1]$, its $r-$level set. By $\operatorname{supp} u$, we denote the support of u, i.e. the set $\{x\in R: u(x)>0\}$. By $[u]^0$ we denote the closure of $\operatorname{supp} u$, i.e., $[u]^0=\overline{\{x\in R: u(x)>0\}}$.

Let u,v are two fuzzy set of R. It is known that $u=v$ if and only if $[u]^r=[v]^r$ for all $r\in[0,1]$.

For any fuzzy sets u, v and real number k, we define the addition and the multiplication of u and v, and the scalar product of k and u via the following:

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$$(u+v)(x)=\sup_{y+z=x}\min\{u(y),v(z)\}$$

$$(uv)(x)=\sup_{yz=x}\min\{u(y),v(z)\}$$

$$(ku)(x)=\begin{cases}u(x/k) & \text{if} \quad k\neq 0\\ \hat{0} & \text{if} \quad k=0\end{cases}$$

where, $\hat{0}(x)=\begin{cases}1 & \text{if} \quad x=0\\ 0 & \text{if} \quad x\neq 0\end{cases}$.

Definition 2.1[6]***.*** *A fuzzy set $u: R\to[0.1]$ is a discrete fuzzy number if the support of u is finite, i.e. there exist $x_1,x_2,\cdots,x_n\in R$ with $x_1<x_2<\cdots<x_n$ such that $[u]^0=\{x_1,x_2,\cdots,x_n\}$, and there exist natural numbers s,t with $1\le s\le t\le n$ such that*

(1) $u(x_i)=1$ for any natural number i with $s\le i\le t$;

(2) $u(x_i)\le u(x_j)$ for any natural numbers i,j with $1\le i\le j\le s$, $u(x_i)\ge u(x_j)$ for any natural numbers i,j with $t\le i\le j\le n$.

We denote the collection of all discrete fuzzy numbers by F_D.

About the contents of the representation, additions, the multiplications and the scalar product of discrete fuzzy numbers, we can see [7,9,11,12].

Definition 2.2[7]***.*** *Let $u\in F_D$. If there exist $s_0,t_0\in I$ with $s_0\le t_0$, such that $[u]^0=\{x\in I: s_0\le x\le t_0\}$, then we call u a fuzzy integer, and denote the collection of all fuzzy integers by F_I, where I is the collection of all integers.*

III. MAIN RESULTS

Let C be a countable (including finite) subset of real number field R.

In the following, we define a special kind of discrete fuzzy numbers--discrete fuzzy numbers on the countable set R so that we can study their operation of addition, the multiplication and the scalar product, but also can study their difference and metric.

Definition 3.1. *Let C be a countable (including finite) subset of real number field R. A fuzzy set $u: R\to[0.1]$ is a discrete fuzzy number on C if it satisfies the following conditions:*

(1) $[u]^0\subset C$ and is finite;

(2) there exists $x_0\in C$ such that $u(x_0)=1$;

(3) $u(x_s)\le u(x_t)$ for any $x_s,x_t\in C$ with $x_s\le x_t\le x_0$;

(4) $u(x_s)\ge u(x_t)$ for any $x_s,x_t\in C$ with $x_0\le x_s\le x_t$.

And we denote the collection of all discrete fuzzy numbers by F_{DC}.

It is obvious that the following Theorem 3.1 holds.

Theorem 3.1. *Let C be a countable (including finite) subset of real number field R. Then*

$$F_I\subset F_{DC}\subset F_D.$$

Let C be a countable (including finite) subset of real number field R. For any $x_0,y_0\in R$, we denote $[x_0,y_0]_C=\{x\in C: x_0\le x\le y_0\}$. In the following, we give out a cut-sets form representation of discrete fuzzy numbers on C.

Theorem 3.2. *Let C be a countable (including finite) subset of real number field R, and $u\in F_{DC}$. Then*

(1) for any $r\in[0,1]$, there exist $x_r,y_r\in C$ with $x_r\le y_r$ such that $[u]^r=[x_r,y_r]_C$, and $[x_0,y_0]_C$ is finite;

(2) $[u]^{r_2}\subset[u]^{r_1}$ for any $r_1,r_2\in[0,1]$ with $0\le r_1\le r_2\le 1$;

(3) for any $r_0\in(0,1]$, there exists real number r_0' with $0<r_0'<r_0$ such that $[u]^{r_0'}=[u]^{r_0}$ (i.e. $[u]^r=[u]^{r_0}$ for any $r\in[r_0',r_0]$).

Conversely, if for any $r\in[0,1]$, there exists $A_r\subset R$ satisfying

(i) there exist $x_r,y_r\in C$ with $x_r\le y_r$ such that $A_r=[x_r,y_r]_C$, and $[x_0,y_0]_C$ is finite;

(ii) $A_{r_2}\subset A_{r_1}$ for any $r_1,r_2\in[0,1]$ with $0\le r_1\le r_2\le 1$;

(iii) for any $r_0\in(0,1]$, there exists real number r_0' with $0<r_0'<r_0$ such that $A_{r_0'}=A_{r_0}$ (i.e. $A_r=A_{r_0}$ for any $r\in[r_0',r_0]$).

Then there exists a unique $u\in F_{DC}$ such that $[u]^r=A_r$ for any $r\in[0,1]$.

Proof. At first, we show that $u\in F_{DC}$ implies the conclusions (1)-(3) of the theorem hold.

By the definition of $[u]^r$, we can easily see that the conclusion (2) holds. Therefore, for any $r\in[0,1]$, we have $[u]^r\subset[u]^0$, so we see that $[u]^r$ is finite for any $r\in[0,1]$ by the condition (1) in the definition of discrete fuzzy numbers on C. Let $x_r=\min[u]^r$ and $y_r=\max[u]^r$ for any $r\in[0,1]$. In order to show that the conclusion (1) holds, we only show $[u]^r=[x_r,x_y]_C$. Let $r\in[0,1]$ and $x\in[u]^r$. By $[u]^r\subset[u]^0\subset C$, we know $x\in C$. In addition, by the definitions of x_r and y_r, we know $x_r\le x\le y_r$, so obtain $x\in[x_r,y_r]_C$. Thus we have proved $[u]^r\subset[x_r,x_y]_C$. On the other hand, let $x\in[x_r,y_r]_C$. By the definition of $[x_r,y_r]_C$, we know $x_r\le x\le y_r$ and $x\in C$. By the condition (2) in the definition of discrete fuzzy numbers on C, we know that there exists $x_0\in C$ such that $u(x_0)=1$. From $x_0\in[u]^1\subset[u]^r$ and $[u]^r\subset[x_r,x_y]_C$, we see $x_0\in[x_r,y_r]_C$, so $x_r\le x_0\le y_r$. If $x_r\le x\le x_0$, by the condition (3) in the definition of discrete fuzzy numbers on C, we obtain $u(x) \ge u(x_r) \ge r$, so we have $x \in [u]^r$. Similarly, when $x_0 \le x \le y_r$, we can also obtain $x_0 \in [x_r, y_r]_C$ from the condition (4) in the definition of discrete fuzzy numbers on C. Therefore, we have shown $[x_r, x_y]_C \subset [u]^r$, so we have $[u]^r = [x_r, x_y]_C$, i.e., the conclusion (1) of the theorem holds.

Since $[u]^0 \subset C$ is finite, there exist $x_1, x_2, \cdots, x_n \in C$ such that $[u]^0 = \{x_1, x_2, \cdots, x_n\}$. No losing generality, we can suppose $x_1 < x_2 < \cdots < x_n$, and denote $r_i = u(x_i)$, $i = 1,2,\cdots,n$. Then we know that there exist natural numbers s and t with $1 \le s \le t \le n$ such that $0 < r_1 \le r_2 \le \cdots \le r_{s-1} \le 1 = r_s = r_{s+1} = \cdots = r_t$ $= 1 \ge r_{t+1} \ge \cdots \ge r_n > 0$ from the conditions (2), (3) and (4) in the definition of discrete fuzzy numbers on C. Suppose that there are only n_0 $(1 \le n_0 \le n)$ real numbers $r_1', r_2', \cdots, r_{n_0}'$ that are not equal to each other in the real numbers $r_1, r_2, \cdots, r_n$ and satisfying $r_1' < r_2' < \cdots < r_{n_0}' = 1$. Let $r_0 \in (0,1]$. If $r_0 \le r_1'$, then there exists a real number r_0' such that $0 < r_0' < r_0 \le r_1'$, so that we have $[u]^r = [u]^{r_0} = [u]^0$ for any $r \in [r_0', r_0]$. If $r_0 > r_0'$, then there exists a natural number i_0 with $1 < i_0 \le n_0$ such that $r_{i_0} < r_0 \le r_{i_0+1}$. Therefore, there exists a real number r_0' such that $r_{i_0} < r_0' < r_0 \le r_{i_0+1}$, and so for any $r \in [r_0', r_0]$, we have $[u]^r = [u]^{r_0} = [u]^{r_0+1}$. This proves conclusion (3) of the theorem. Thus, we have completed the proof of the first section of the theorem.

In the following, we show the second section of the theorem.

Let A_r satisfy the conditions (i)--(iii) of the theorem for any $r \in [0,1]$. Let

$$u(x) = \begin{cases} \sup\{r \in [0,1] : x \in A_r\} & \text{if} \quad \{r \in [0,1] : x \in A_r\} \ne \phi \\ 0 & \text{if} \quad \{r \in [0,1] : x \in A_r\} = \phi \end{cases}$$

We first show $[u]^r = A_r$ for any $r \in [0,1]$. Let $r_0 \in [0,1]$. If $x \in A_{r_0}$, then $r_0 \in \{r \in [0,1] : x \in A_r\}$. Hence, $u(x) = \sup\{r \in [0,1] : x \in A_r\} \ge r_0$, i.e., $x \in [u]^{r_0}$. Therefore, we obtain $A_{r_0} \subset [u]^{r_0}$ for any $r \in [0,1]$.

Conversely, if $r_0 \in (0,1]$ and $x \in [u]^{r_0}$, then $u(x) \ge r_0 > 0$, i.e., $\sup\{r \in [0,1] : x \in A_r\} \ge r_0$. By using the condition (iii), we know that there exists $r_0' \in (0, r_0)$ such that $A_{r_0'} = A_{r_0}$. Observe that $\sup\{r \in [0,1] : x \in A_r\} \ge r_0 > r_0'$ and by the definition of supremum, we know that there exists $r_0'' \in \sup\{r \in [0,1] : x \in A_r\}$ such that $r_0'' > r_0'$, i.e., there exists $r_0'' \in [0,1]$ such that $x \in A_{r_0''}$ and $r_0'' > r_0'$, so $x \in A_{r_0'} = A_{r_0}$. In addition, if $x \in [u]^0$, then we have $u(x) > 0$, and so there exists a real number r_x such that $0 < r_0 < u(x)$, so $x \in [u]^{r_x} = A_{r_x} \subset A_0$. Therefore, we have known that $[u]^{r_0} = A_{r_0}$ for any $r_0 \in [0,1]$.

Secondly, we prove $u \in F_{DC}$. From $[u]^0 = A_0$ and the condition (i) of the theorem, we see that $[u]^0 \subset C$ and is finite, i.e., the condition (1) in the definition of discrete fuzzy numbers on C holds. In addition, the condition (i) of the theorem also implies $A_1 \ne \phi$, i.e., $[u]^1 \ne \phi$, so there exists $x_0 \in [u]^1$, i.e., $u(x_0) = 1$. Thus, we know that u satisfies the condition (2) in the definition of discrete fuzzy numbers on C. And then, in the following, we show that u satisfies the condition (3) in the definition of discrete fuzzy numbers on C. If the condition (3) is not satisfied, then there exist $x_{s_0}, x_{t_0} \in C$ with $x_{s_0} \le x_{t_0} \le x_0$ (where $u(x_0) = 1$) such that $u(x_{s_0}) > u(x_{t_0})$. Denote $h_0 = u(x_{s_0})$, then $h_0 \in (0,1]$ and $u(x_{t_0}) < h_0$, i.e., $x_{t_0} \notin [u]^{h_0} = A_{h_0}$. By the condition (i) of the theorem, we know that there exist $x_{h_0}, y_{h_0} \in C$ with $x_{h_0} \le y_{h_0}$ such that $x_{t_0} \notin [x_{h_0}, y_{h_0}]_C = A_{h_0}$. On the other hand, from $h_0 = u(x_{s_0})$, we see that $x_{s_0} \in [u]^{h_0} = A_{h_0} = [x_{h_0}, y_{h_0}]_C$, so $[x_{s_0}, x_0]_C \subset [x_{h_0}, y_{h_0}]_C$ (note $u(x_0) = 1$). Therefore, we know $x_{t_0} \in [x_{s_0}, x_0]_C \subset [x_{h_0}, y_{h_0}]_C$ from $x_{s_0}, x_{t_0} \in C$ and $x_{s_0} \le x_{t_0} \le x_0$, which contradicts to $x_{t_0} \notin [x_{h_0}, y_{h_0}]_C$. This shows that condition (3) in the definition of discrete fuzzy numbers on C holds. Similarly, we can prove that condition (4) in the definition of discrete fuzzy numbers on C also holds. Therefore, we have $u \in F_{DC}$.

Thus we completed the proof of the theorem.

By Theorem 3.1 and Theorem 3.1 in [9], we can directly obtain the following Theorem 3.3.

Theorem 3.3. *Let C be a countable (including finite) subset of real number field R, and $u, v \in F_{DC}$, $k \in R$. Then for any $r \in [0,1]$,*

(1) $[u+v]^r = [u]^r + [v]^r$;

(2) $[ku]^r = k[u]^r$;

(3) $[uv]^r = [u]^r [v]^r$.

Theorem 3.4. *Let C be a countable subset of real number field R. If $u, v \in F_{DC}$, $k \in R$. Then*

(1) $ku \in F_{DC}$ if C satisfies $kx \in C$ for any $x \in C$;

(2) $u + v \in F_{DC}$ if C preserve the closeness of the operations of addition and difference.

Proof. In order to show $ku,\ u+v,\ uv\in F_{DC}$, we only need to show $[ku]^r$, $[u+v]^r$ and $[uv]^r$ satisfy conditions (i), (ii) and (iii) of Theorem 3.2. From $[ku]^r=k[u]^r$, we can easily see that $[ku]^r$ satisfies conditions (i), (ii) and (iii) of Theorem 3.2 since $[u]^r$ satisfies these conditions and $kx\in C$ for any $x\in C$.

In the following, we show that $[u+v]^r$, i.e. $[u]^r+[v]^r$ (by (1) of Theorem 3.3) satisfies conditions (i), (ii) and (iii) of Theorem 3.2. For any $r\in[0,1]$, from $u,v\in F_{DC}$, we know that exist $x_{u,r},y_{u,r},x_{v,r},y_{v,r}\in C$ with $x_{u,r}\le y_{u,r}$ and $x_{v,r}\le y_{v,r}$ such that $[u]^r=[x_{u,r},y_{u,r}]_C$, $[v]^r=[x_{v,r},y_{v,r}]_C$, and $[x_{u,0},y_{u,0}]_C$, $[x_{v,0},y_{v,0}]_C$ are finite. Let $x\in[x_{u,r},y_{u,r}]_C+[x_{v,r},y_{v,r}]_C$. Then there exist $y\in[x_{u,r},y_{u,r}]_C$ and $z\in[x_{v,r},y_{v,r}]_C$ such that $x=y+z$. From $x,y,x_{u,r},y_{u,r},x_{v,r},y_{v,r}\in C$, $x_{u,r}\le y\le y_{u,r}$ and $x_{v,r}\le z\le y_{v,r}$, we see $x,\ x_{u,r}+x_{v,r},\ y_{u,r}+y_{v,r}\in C$ and $x_{u,r}+x_{v,r}\le x\le y_{u,r}+y_{v,r}$, so $x\in[x_{u,r}+x_{v,r},\ y_{u,r}+y_{v,r}]_C$. Hence we obtain $[u]^r+[v]^r\subset[x_{u,r}+x_{v,r},y_{u,r}+y_{v,r}]_C$. Conversely, if $x\in[x_{u,r}+x_{v,r},y_{u,r}+y_{v,r}]_C$, then we have $x,\ x_{u,r}+x_{v,r},\ y_{u,r}+y_{v,r}\in C$ and $x_{u,r}+x_{v,r}\le x\le y_{u,r}+y_{v,r}$. If $x_{u,r}+y_{v,r}\ge x_{v,r}+y_{u,r}$, then $x_{u,r}+x_{v,r}\le x\le x_{u,r}+y_{v,r}$ or $x_{v,r}+y_{u,r}\le x\le y_{u,r}+y_{v,r}$, i.e., $x-x_{u,r}\in[x_{v,r},y_{v,r}]_C$ or $x-y_{u,r}\in[x_{v,r},y_{v,r}]_C$, so $x=x_{u,r}+(x-x_{u,r})\in[u]^r+[v]^r$ or $x=y_{u,r}+(x-y_{u,r})\in[u]^r+[v]^r$. Similarly, when $x_{u,r}+y_{v,r}\ge x_{v,r}+y_{u,r}$, we can also obtain $x\in[u]^r+[v]^r$. Hence we have that $[u]^r+[v]^r\supset[x_{u,r}+x_{v,r},y_{u,r}+y_{v,r}]_C$, so $[u]^r+[v]^r=[x_{u,r}+x_{v,r},y_{u,r}+y_{v,r}]_C$, i.e., the conditions (i) of Theorem 3.2 holds. For any $r_1,r_2\in[0,1]$ with $0\le r_1\le r_2\le 1$, from $[u]^{r_2}\subset[u]^{r_1}$ and $[u]^{r_2}\subset[u]^{r_1}$, we see that $[u+v]^{r_2}=[u]^{r_2}+[v]^{r_2}\subset[u]^{r_1}+[v]^{r_1}=[u+v]^{r_1}$, i.e., the conditions (ii) of Theorem 3.2 holds. Since $u\in F_{DC}$, for any $r_0\in(0,1]$, there exists real number r_u with $0<r_u<r_0$ such that $A_{r_u}=A_{r_0}$ (i.e. $A_r=A_{r_0}$ for any $r\in[r_u,r_0]$). Similarly, there exists real number r_v with $0<r_v<r_0$ such that $A_{r_v}=A_{r_0}$ (i.e. $A_r=A_{r_0}$ for any $r\in[r_v,r_0]$). Let $r_0'=\min(r_u,r_v)$, then we have that $0<r_0'<r_0$ and $A_{r_0'}=A_{r_0}$ (i.e. $A_r=A_{r_0}$ for any $r\in[r_0',r_0]$), i.e., the conditions (iii) of Theorem 3.2 holds. Therefore, we completed the proof of the theorem.

Remark 3.1. *Although, on the surface, we can also obtain the similar result ($u,v\in F_{DC}$ and C preserves the closeness of the operations of multiplication and division $\Rightarrow$ $uv\in F_{DC}$) with (2) of Theorem 3.4 for the operation of multiplication, it has any meaning because there no exists countable (including finite) subset (of R) which not only preserves the closeness of the operations of multiplication and division but also exist $u\in F_{DC}$ such that $[u]^0\subset C$ and is finite.*

Remark 3.2. *Generally speaking, the condition in (1) of Theorem 3.4: "C satisfies $kx\in C$ for any $x\in C$" and the condition in (2) of Theorem 3.4: "C preserve the closeness of the operations of addition and difference" can not be omitted (see the following Example 3.1).*

Example 3.1. Let $C=\{1,2,3,4,5,6,8,10,12,16\}$, and $u:R\rightarrow[0,1]$ and $v:R\rightarrow[0,1]$ be defined respectively by

$$u(x)=\begin{cases}1 & if\quad x=3\\ \frac{1}{2} & if\quad x=2,4\\ \frac{1}{3} & if\quad x=1,5\\ 0 & if\quad x\neq 1,2,3,4,5,\end{cases}$$

and

$$v(x)=\begin{cases}1 & if\quad x=4\\ \frac{1}{2} & if\quad x=3,5\\ \frac{1}{3} & if\quad x=2,6\\ 0 & if\quad x\neq 2,3,4,5,6\end{cases}$$

then $u,v\in F_{DC}$, but from Theorem 4.3, we see that $[u]^{\frac{1}{2}}+[v]^{\frac{1}{2}}=\{5,6,7,8.9\}$, so $[u]^{\frac{1}{2}}+[v]^{\frac{1}{2}}\not\subset C$. Hence by Theorem 4.2, we know $u+v\notin F_{DC}$. This shows that the condition in (2) of Theorem 3.4: "C preserve the closeness of the operations of addition and difference" cannot be omitted. Similarly, we can also set up an example to show that the condition in (1) of Theorem 3.4: "C satisfies $kx\in C$ for any $x\in C$" cannot be omitted.

Remark 3.3. *Generally speaking, directly using the definition of the operations of addition and multiplication to carry the concrete operations addition and multiplication of discrete fuzzy numbers on C is not easy. Theorem 3.2 and Theorem 3.3 provide us a feasible method about the concrete operations addition of discrete fuzzy numbers on C. The following Example 3.2 shall show how to carry the concrete operations using Theorem 3.2 and Theorem 3.3.*

Example 3.2. Let $C=\{2k:k=0,\pm1,\pm2,\cdots\}$, and $u:R\rightarrow[0,1]$ and $v:R\rightarrow[0,1]$ be defined respectively by

$$u(x)=\begin{cases}1 & if\quad x=8\\ \frac{1}{2} & if\quad x=4,6,10\\ \frac{1}{3} & if\quad x=2,12,14\\ 0 & if\quad x\neq 2,4,6,8,10,12,14\end{cases}$$

and

$$v(x)=\begin{cases}1 & if \quad x=0\\ \frac{1}{2} & if \quad x=-2,2,4\\ \frac{1}{3} & if \quad x=-6,-4,6\\ 0 & if \quad x\neq -6,-4,-2,0,2,4,6\end{cases}$$

so $u,v\in F_{DC}$ and

$$[u]^r=\begin{cases}\{8\} & \text{if} \quad \frac{1}{2}<r\le 1\\ [4,10]_C & \text{if} \quad \frac{1}{3}<r\le\frac{1}{2}\\ [2,14]_C & \text{if} \quad 0\le r\le\frac{1}{3}\end{cases}$$

$$[v]^r=\begin{cases}\{0\} & \text{if} \quad \frac{1}{2}<r\le 1\\ [-2,4]_C & \text{if} \quad \frac{1}{3}<r\le\frac{1}{2}\\ [-6,6]_C & \text{if} \quad 0\le r\le\frac{1}{3}\end{cases}$$

Then we have that

$$[u+v]^r=\begin{cases}\{8\} & \text{if} \quad \frac{1}{2}<r\le 1\\ [2,14]_C & \text{if} \quad \frac{1}{3}<r\le\frac{1}{2}\\ [-4,20]_C & \text{if} \quad 0\le r\le\frac{1}{3}\end{cases}$$

so

$$(u+v)=\begin{cases}1 & \text{if} \quad x=8\\ \frac{1}{2} & \text{if} \quad x\in[2,14]_C\\ \frac{1}{3} & \text{if} \quad x\in[-4,20]_C\\ 0 & \text{if} \quad x\notin[-4,20]_C\end{cases}$$

For $u\in F_{DC}$, we denote $\underline{u}_r=\min[u]^r$ and $\overline{u}_r=\max[u]^r$. Then by Theorem 3.2, we have $[u]^r=[\underline{u}_r,\overline{u}_r]_C$.

Definition 3.2. *Let C be a countable (including finite) subset of real number field R, and $u,v\in F_{DC}$, $k\in R$. We define*

$$k\circ[u]^r=\begin{cases}[k\underline{u}_r,\ k\overline{u}_r]_C & if \quad k\ge 0\\ [k\overline{u}_r,\ k\underline{u}_r]_C & if \quad k<0\end{cases}$$

$$[u]^r\oplus[v]^r=[\underline{u}_r+\underline{v}_r,\ \overline{u}_r+\overline{v}_r]_C$$

and

$$[u]^r\otimes[v]^r=[\underline{u}_r\otimes\underline{v}_r,\ \overline{u}_r\otimes\overline{v}_r]_C$$

where

$$\underline{u}_r\otimes\underline{v}_r=\min\{\underline{u}_r\cdot\underline{v}_r,\ \underline{u}_r\cdot\overline{v}_r,\ \overline{u}_r\cdot\underline{v}_r,\ \overline{u}_r\cdot\overline{v}_r\}$$

And

$$\overline{u}_r\otimes\overline{v}_r=\max\{\underline{u}_r\cdot\underline{v}_r,\ \underline{u}_r\cdot\overline{v}_r,\ \overline{u}_r\cdot\underline{v}_r,\ \overline{u}_r\cdot\overline{v}_r\}.$$

Theorem 3.5. *Let C be a countable subset of real number field R. If $u,v\in F_{DC}$, $k\in R$. Then $k\circ[u]^r$, $[u]^r\oplus[v]^r$ and $[u]^r\otimes[v]^r$ ($r\in[0,1]$) satisfy the conditions conditions (i), (ii) and (iii) of Theorem 3.2.*

Definition 3.2. *Let C be a countable subset of real number field R. If $u,v\in F_{DC}$, $k\in R$. By Theorem 3.5 and 3.2, we know that $k\circ[u]^r$, $[u]^r\oplus[v]^r$ and $[u]^r\otimes[v]^r$ ($r\in[0,1]$) determine, respectively, unique discrete fuzzy number on C. We define, respectively, the unique discrete fuzzy numbers on C are $k\circ u$, $u\oplus v$ and $u\otimes v$.*

Therefore we have the following result.

Theorem 3.6. *Let C be a countable subset of real number field R. If $u,v\in F_{DC}$, $k\in R$. Then $k\circ u$, $u\oplus v$ and $u\otimes v\in F_{DC}$, and*

$$\underline{(k\circ u)}_r=\begin{cases}k\underline{u}_r & if \quad k\ge 0\\ k\overline{u}_r & if \quad k<0\end{cases}$$

$$\overline{(k\circ u)}_r=\begin{cases}k\overline{u}_r & if \quad k\ge 0\\ k\underline{u}_r & if \quad k<0\end{cases}$$

$$\underline{(u\oplus v)}_r=\underline{u}_r+\underline{v}_r$$

$$\overline{(u\oplus v)}_r=\overline{u}_r+\overline{v}_r$$

$$\underline{(u\otimes v)}_r=\min\{\underline{u}_r\cdot\underline{v}_r,\ \underline{u}_r\cdot\overline{v}_r,\ \overline{u}_r\cdot\underline{v}_r,\ \overline{u}_r\cdot\overline{v}_r\}$$

and

$$\overline{(u\otimes v)}_r=\max\{\underline{u}_r\cdot\underline{v}_r,\ \underline{u}_r\cdot\overline{v}_r,\ \overline{u}_r\cdot\underline{v}_r,\ \overline{u}_r\cdot\overline{v}_r\}.$$

IV. Conclusion

In this paper, in order to overcome the defect that the metric and the difference of two general discrete fuzzy numbers cannot reasonably be defined, we introduce the concept of discrete fuzzy number on a fixed set, give a representation theorem of such discrete fuzzy numbers, and obtain some properties of the operations of scalar product, addition and multiplication for such discrete fuzzy numbers. We also point out the conditions that make the usual operations of scalar product, addition and multiplication keep the closeness, define a new kind of operations of scalar product, addition and multiplication, and set up the relationship of the usual operations and the new operations. This make us can study conveniently the problems in relation to metric and difference.

References

[1] J.J. Buckley, "Fuzzy complex numbers", *Fuzzy Sets and Systems*, Vol 33, pp. 333-345, 1989.

[2] S.S.L. Chang and L.A. Zadeh, "On fuzzy mapping and control", *IEEE Trans. Systems Man Cybernet.,* Vol 2, pp. 30-34, 1972.

[3] P. Diamond and P. Kloeden, "Metric space of fuzzy sets", *Fuzzy Sets and Systems,* Vol 100 (suppl.), pp. 63-71, 1999.

[4] D. Dubois and H. Prade, "Operations on fuzzy numbers", *Internat. J. Systems Sci.,* Vol 9, pp. 613-626, 1978.

[5] R. Goetschel and W. Voxman, "Elementary calculus", *Fuzzy Sets and Systems,* Vol 18, pp. 31-43, 1986.

[6] W. Voxman, "Canonical representations of discrete fuzzy numbers", *Fuzzy Sets and Systems,* Vol 118, pp. 457-466, 2001.

[7] Wang Gui-Xiang, Li You-Ming, "The representations and operations of fuzzy integers", Proceeding of ICMLC2006 Conference, Dalian, pp. 1673-1678, August 2006.

[8] Wang Guixiang, Li Youming and Wen Chenglin, "On fuzzy $n-$cell numbers and $n-$dimension fuzzy vectors", *Fuzzy Sets and Systems,* Vol 158, pp. 71-84, 2007.

[9] Wang Guixiang, Wu Cong and Chunhui Zhao, "Representation and Operations of Discrete Fuzzy Numbers", *Southeast Asian Bulletin of Mathematics,* Vol 28, pp. 1003-1010, 2005.

[10] Wang Guixiang and Wu Congxin, "Fuzzy $n-$cell numbers and the differential of fuzzy $n-$cell number value mappings", *Fuzzy Sets and Systems,* Vol 130, pp. 367-381, 2002.

[11] Wang Guixiang, Wen Chenglin, A New Fuzzy Arithmetic for Discrete Fuzzy Numbers, Proc. The Fourth International Conference on Fuzzy Systems and Knowledge Discovery, Haikou, PP. 52-56, August 2007.8.