Activity Invariant Sets and Stable Attractors of Lotka-Volterra Recurrent Neural Networks

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***Abstract*— This paper proposes to study the activity invariant sets and exponentially stable attractors of Lotka-Volterra recurrent neural networks. The concept of activity invariant sets deeply describes the property of an invariant set by that the activity of some neurons keeps invariant all the time. Conditions are obtained for locating activity invariant sets. Under some conditions, it shows that an invariant set can have one equilibrium point which is exponentially stable. Since the attractors are located in activity invariant sets, each attractor has binary pattern and also carries analog information. Such results can provide new perspective to apply attractor networks for applications such as group winner-take-all, associative memory, etc..**

I. Introduction

The Lotka-Volterra model of recurrent neural networks was first proposed in [4]. It was derived from the conventional membrane dynamics of neurons with a sigmoid response function and its dynamic properties were analytically studied. Moreover, it has found successful applications in winner-take-all, winner-share-all and k-winner-take-all problems, see [1], [2], [8]. Due to the application potential of this class of neural networks, it is necessary and useful to study its general dynamic properties.

In this paper, we propose to study activity invariant sets and exponentially stable attractors of Lotka-Volterra recurrent neural networks. Invariant sets play important roles in dynamics study of recurrent neural networks. An invariant set restricts trajectories starting from the set stay in the set. The concept of activity invariant set more deeply describes the dynamic properties of invariant sets: the activity of some neurons keeps invariant during the time evolution. Thus, neurons can be divided into two classes of active neurons and inactive neurons. We will derive conditions for locating activity invariant sets.

In applying of recurrent neural networks to the application of associative memory, it is crucial that the networks have stable attractors. Stable attractors stored as memories to the networks are often used to implement associative memory [7], the memories can be recalled by encoding initial conditions as computational inputs to the network. If a neural network has more than one stable attractor, then it is multistable. In a sense, multistability is a necessary property in neural networks in order to enable certain applications where monostable networks could be computationally restrictive [5]. Recently, there has been increasing interest in multistability analysis for neural networks [9], [10], [11], [12], [13], [14], [15].

We will show that under some conditions, an activity invariant set has one equilibrium point which is exponentially stable. Such attractors are located in activity invariant sets, thus each attractor has binary pattern and also carries analog information. This is quite interesting since these attractors could be used to store memories with both binary and analog information. It is believed that these results can have potential applications such as group winner-take-all, associative memory, etc.. In the application of group winner-take-all, the network outputs are required to have binary pattern, i.e., the winners and the losers, on the other hand, differences may exist in different winners, such differences can be described by analogy information of each winner.

The rest of this paper is organized as follows. In Section II, we present some preliminaries. Main results about activity invariant sets and exponentially stable attractors are given in Section III. Simulations are carried out in Section IV to illustrate the theory. Conclusions are given in Section V.

II. Preliminaries

Consider the following Lotka-Volterra recurrent neural networks:

$$\dot{x}_i(t) = x_i(t)\left[h_i - x_i(t) + \sum_{j=1}^{n} w_{ij}x_j(t)\right] \tag{1}$$

for $t \geq 0$ and $i = 1, 2, \cdots, n$, where each $x_i (i = 1, 2, \cdots, n)$ denotes the activity of neuron i, and $x = (x_1, x_2, \cdots, x_n)^T \in R^n$. $w_{ij}(i, j = 1, 2, \cdots, n)$ are connection weights which are constants.

Definition 1: A neuron with index i is called active at time t if $x_i(t) > 0$, while a neuron with index i is called inactive at time t if $x_i(t) = 0$.

Lemma 1: Given any neuron i, if it is active initially then it will be active all the time; if it is inactive initially then it keeps inactive thereafter.

Proof: By Definition 1, we will prove that given any $i(1 \leq i \leq n)$, if $x_i(0) > 0$ then $x_i(t) > 0$ for all $t \geq 0$; if $x_i(0) = 0$ then $x_i(t) = 0$ for all $t \geq 0$. Denote

$$r_i(t) = h_i - \sum_{j=1}^{n} w_{ij}x_i(t).$$

Then,

$$x_i(t) = x_i(0) \cdot \exp\left(\int_0^t r_i(s)ds\right) \begin{cases} > 0, & \text{if } x_i(0) > 0 \\ = 0, & \text{if } x_i(0) = 0 \end{cases}$$

for all $t \geq 0$. The results now follows and the proof is completed. ■

Definition 2: A set $D \subset R^n$ is called an invariant set of (1), if each trajectory starting in D will remain in D forever.

Definition 3: A set $D \subset R^n$ is called an activity invariant set of (1), if D is an invariant set, and given any $x(0) \in D$ it holds that

$$\begin{cases} x_i(t) > 0, & \text{if } x_i(0) > 0 \\ x_i(t) = 0, & \text{if } x_i(0) = 0 \end{cases}$$

for all $t \geq 0$.

Activity invariant set says that in an invariant set, the activity of a neuron keeps invariant, i.e., if a neuron is initially active then it keeps active for all the time, if a neuron is initially inactive then it keeps inactive thereafter.

A point $x^* \in R^n$ is called an equilibrium point of (1) if

$$x_i^* \left[h_i - x_i^* + \sum_{j=1}^n w_{ij} x_j^* \right] = 0, \quad (i = 1, \cdots, n). \quad (2)$$

Given any $x \in R^n$, denote $\|x\| = \max_{1 \leq i \leq n}\{|x_i|\}$.

Definition 4: An invariant set D of (1) is said to have an exponentially stable attractor x^*, if $x^* \in D$ is an equilibrium point of (1), and there exits a constant $\epsilon > 0$ and for any $C > 0$, there exists a constant $\delta > 0$ such that $\|x(0) - x^*\| < \delta$ implies that

$$\|x(t) - x^*\| \leq C \cdot e^{-\epsilon t}$$

for all $t \geq 0$.

Definition 5: Suppose that M is an $n \times n$ matrix, and let $P \subseteq \{1, 2, \cdots, n\}$ be an index set. The matrix M_P is said to be a submatrix of M if the matrix M_P can be constructed from M simply by removing from M all rows and columns not indexed by P.

Given a constant c, denote by

$$\begin{cases} c^- = \min\{c, 0\}, \\ c^+ = \max\{c, 0\}. \end{cases}$$

Clearly, $c^- \leq 0$ and $c^+ \geq 0$.

Lemma 2: It holds that

$$c^+ - c^- = |c|, \quad c^+ \times c^- = 0.$$

Proof: If $c \geq 0$, then, $c^+ = c$ and $c^- = 0$, thus,

$$c^+ - c^- = c = |c|, \quad c^+ \times c^- = 0.$$

If $c < 0$, then, $c^+ = 0$ and $c^- = c$, thus,

$$c^+ - c^- = -c = |c|, \quad c^+ \times c^- = 0.$$

This proof is completed. ■

III. Activity Invariant Sets and Exponentially Stable Attractor

In this section, we will address the questions: under what conditions the network (1) can have invariant sets, and can an invariant set have an exponentially stable attractor? The two questions can be solved by the following theorems, respectively. Theorem 1 firstly establishes the conditions to locate activity invariant sets.

Theorem 1: Suppose that $P \cup Z = \{1, 2, \cdots, n\}$ and $P \cap Z$ is empty. If there exist constants $0 < \xi_i < \eta_i (i \in P)$ such that

$$\begin{cases} h_i + (w_{ii} - 1)\xi_i + \sum_{j \in P, j \neq i} \left(w_{ij}^+ \xi_j + w_{ij}^- \eta_j\right) \geq 0, \\ h_i + (w_{ii} - 1)\eta_i + \sum_{j \in P, j \neq i} \left(w_{ij}^+ \eta_j + w_{ij}^- \xi_j\right) \leq 0, \end{cases} \quad (3)$$

then the set

$$D = \{x | \xi_i \leq x_i \leq \eta_i (i \in P); \quad x_l = 0 (l \in Z)\}$$

is an activity invariant set of the network (1), and the neurons with index in P are active invariant, the neurons with index in Z are inactive invariant.

Proof: Given any initial $x(0) \in D$, we will prove that it holds that the trajectory $\xi_i \leq x_i(t) \leq \eta_i$ for all $t \geq 0$ if $i \in P$, and $x_l(t) = 0$ for all $t \geq 0$ if $l \in Z$.

By Lemma 1, given any initial $x_l(0) = 0 (l \in Z)$, it must hold $x_l(t) = 0$ for all $t \geq 0$ and $l \in Z$. Thus, it will be sufficient to prove that any initial $\xi_i \leq x_i(0) \leq \eta_i (i \in P)$ implies $\xi_i \leq x_i(t) \leq \eta_i$ for all $t \geq 0$ and $i \in P$. We will show this by counterproof. Suppose this is not true, then two cases can happen.

Case 1. There exist $\tilde{t}_1 > t_1 > 0$ and some $i \in P$ such that $x_i(t)$ is strictly decreasing on the interval $[t_1, \tilde{t}_1]$, and

$$\begin{cases} \xi_i \leq x_i(t) \leq \eta_i, & 0 \leq t < t_1, \\ \xi_j \leq x_j(t) \leq \eta_j, & 0 \leq t \leq \tilde{t}_1, \quad j \neq i, \quad j \in P, \\ x_l(t) = 0, & t \geq 0, \quad l \in Z. \end{cases}$$

Thus, it must hold that $\dot{x}_i(t) < 0$ for $t \in [t_1, \tilde{t}_1]$. However, from (1) and by condition (3), it follows that for $t \in [t_1, \tilde{t}_1]$

$$\begin{aligned} \dot{x}_i(t) &= x_i(t) \left[h_i - x_i(t) + \sum_{j=1}^n w_{ij} x_j(t) \right] \\ &= x_i(t) \left[h_i - x_i(t) + \sum_{j \in P} w_{ij} x_j(t) \right] \\ &\geq \xi_i \cdot \left[h_i + (w_{ii} - 1)\xi_i + \sum_{j \in P, j \neq i} \left(w_{ij}^+ \xi_j + w_{ij}^- \eta_j\right) \right] \\ &\geq 0. \end{aligned}$$

This is a contradiction.

Case 2. There exist $\tilde{t}_2 > t_2 > 0$ and some $i \in P$ such that $x_i(t)$ is strictly increasing on the interval $[t_2, \tilde{t}_2]$, and

$$\begin{cases} \xi_i \leq x_i(t) \leq \eta_i, & 0 \leq t < t_2, \\ \xi_j \leq x_j(t) \leq \eta_j, & 0 \leq t \leq \tilde{t}_2, \quad j \neq i, \quad j \in P, \\ x_l(t) = 0, & t \geq 0, \quad l \in Z. \end{cases}$$

Thus, it must hold that $\dot{x}_i(t) > 0$ for $t \in [t_2, \tilde{t}_2]$. However, from (1) and by condition (3), it follows for $t \in [t_2, \tilde{t}_2]$ that

$$\begin{aligned}\dot{x}_i(t) &= x_i(t)\left[h_i - x_i(t) + \sum_{j=1}^{n} w_{ij}x_j(t)\right]\\ &= x_i(t)\left[h_i - x_i(t) + \sum_{j\in P} w_{ij}x_j(t)\right]\\ &\leq \eta_i \cdot \left[h_i + (w_{ii}-1)\eta_i + \sum_{j\in P, j\neq i}\left(w_{ij}^{+}\eta_j + w_{ij}^{-}\xi_j\right)\right]\\ &\leq 0.\end{aligned}$$

This is also a contradiction.

The above contradictions clearly imply that D must be a local invariant set. Moreover, D is an activity invariant set. So the set of neurons with index P is an active invariant set of the network while the set of neurons with index Z is an inactive invariant set. The proof is completed. ■

Theorem 1 gives the conditions to locate the activity invariant sets. Before addressing the second question whether an invariant set can have an exponentially stable attractor, we will firstly give the following lemma which is useful to prove Theorem 2.

Lemma 3: Let x^* be an equilibrium, then, the linearization of the network at x^* is given by:

$$\begin{aligned}&\frac{d\left[x_i(t) - x_i^*\right]}{dt}\\ &= (x_i(t) - x_i^*)\cdot\left(h_i - x_i^* + \sum_{j=1}^{n} w_{ij}x_j^*\right)\\ &\quad + x_i^*\left[-(x_i(t) - x_i^*) + \sum_{j=1}^{n} w_{ij}\left(x_j(t) - x_j^*\right)\right] \quad (4)\end{aligned}$$

for $t \geq 0$ and $i = 1, \cdots, n$.

Proof: From (1), it gives that

$$\begin{aligned}&\frac{d\left[x_i(t) - x_i^*\right]}{dt}\\ &= (x_i(t) - x_i^*)\cdot\left[h_i - x_i^* + \sum_{j=1}^{n} w_{ij}x_j^* - (x_i(t) - x_i^*)\right.\\ &\quad \left. + \sum_{j=1}^{n} w_{ij}\left(x_j(t) - x_j^*\right)\right]\\ &\quad + x_i^*\left(h_i - x_i^* + \sum_{j=1}^{n} w_{ij}x_j^*\right)\\ &\quad + x_i^*\left[-(x_i(t) - x_i^*) + \sum_{j=1}^{n} w_{ij}\left(x_j(t) - x_j^*\right)\right]\\ &= (x_i(t) - x_i^*)\cdot\left(h_i - x_i^* + \sum_{j=1}^{n} w_{ij}x_j^*\right)\\ &\quad + x_i^*\left[-(x_i(t) - x_i^*) + \sum_{j=1}^{n} w_{ij}\left(x_j(t) - x_j^*\right)\right]\\ &\quad + o\left(x_i(t) - x_i^*\right),\end{aligned}$$

where $o(s)$ denotes the higher-degree polynomial of variable $s \in R$. The result now follows by removing the high order term. The proof is completed. ■

Theorem 2: Suppose the conditions of Theorem 1 are satisfied. Moreover, if it holds that

$$h_l + \sum_{j\in P}\left(w_{lj}^{+}\eta_j + w_{lj}^{-}\xi_j\right) < 0, \quad (l \in Z), \quad (5)$$

then D has an exponentially stable attractor.

Proof: The proof will be divided into two parts. In the first part, we will prove that there exists a unique equilibrium point $x^* \in D$, i.e.,

$$\begin{cases} 0 < \xi_i \leq x_i^* \leq \eta_i, & i \in P,\\ x_i^* = 0, & i \in Z.\end{cases}$$

If x^* is an equilibrium point, then from (2), it must hold that

$$\begin{cases} h_P - x_P^* + W_P x_P^* = 0_P,\\ x_Z^* = 0_Z,\end{cases} \quad (6)$$

where h_P, x_P^* are the vectors constructed by removing the elements not indexed by P from h and x^*, respectively. x_Z^* is the vector constructed by removing from x^* the elements not indexed by Z. W_P is the submatrix of W constructed by removing from W all rows and columns not indexed by P. 0_P and 0_Z are vectors with each elements as 0. We will prove that (6) has a unique solution. By condition (3), for $i \in P$, we have

$$(1 - w_{ii})(\eta_i - \xi_i) - \sum_{j\in P, j\neq i}\left(w_{ij}^{+} - w_{ij}^{-}\right)(\eta_j - \xi_j) > 0,$$

i.e.,

$$(1 - w_{ii})(\eta_i - \xi_i) - \sum_{j\in P, j\neq i}|w_{ij}|(\eta_j - \xi_j) > 0 \quad (7)$$

for $i \in P$. Then it must hold that

$$(1-w_{ii})(\eta_i-\xi_i)-\sum_{j\in P, j\neq i} w_{ij}(\eta_j-\xi_j)>0, \quad (i\in P).$$

This implies that $(I-W)_P$ is a nonsingular *M*-matrix [3]. Then from (6), it follows that

$$x_P^*=(I-W)_P^{-1}h_P.$$

Thus,

$$\begin{cases} x_P^*=(I-W)_P^{-1}h_P \\ x_Z^*=0_Z \end{cases}$$

is the unique solution of (6), i.e, x^* is a unique equilibrium point in D.

Next, in the second part, we will prove x^* is exponentially stable. By the definition of stability in Lyapunov sense, it is sufficient to prove the stability of the linearization of the network (1) at x^*.

By Lemma 3, we have

$$\begin{cases} \dfrac{d[x_i(t)-x_i^*]}{dt} &= x_i^*\Big[-(x_i(t)-x_i^*) \\ & \quad +\sum_{j\in P} w_{ij}\left(x_j(t)-x_j^*\right) \\ & \quad +\sum_{j\in Z} w_{ij}x_j(t)\Big], \qquad (i\in P) \\ \dfrac{dx_i(t)}{dt} &= x_i(t)\Big[h_i+\sum_{j\in P} w_{ij}x_j^*\Big], \quad (i\in Z) \end{cases} \tag{8}$$

for $t\geq 0$. Firstly, we will prove that $x_i(t)(i\in Z)$ converge to $x_i^*(i\in Z)$ exponentially. By condition (5), it holds for $i\in Z$ that

$$\begin{aligned} \lambda_i &\triangleq -\left[h_i+\sum_{j\in P} w_{ij}x_j^*\right] \\ &\geq -\left[h_i+\sum_{j\in P}\left(w_{ij}^+\eta_j+w_{ij}^-\xi_j\right)\right] \\ &> 0. \end{aligned}$$

Then the trajectories starting from $x_i(0)>0(i\in Z)$ satisfy that

$$\|x_Z(t)\|\leq\|x_Z(0)\|\cdot e^{-\lambda t} \tag{9}$$

for all $t\geq 0$, where $\lambda=\min_{i\in Z}\{\lambda_i\}$. Clearly, $x_i(t)(i\in Z)$ will converge to zero exponentially.

Next, we consider another subsystem of (8)

$$\begin{aligned} &\frac{d[x_i(t)-x_i^*]}{dt} \\ &= x_i^*\left[-(x_i(t)-x_i^*)+\sum_{j\in P} w_{ij}\left(x_j(t)-x_j^*\right)\right. \\ &\quad \left.+\sum_{j\in Z} w_{ij}x_j(t)\right] \end{aligned} \tag{10}$$

for $t\geq 0$ and $i\in P$.

Denote by $\alpha_i=\eta_i-\xi_i>0(i\in P)$, it follows from (7) that

$$\beta\triangleq\min_{i\in P}\left\{x_i^*\left(1-w_{ii}-\frac{1}{\alpha_i}\sum_{j\in P,j\neq i}\alpha_j|w_{ij}|\right)\right\}>0.$$

Construct a Lyapunov function as follows

$$V(t)=\sum_{i\in P}\alpha_i\frac{|x_i(t)-x_i^*|}{x_i^*} \tag{11}$$

for $t\geq 0$. It follows from (10) and (11) that

$$\begin{aligned} D^+V(t) &\leq \sum_{i\in P}\alpha_i\Big[-(1-w_{ii})|x_i(t)-x_i^*| \\ &\quad +\sum_{j\in P,j\neq i}|w_{ij}|\left|x_j(t)-x_j^*\right|\Big] \\ &\quad +\sum_{i\in P}\alpha_i\sum_{j\in Z}|w_{ij}|x_j(t) \\ &\leq -\sum_{j\in P}\alpha_j\left(1-w_{jj}-\frac{1}{\alpha_j}\sum_{i\in P,i\neq j}\alpha_i|w_{ji}|\right) \\ &\quad \times\left|x_j(t)-x_j^*\right|+\sum_{i\in P}\alpha_i\sum_{j\in Z}|w_{ij}|x_j(t) \\ &\leq -\beta V(t)+\sum_{i\in P}\alpha_i\sum_{j\in Z}|w_{ij}|x_j(t) \\ &\leq -\beta V(t)+M\|x_Z(t)\| \end{aligned}$$

for $t\geq 0$, where

$$M=\max_{j\in Z}\left\{\sum_{i\in P}\alpha_i|w_{ij}|\right\}.$$

Then it gives that

$$\begin{aligned} V(t) &\leq V(0)e^{-\beta t}+M\int_0^t\|x_Z(s)\|\cdot e^{-\beta(t-s)}ds \\ &\leq V(0)e^{-\beta t}+M\|x_Z(0)\|\cdot e^{-\beta t}\cdot\int_0^t e^{(\beta-\lambda)s}ds \\ &\leq \overline{Q}\|x_P(0)-x_P^*\|\cdot e^{-\beta t} \\ &\quad +M\|x_Z(0)\|\cdot e^{-\beta t}\cdot\int_0^t e^{(\beta-\lambda)s}ds \end{aligned}$$

for $t\geq 0$, where

$$\overline{Q}=\max_{i\in P}\left\{\frac{\alpha_i}{x_i^*}\right\}.$$

Since

$$\int_0^t e^{(\beta-\lambda)s}ds=\begin{cases}\dfrac{1}{\beta-\lambda}\cdot\left[e^{(\beta-\lambda)t}-1\right], & \text{if } \beta\neq\lambda \\ t, & \text{if } \beta=\lambda\end{cases}$$

for $t\geq 0$, if $\beta\neq\lambda$, it gives that

$$\begin{aligned} V(t) &\leq \overline{Q}\|x_P(0)-x_P^*\|\cdot e^{-\beta t} \\ &\quad +M\|x_Z(0)\|\cdot\frac{1}{\beta-\lambda}\cdot\left(e^{-\lambda t}-e^{-\beta t}\right) \\ &= \Pi e^{-\beta t}+M\|x_Z(0)\|\cdot\frac{1}{\beta-\lambda}\cdot e^{-\lambda t} \end{aligned}$$

for $t \geq 0$, where

$$\Pi = \overline{Q}\|x_P(0) - x_P^*\| - M\|x_Z(0)\| \cdot \frac{1}{\beta - \lambda}.$$

And if $\beta = \lambda$, it gives that

$$V(t) \leq \overline{Q}\|x_P(0) - x_P^*\|e^{-\beta t} + M\|x_Z(0)\| \cdot te^{-\beta t}$$

for $t \geq 0$. Since

$$V(t) \geq \underline{Q}\,\|x_P(t) - x_P^*\|,$$

where

$$\underline{Q} = \min_{i\in P}\left\{\frac{\alpha_i}{x_i^*}\right\},$$

there must exist constants $\Phi > 0$ and $\epsilon > 0$ such that

$$\|x_P(t) - x_P^*\| \leq \Phi \cdot e^{-\epsilon t} \tag{12}$$

for all $t \geq 0$. It implies that x_P will converge exponentially to x_P^*. Then from (9) and (12), x^* is a exponentially stable attractor located in D. The proof is completed. ■

The theorem above shows that for some division of neurons of the network (1), i.e., $P \cup Z = \{1, 2, \cdots, n\}$, and $P \cap Z = \emptyset$, if there exists a pair of constant vector (ξ, η) such that (3) which locates an activity invariant set D, then the activity of each neuron in D keeps invariant, i.e., the set of neurons with index P will keep active while the set of neurons with index Z will keep inactive all the time.

Moreover, it also shows that under the conditions of Theorem 2, the activity invariant set D has one exponentially stable attractor which is regarded as memory stored in the synaptic connections of the networks. Since the activity invariant set D is composed of two parts, active and inactive invariant set, each attractor has binary pattern. Furthermore, in the active invariant part, the neurons carry analogy information. Thus, the networks implement a form of hybrid analog-digital computation. In other words, the attractors of the network (1) could be used to store memories with both binary and analog information. Thus it can provide new perspective for some potential applications. For example, in the application of group winner-take-all, the network outputs are required to have binary pattern, i.e., the winner group and the other losers. In addition, there may exist differences among neurons in the winner group, such differences can be depicted by analogy information of each neuron in the winner group.

From Theorems 1 and 2, we can have the following corollary.

Corollary 1: If there exist constants $0 < \xi_i < \eta_i (i = 1, 2, \cdots, n)$ such that

$$\begin{cases} h_i + (w_{ii} - 1)\,\xi_i + \displaystyle\sum_{j\in P, j\neq i}\left(w_{ij}^+\xi_j + w_{ij}^-\eta_j\right) \geq 0 \\ h_i + (w_{ii} - 1)\,\eta_i + \displaystyle\sum_{j\in P, j\neq i}\left(w_{ij}^+\eta_j + w_{ij}^-\xi_j\right) \leq 0 \end{cases}$$

for $i = 1, 2, \cdots, n$, then the set

$$D = \{x | x_i \in [\xi_i, \eta_i], (i = 1, 2, \cdots, n)\}$$

is an activity invariant set of the network (1). Moreover, D has an exponentially stable attractor.

Proof: Let $P = \{1, 2, \cdots, n\}$ and Z be empty, the result follows from Theorems 1 and 2. ■

IV. Simulation Results

In this section, simulations will be carried out to illustrate how to locate the activity invariant sets. A simple two dimensional network will be employed for illustrations.

Let us consider the following two dimensional network:

$$\begin{cases} \dot{x}_1(t) = & x_1(t)\left[1 - x_1(t) + 0.5x_1(t) - 5x_2(t)\right] \\ \dot{x}_2(t) = & x_2(t)\left[1 - x_2(t) - 5x_1(t) + 0.5x_2(t)\right] \end{cases} \tag{13}$$

for $t \geq 0$. Clearly, $w_{11} = w_{22} = 0.5, w_{12} = w_{21} = -5, h_1 = h_2 = 1$.

Taking $P = \{1\}, Z = \{2\}$, by conditions (3) and (5) of Theorems 1 and 2, we have inequalities for possible invariant set as

$$\begin{cases} 1 - 0.5\xi_1 \geq 0 \\ 1 - 0.5\eta_1 \leq 0 \\ 1 - 5\xi_1 \leq 0 \\ 0 < \xi_1 < \eta_1. \end{cases}$$

Solving the inequalities, one can have that $\xi_1 = 0.2$ and $\eta_1 \geq 2$. Thus,

$$D_1 = \left\{x = (x_1, x_2)^T \middle| \, 0.2 \leq x_1 < +\infty; \quad x_2 = 0\right\}$$

is an activity invariant set, and the neuron with index $i = 1$ is active invariant in D_1 while the neuron with index $i = 2$ is inactive invariant in D_1, respectively. Moreover, by Theorem 2, D_1 has an exponentially stable attractor.

Next, taking $P = \{2\}, Z = \{1\}$ and solving the inequalities

$$\begin{cases} 1 - 0.5\xi_1 \geq 0 \\ 1 - 0.5\eta_1 \leq 0 \\ 1 - 5\xi_1 \leq 0 \\ 0 < \xi_1 < \eta_1. \end{cases}$$

It can be found that $\xi_2 = 0.2$, $\eta_2 \geq 2$ is a solution. Thus,

$$D_2 = \left\{x = (x_1, x_2)^T \middle| \, x_1 = 0; \quad 0.2 \leq x_2 < +\infty\right\}$$

is an activity invariant set, and the neuron with index $i = 2$ is active invariant in D_2 while the neuron with index $i = 1$ is inactive invariant in D_2, respectively. Moreover, by Theorem 2, D_2 has an exponentially stable attractor.

We further consider the case: $P = \{1, 2, \cdots, n\}$ and Z to be empty. In this case, the inequalities (3) and (5) cannot have solutions.

Figure 1 shows the activity invariant sets and the exponentially stable attractors of the network (13). The dashed lines denote D_1 and D_2 which are two rays located in x-axis and y-axis, respectively. Furthermore, there are two local stable equilibrium points $(2, 0)^T$ and $(0, 2)^T$ located in D_1 and D_2, respectively, and attract all trajectories in the corresponding regions.

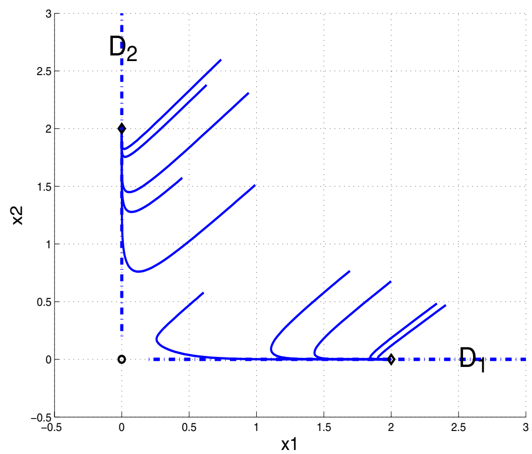

Fig. 1. Activity invariant sets and exponentially stable attractors of the network (13). There are two local stable equilibrium points $(2,0)^T$ and $(0,2)^T$ located in two activity invariant sets $D_1 = \{x\,|0.2 \le x_1 < +\infty; \quad x_2 = 0\}$ and $D_2 = \{x\,|x_1 = 0; \quad 0.2 \le x_2 < +\infty\}$, respectively.

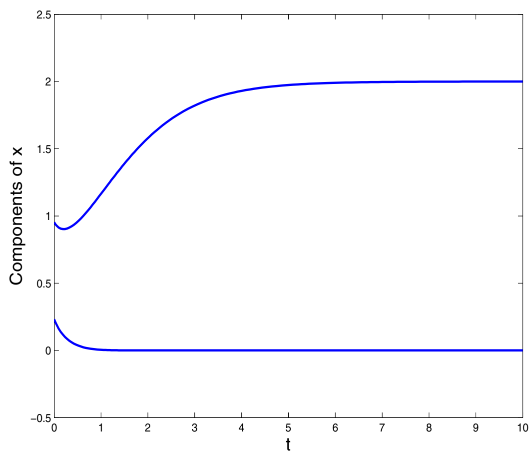

Fig. 2. Convergence of the network (13). The trajectory starts from a randomly initial point $x(0) = (2.3477, 0.4665)^T$. It converges to the stable point $x^* = (2,0)^T \in D_1$.

V. Conclusion

In this paper, the activity invariant sets and exponentially stable attractors of Lotka-Volterra recurrent neural networks have been studied. Conditions have been derived to locate the activity invariant sets. It shows that under some conditions an invariant set can process an exponentially stable attractor. Such an attractor carries both binary and analog information. We believe these interesting properties can give new perspective for applications of attractor networks to group winner-take-all, associative memory, etc.. More researches in this direction will be carried out in future.

Acknowledgment

This work was supported by Chinese 863 High-Tech Program under Grant 2007AA01Z321.

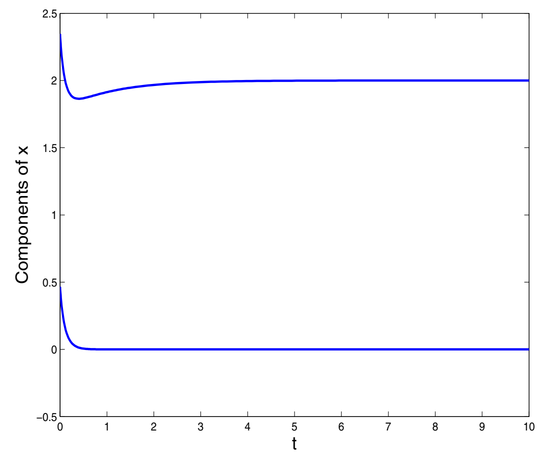

Fig. 3. Convergence of the network (13). The trajectory starts from a randomly initial point $x(0) = (0.2323, 0.9536)^T$. It converges to the stable point $x^* = (0,2)^T \in D_2$.

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