A Symbolic Approach to Reconstruct Polyhedral Scene from Single 2D Line Drawing

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***Abstract*—Scene reconstruction based on polyhedral solids is an important problem in computer vision. Generally speaking, an efficient representation is to use a line drawing which is regarded as a projection of the 3D object. In this paper, we consider a single line drawing whose face topology is already known and all the vertices are given in the 2D projection. To recover the possible 3D polyhedra which project to the line drawing, we present a symbolic geometric algorithm based on the Grassmann-Cayley algebra. A number of examples demonstrate its outstanding performance which can lead to an very significant simplifications in symbolic manipulation of geometric data.**

I. Introduction

How to characterize and perceive a 3D polyhedral solid has been a very interesting problem in both science and art [14]. A simple and straightforward way is to use a line drawing which is the 2D planar projection of the wireframe of the solid in a generic view, like draft or sketch in geometric model design and so on.

The reconstruction problem usually includes topological reconstruction (i.e. 2D face identification) and geometric reconstruction (i.e. 3D coordinate parameterization). Since the former can be solved by means of heuristic or optimized detection [1], [5], [6], [7], throughout this paper, it will be assumed that we have the topological information from the single line drawing. We focus on geometric reconstruction, in order to judge whether the given line drawing is the projection of a 3D object, and if so, what are the relative depths of its vertices. Obviously, the process is one of "upgrading" a lower dimensional object into higher dimensions. Consider the picture of a cheese in Fig. 1a and the corresponding line drawing extracted from it in Fig. 1b where there is one truncated pyramid formed by 2 triangular faces and 3 quadrilateral faces. However, this truncated pyramid can be realized in 3D space if and only if the three lines $\mathbf{14}$, $\mathbf{25}$, and $\mathbf{36}$ intersect in the 2D image plane (Fig. 1c). Such conditions that must be satisfied for the line drawing to be upgraded in the 3D space are called *realizability conditions*. [3] and [8] first studied the realizability conditions. Their methods can provide necessary but not sufficient conditions. An algebraic and combinatorial approach [11] was presented to establish a necessary and sufficient condition for the interpretation of line drawing. [13] further proposed the concept —- *resolvable sequences* for polyhedra and proved that they always exist for spherical polyhedra, that is to say, polyhedra which are homeomorphic to a sphere. This result is directly used by [9] to correct incorrect line drawings —- incorrect projections of a polyhedral scene. A drawback of these methods is that they are non-robust and require heavy computation in practice, moreover, in some cases they do not find all the realizability conditions. The Sugihara-torus [12] is an example as shown in Fig. 2. For this example, there are 36 fundamental equations in 36 unknowns and the torus does not have any resolvable sequence. [2] based on invariant approach reformulated the above system and established a set of equations called *syzygy equations* whose number is reduced to 9. Although the system of [2] is linear and sparse, the syzygy equations are still too difficult to be solved symbolically. Especially, traditional elimination techniques to solve these equations for deriving the 3D coordinates of all vertices often fail to be efficient in complicated scenes. It remains an open problem to find a general and feasible framework for the parameterization of the 3D-coordinate solution spaces.

The main contribution of this paper are two-fold:

- We write the reconstruction equations in terms of vectors, bivectors and their brackets for easy manipulation by using Grassmann-Cayley algebra.
- Then we develop the powerful *vectorial equation-solving strategy* to solve this system. In solving reconstruction equations, some new parameters locally are introduced and a simplified expression is obtained after propagating these local parametric solutions.

Compared with the previous methods, our vectorial equa- tion-solving based on parametric propagation has the following advantages: (1) the order of propagation is easy to be determined automatically, (2) the solution procedure becomes more compact, (3) all possible 3D polyhedral reconstructions and realizability conditions are found, (4) some realizability conditions can be expressed in a factorable form which indicate the intuitionistic geometric meanings.

The rest of the paper is arranged as follows. Next section lists some brief preliminaries. Section 3 reports a novel algebraic formulation of polyhedral scene analysis. Section 4 solves the reconstruction equations by parametric propagation. Section 5 discusses the classification problem and Section 6 contains some further experimental examples. At last, Section

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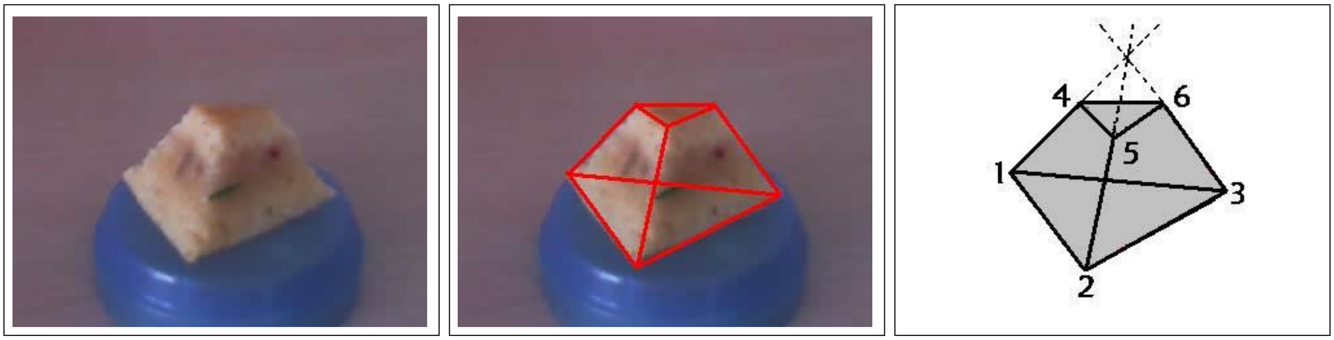

(a) (b) (c)

Fig. 1. (a) A picture of cheese; (b) The detected line drawing; (c) The realizability condition.

TABLE I
FORMULAS IN THE GRASSMANN-CAYLEY ALGEBRA.

$\mathbf{12} = -\mathbf{21}$
$\mathbf{1} \wedge \mathbf{23} = \mathbf{12} \wedge \mathbf{3} = [\mathbf{123}]$
$\mathbf{12} \wedge \mathbf{34} = [\mathbf{134}]\mathbf{2} - [\mathbf{234}]\mathbf{1} = -\mathbf{34} \wedge \mathbf{12}$
$[\mathbf{123}]\mathbf{4} = [\mathbf{124}]\mathbf{3} - [\mathbf{134}]\mathbf{2} + [\mathbf{234}]\mathbf{1}$
$[\mathbf{123}]\mathbf{45} = [\mathbf{145}]\mathbf{23} - [\mathbf{245}]\mathbf{13} + [\mathbf{345}]\mathbf{12}$
$[\mathbf{124}][\mathbf{135}] - [\mathbf{123}][\mathbf{145}] = [\mathbf{125}][\mathbf{134}]$
$[\mathbf{134}][\mathbf{256}] - [\mathbf{234}][\mathbf{156}] = \mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}$

7 concludes our work.

II. PRELIMINARIES

Grassmann-Cayley algebra is a structure with two operators: the *out product* "$\vee$" and the *meet product* "$\wedge$" in which projective properties are described in a coordinate-free way. We only list the needed notations in 2D projective geometry:

- For 2D projective space P^2, the homogeneous coordinates of a point $A = (a_1, a_2)$ are $(a_1, a_2, 1)$, and the *bracket* of three points A_1, A_2, A_3 is defined by:

$$[A_1 A_2 A_3] = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ 1 & 1 & 1 \end{vmatrix}. \qquad (1)$$

  In a general way, for any three points A_1, A_2, A_3 in P^n, $[A_1 A_2 A_3]$ can be regarded as twice the signed area of $S_{\triangle A_1 A_2 A_3}$. Out of habit, we use a bold number to denote a point which is represented by a nonzero vector.
- A line passing through points $\mathbf{1}$, $\mathbf{2}$ is represented by a *bivector* $\mathbf{1} \vee \mathbf{2}$, which can be written as $\mathbf{12}$ for simplification. Then the intersection of two lines $\mathbf{12}$, $\mathbf{34}$ is represented by $\mathbf{12} \wedge \mathbf{34}$.
- Three points $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ are collinear if and only if their bracket is zero, that is to say, $[\mathbf{123}] = 0$.
- Three lines $\mathbf{12}$, $\mathbf{34}$, $\mathbf{56}$ meet at a point if and only if their meet product is zero, that is to say, $\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56} = 0$.

Some basic formulas are given in [4], [10], [16] where you can find more details about Grassmann-Cayley algebra, and also listed in Table II.

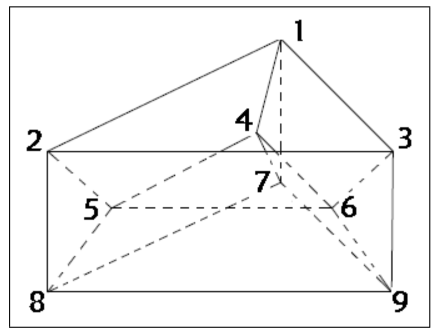

Fig. 2. The 2D image of the 3D Sugihara-torus which is composed of 9 vertices and 9 faces (**1245**, **1346**, **2356**, **1278**, **1379**, **2389**, **4578**, **4679**, **5689**) with the assumption that no three vertices in a face are collinear.

III. PROJECTIVE RECONSTRUCTION

Consider a polyhedron $\mathcal{P} = \{(v_i,\ f_j) | v_i \in \mathcal{V},\ f_j \in \mathcal{F}\}$, where $\mathcal{V}$ and $\mathcal{F}$ denote the sets of vertices and faces respectively, and $(v_i,\ f_j)$ denotes the associated structure that vertex v_i is in face f_j. Let $\mathbf{i}$ be the image of vertex v_i and let e_3 outside the image plane $\mathcal{I}$ be the projective center of perspective projection (or the projective direction of parallel projection). Because some qualification is needed to avoid an ambiguity in reconstruction from a single image, we adopt the following assumption: *no three neighboring vertices of a face are collinear in the line drawing*.

Let e_1, e_2, e_4 be a basis of $\mathcal{I}$ satisfied with $[e_1 e_2 e_4] = 1$, then e_1, e_2, e_3, e_4 form a basis of the 3D projective space $\mathbb{P}^3$. The homogeneous coordinates of image $\mathbf{i}$ with respect to the basis e_1, e_2, e_4 are $(x_i, y_i, 1)$, i.e.,

$$\mathbf{i} = x_i e_1 + y_i e_2 + e_4, \qquad (2)$$

so the homogeneous coordinates of point v_i with respect to the basis e_1, e_2, e_3, e_4 are $(x_i, y_i, z_i, 1)$, i.e.,

$$v_i = \mathbf{i} + z_i e_3 = x_i e_1 + y_i e_2 + z_i e_3 + e_4 \qquad (3)$$

where z_i is the unknown "depth" of v_i. Let the homogeneous coordinates which specify the plane containing the face f_j be $(a_j, b_j, -1, c_j)$. The point v_i is in face f_j if and only if

$$a_j x_i + b_j y_i + c_j = z_i. \qquad (4)$$

Let

$$\mathbf{B}_{\mathbf{f_j}} = a_j e_2 e_4 - b_j e_1 e_4 + c_j e_1 e_2, \tag{5}$$

where the bivector $\mathbf{B}_{\mathbf{f_j}}$ is called the *inhomogeneous coordinate* of face f_j. Substitute (2) and (4) into (5), easy to see that

$$z_i = [\mathbf{i}\mathbf{B}_{\mathbf{f_j}}], \tag{6}$$

If v_i is contained in the set of faces $\{f_{j_1}, \cdots, f_{j_m}\}$, then we have the following $\mathbf{B}$-*system*:

$$z_i = [\mathbf{i}\mathbf{B}_{\mathbf{f_{j_1}}}] = [\mathbf{i}\mathbf{B}_{\mathbf{f_{j_2}}}] = \cdots = [\mathbf{i}\mathbf{B}_{\mathbf{f_{j_m}}}]. \tag{7}$$

By [2], a sufficient and necessary condition for any 4-tuple of vertices v_1, v_2, v_3, v_4 to be coplanar can be expressed as:

$$z_1[\mathbf{234}] - z_2[\mathbf{134}] + z_3[\mathbf{124}] - z_4[\mathbf{123}] = 0. \tag{8}$$

On the other hand, for any bivector $\mathbf{B}$, we have the *Grassmann-Plücker relation* [15]:

$$[\mathbf{1B}][\mathbf{234}] - [\mathbf{2B}][\mathbf{134}] + [\mathbf{3B}][\mathbf{124}] - [\mathbf{4B}][\mathbf{123}] = 0, \tag{9}$$

which shows the solutions of (7) is a set of parameterized solutions of (8), that is to say, all constraints on the reconstruction are included in the $\mathbf{B}$-system. To make this more impressive let us pay our attention on Fig. 1c, where its topological structure is determined by 5 coplanarity constraints: $\mathbf{1245}$, $\mathbf{1346}$, $\mathbf{2356}$, $\mathbf{123}$ and $\mathbf{456}$. So its $\mathbf{B}$-system is written as:

$$\begin{cases} [\mathbf{1B_{1245}}] = [\mathbf{1B_{1346}}] = [\mathbf{1B_{123}}] = z_1 \\ [\mathbf{2B_{1245}}] = [\mathbf{2B_{2356}}] = [\mathbf{2B_{123}}] = z_2 \\ [\mathbf{3B_{1346}}] = [\mathbf{3B_{2356}}] = [\mathbf{3B_{123}}] = z_3 \\ [\mathbf{4B_{1245}}] = [\mathbf{4B_{1346}}] = [\mathbf{4B_{456}}] = z_4 \\ [\mathbf{5B_{1245}}] = [\mathbf{5B_{2356}}] = [\mathbf{5B_{456}}] = z_5 \\ [\mathbf{6B_{1346}}] = [\mathbf{6B_{2356}}] = [\mathbf{6B_{456}}] = z_6. \end{cases} \tag{10}$$

Henceforth, $\mathbf{B}_{*\cdots*}$ always denotes the inhomogeneous coordinates of faces composed of the points $\{*\cdots*\}$.

IV. Parametric Propagation to Solve $\mathbf{B}$-System

Once $\mathbf{B}_{\mathbf{f_j}}$ is known, we can use (6) to obtain the actual height of vertex $\mathbf{i}$. First of all, the needed three formulas will be listed, whose proofs can be easily finished by *Cramer rule* and are omitted. In the list, the $\mathbf{V}$'s are vectors, the $\mathbf{B}$'s are bivectors, the μ's are scalars, and the ω's are new parameters.

Type B.1.

$$\begin{cases} [\mathbf{V_1B}] & = & [\mathbf{V_1B'}] \\ [\mathbf{V_2B}] & = & [\mathbf{V_2B'}] \\ \mathbf{V_1V_2} & \neq & 0 \end{cases}$$

Solution:

$$\mathbf{B} = \mathbf{B'} + \omega\mathbf{V_1V_2}. \tag{11}$$

Type B.2.

$$\begin{cases} [\mathbf{V_1B}] & = & \mu_1 \\ [\mathbf{V_2B}] & = & \mu_2 \\ [\mathbf{V_3B}] & = & \mu_3 \\ [\mathbf{V_1V_2V_3}] & \neq & 0 \end{cases}$$

Solution:

$$[\mathbf{V_1V_2V_3}]\mathbf{B} = \mu_1\mathbf{V_2V_3} - \mu_2\mathbf{V_1V_3} + \mu_3\mathbf{V_1V_2}. \tag{12}$$

Type B.3.

$$\begin{cases} [\mathbf{V_1B}] & = & \mu_1 \\ [\mathbf{V_2B}] & = & \mu_2 \\ \cdots & & \cdots \\ [\mathbf{V_kB}] & = & \mu_k \quad \text{where } k > 3 \\ [\mathbf{V_1V_2V_3}] & \neq & 0 \end{cases}$$

Solution:

$$\begin{cases} [\mathbf{V_1V_2V_3}]\mathbf{B} = \mu_1\mathbf{V_2V_3} - \mu_2\mathbf{V_1V_3} + \mu_3\mathbf{V_1V_2}, \\ \mu_1[\mathbf{V_2V_3V_j}] - \mu_2[\mathbf{V_1V_3V_j}] + \mu_3[\mathbf{V_1V_2V_j}] \\ -\mu_j[\mathbf{V_1V_2V_3}] = 0 \quad \text{for} \quad 3 < j \leq k. \end{cases} \tag{13}$$

The difference between **B.2.** and **B.3.** is the latter type has $k-3$ constraints, which happen to correspond to the realizability conditions.

We develop a technique called *parametric propagation* to solve the $\mathbf{B}$-system. The algorithm is shown below whose basic idea is to choose a $\mathbf{B}$ as the "origin" and solve for other $\mathbf{B}$'s neighboring to it by introduce new parameters. The solved $\mathbf{B}$'s are then put into the origin as the propagation continuing. In the end, the $\mathbf{B}$-system is transformed into two subsystems, one for the height expressed by parameters, the other for the realizability conditions.

Algorithm. (Parametric propagation)

Input: A set of faces $\mathcal{F} = \{f_1, \cdots, f_n\}$ and n sets of vertices for each face $\{\mathcal{V}(f_1)\}, \cdots, \{\mathcal{V}(f_n)\}$ with its $\mathbf{B}$-system.

Output: The parameterized expression for all reconstruction results.

1. Initialization: Set the inhomogeneous coordinate of f_1 be a free parameter **and** let $\mathcal{V} = \{\mathcal{V}(f_1)\}$ **and** let $\mathcal{F} = \mathcal{F} - \{f_1\}$

2. If $\mathcal{F}$ is empty **then break**

3. Else choose f_i in $\mathcal{F}$ satisfied with $\sharp(\{\mathcal{V}(f_i)\} \cap \mathcal{V})$ is maximal **then** let $\mathcal{F} = \mathcal{F} - \{f_i\}$

4. If $\sharp(\{\mathcal{V}(f_i)\} \cap \mathcal{V}) = 2$ **then** use $\mathbf{B.1}$. to obtain the inhomogeneous coordinate of f_i by introducing a new scale parameter

5. If $\sharp(\{\mathcal{V}(f_i)\} \cap \mathcal{V}) = 3$ **then** use $\mathbf{B.2}$. to obtain the inhomogeneous coordinate of f_i

6. If $\sharp(\{\mathcal{V}(f_i)\} \cap \mathcal{V}) \geq 4$ **then** use $\mathbf{B.3}$. to obtain the inhomogeneous coordinate of f_i and the realizability conditions

7. Goto 2

The computational complexities of parametric propagation are divided into two parts: one is for the solution of all inhomogeneous coordinates, while the other one is the computations required for the realizability conditions. Since the operations of choosing f_i in the step **3** at a time is in the order of $O(n)$, the operation for the first part is in the order of $O(n^2)$. For the second part, $O(m)$ is needed. Therefore, the total computation required is $O(n^2) + O(m)$.

As an example, we use parametric propagation to solve the system (10):

Round 1. Let $\mathcal{F} = \{f_{\mathbf{1245}}, f_{\mathbf{1346}}, f_{\mathbf{2356}}, f_{\mathbf{123}}, f_{\mathbf{456}}\}$, then set $\mathbf{B_{1245}}$ be a free new parameter and $\mathcal{F}$ becomes $\{f_{\mathbf{1346}}, f_{\mathbf{2356}}, f_{\mathbf{123}}, f_{\mathbf{456}}\}$.

Round 2. Propagate towards a neighbor of $f_{\mathbf{1245}}$, say $f_{\mathbf{1346}}$. Algebraically this is equivalent to solving a system:

$$\begin{cases} [\mathbf{1B_{1346}}] = [\mathbf{1B_{1245}}] \\ [\mathbf{4B_{1346}}] = [\mathbf{4B_{1245}}] \end{cases} \xrightarrow{B.1} \mathbf{B_{1346}} = \omega\mathbf{14} + \mathbf{B_{1245}}.$$

Here $\mathcal{F}$ becomes $\{f_{\mathbf{2356}}, f_{\mathbf{123}}, f_{\mathbf{456}}\}$.

Round 3. Propagate towards $f_{\mathbf{2356}}$ without introducing new parameters,

$$\begin{cases} [\mathbf{2B_{2356}}] = [\mathbf{1B_{1245}}] \\ [\mathbf{3B_{2356}}] = [\mathbf{3B_{1346}}] = -\omega[\mathbf{134}] + [\mathbf{3B_{1245}}] \\ [\mathbf{5B_{2356}}] = [\mathbf{5B_{1245}}] \\ [\mathbf{6B_{2356}}] = [\mathbf{6B_{1346}}] = \omega[\mathbf{146}] + [\mathbf{6B_{1245}}] \end{cases}$$

$$\xrightarrow{B.3} \begin{cases} [\mathbf{235}]\mathbf{B_{2356}} = \omega[\mathbf{134}]\mathbf{25} \\ \omega([\mathbf{134}][\mathbf{256}] - [\mathbf{146}][\mathbf{235}]) = \mathbf{0}. \end{cases}$$

Round 4. Propagate towards $f_{\mathbf{123}}$,

$$\begin{cases} [\mathbf{1B_{123}}] = [\mathbf{1B_{1245}}] \\ [\mathbf{2B_{123}}] = [\mathbf{2B_{1245}}] \\ [\mathbf{3B_{123}}] = [\mathbf{3B_{1346}}] = -\omega[\mathbf{134}] + [\mathbf{3B_{1245}}] \end{cases}$$

$$\xrightarrow{B.2} [\mathbf{123}]\mathbf{B_{123}} = -\omega[\mathbf{134}]\mathbf{12} + [\mathbf{123}]\mathbf{B_{1245}}.$$

Round 5. Similarly, propagate towards $f_{\mathbf{456}}$,

$$\begin{cases} [\mathbf{4B_{456}}] = [\mathbf{4B_{1245}}] \\ [\mathbf{5B_{456}}] = [\mathbf{5B_{1245}}] \\ [\mathbf{6B_{456}}] = [\mathbf{6B_{1346}}] = \omega[\mathbf{146}] + [\mathbf{6B_{1245}}] \end{cases}$$

$$\xrightarrow{B.2} [\mathbf{456}]\mathbf{B_{456}} = \omega[\mathbf{146}]\mathbf{45} + [\mathbf{456}]\mathbf{B_{1245}}.$$

Final solution: The original $\mathbf{B}$-system is changed into

$$\begin{cases} \mathbf{B_{1346}} = \omega\mathbf{14} + \mathbf{B_{1245}} \\ \mathbf{B_{2356}} = \dfrac{\omega[\mathbf{134}]}{[\mathbf{235}]}\mathbf{25} + \mathbf{B_{1245}} \\ \mathbf{B_{123}} = -\dfrac{\omega[\mathbf{134}]}{[\mathbf{123}]}\mathbf{12} + \mathbf{B_{1245}} \\ \mathbf{B_{456}} = \dfrac{\omega[\mathbf{146}]}{[\mathbf{456}]}\mathbf{45} + \mathbf{B_{1245}} \\ z_1 = [\mathbf{1B_{1245}}] \\ z_2 = [\mathbf{2B_{1245}}] \\ z_3 = [\mathbf{3B_{1346}}] = -\omega[\mathbf{134}] + [\mathbf{3B_{1245}}] \\ z_4 = [\mathbf{4B_{1245}}] \\ z_5 = [\mathbf{5B_{1245}}] \\ z_6 = [\mathbf{6B_{1346}}] = \omega[\mathbf{146}] + [\mathbf{6B_{1245}}], \end{cases} \tag{14}$$

together with its realizability condition

$$\omega([\underline{\mathbf{1}}\mathbf{3}\underline{\mathbf{4}}][\underline{\mathbf{2}}\mathbf{5}\underline{\mathbf{6}}] - [\underline{\mathbf{14}}\mathbf{6}][\underline{\mathbf{2}}\mathbf{3}\underline{\mathbf{5}}]) = \mathbf{0}. \tag{15}$$

Thus, all the solution spaces for the reconstruction of Fig. 1c have been found. By the way, (15) can be reduced to the following factorization by the last formula in Table II,

$$\omega\mathbf{14} \wedge \mathbf{25} \wedge \mathbf{36} = 0, \tag{16}$$

which indicates the geometric meaning that the three lines $\mathbf{14}$, $\mathbf{25}$, and $\mathbf{36}$ intersect in the 2D image if ω is non-trivial ($\neq 0$).

V. Classification

The realizability conditions are used to classify the reconstructions from the given 2D line drawing and evaluate the "possible maximal dimension" of the reconstructed object. By (16), if $\mathbf{14} \wedge \mathbf{25} \wedge \mathbf{36} = 0$, then Fig. 1c can be upgraded to 5 distinct planes for $\omega \neq 0$. When $\mathbf{14} \wedge \mathbf{25} \wedge \mathbf{36} \neq 0$, $\omega = 0$ must hold which indicates that no 3D reconstruction is possible. In many cases, the classification is much more complex than in Fig. 1c. Consider the Sugihara-torus shown in Fig. 2. With our algorithm, its solution spaces are

$$\begin{cases} \mathbf{B_{4578}} = \mu\mathbf{45} + \mathbf{B_{1245}} \\ \mathbf{B_{1278}} = \dfrac{\mu[\mathbf{457}]}{[\mathbf{127}]}\mathbf{12} + \mathbf{B_{1245}} \\ \mathbf{B_{2356}} = \nu\mathbf{25} + \mathbf{B_{1245}} \\ \mathbf{B_{1346}} = \dfrac{\nu[\mathbf{235}]}{[\mathbf{134}]}\mathbf{14} + \mathbf{B_{1245}} \\ \mathbf{B_{5689}} = \dfrac{\mu[\mathbf{458}]\mathbf{56} - \nu[\mathbf{256}]\mathbf{58}}{[\mathbf{568}]} + \mathbf{B_{1245}} \\ \mathbf{B_{4679}} = \dfrac{\mu[\mathbf{457}]\mathbf{46} - \nu[\mathbf{256}]\mathbf{47}}{[\mathbf{467}]} + \mathbf{B_{1245}} \\ \mathbf{B_{2389}} = \dfrac{\mu[\mathbf{458}]\mathbf{23} + \nu[\mathbf{235}]\mathbf{28}}{[\mathbf{238}]} + \mathbf{B_{1245}} \\ \mathbf{B_{1379}} = \dfrac{\mu[\mathbf{457}]\mathbf{13} + \nu[\mathbf{235}]\mathbf{17}}{[\mathbf{137}]} + \mathbf{B_{1245}} \end{cases} \tag{17}$$

with

$$\begin{cases} \nu\mathbf{14} \wedge \mathbf{25} \wedge \mathbf{36} = 0 \\ \mu\mathbf{12} \wedge \mathbf{45} \wedge \mathbf{78} = 0 \\ (\mu[\mathbf{456}] - \nu[\mathbf{256}])\mathbf{47} \wedge \mathbf{58} \wedge \mathbf{69} = 0 \\ (\mu[\mathbf{458}] - \nu[\mathbf{258}])\mathbf{23} \wedge \mathbf{56} \wedge \mathbf{89} = 0 \\ (\mu[\mathbf{134}][\mathbf{457}]\mathbf{46} - \nu[\mathbf{134}][\mathbf{256}]\mathbf{47} \\ -\nu[\mathbf{467}][\mathbf{235}]\mathbf{14})\mathbf{13} \wedge \mathbf{46} \wedge \mathbf{79} = 0, \end{cases} \tag{18}$$

where μ and ν are new parameters. Let

$$N_1 = \{[\mathbf{123}], [\mathbf{456}], [\mathbf{789}]\},$$

$$N_2 = \{[\mathbf{147}], [\mathbf{258}], [\mathbf{369}]\},$$

$$L_1 = \{\mathbf{12} \wedge \mathbf{45} \wedge \mathbf{78}, \mathbf{13} \wedge \mathbf{46} \wedge \mathbf{79}, \mathbf{23} \wedge \mathbf{56} \wedge \mathbf{89}\},$$

$$L_2 = \{\mathbf{14} \wedge \mathbf{25} \wedge \mathbf{36}, \mathbf{17} \wedge \mathbf{28} \wedge \mathbf{39}, \mathbf{47} \wedge \mathbf{58} \wedge \mathbf{69}\}.$$

And for $\forall x \in L_1 \cup L_2$, let $\check{x}$ be the unique bracket in $N_1 \cup N_2$ whose three elements do not occur in x. The classification is as follows.

1) If $L_1 = L_2 = \{0, 0, 0\}$, then the line drawing can be upgraded to 9 distinct planes, i.e., no two faces are coplanar.
2) If $L_i = \{0, 0, 0\}$ ($i = 1$ or 2), L_{3-i} has only one element x equal to zero, $\check{x} = 0$, and $N_i - \{\check{x}\}$ has no element equal to zero, then the line drawing can be upgraded to 5 distinct planes.
3) If $L_i = \{0, 0, 0\}$ ($i = 1$ or 2), L_{3-i} has only one element x equal to zero, $\check{x} = 0$, and $N_i - \{\check{x}\}$ has at least one element equal to zero, then the line drawing can be upgraded to 3 distinct planes.
4) Otherwise, the line drawing has no upgrade.

Consider the above-cited third case as an example. Without loss of generality, assume that $L_1 = \{0, 0, 0\}$, $x(\in L_2) =$

$\mathbf{47}\wedge\mathbf{58}\wedge\mathbf{69}=0$, $\check{x}=[\mathbf{123}]=0$, $[\mathbf{456}](\in N_1-\{[\mathbf{123}]\})=0$. By the first equation of (18), since $\mathbf{14}\wedge\mathbf{25}\wedge\mathbf{36}\neq 0$, then $\nu=0$, we have $\mathbf{B_{1245}}=\mathbf{B_{2356}}=\mathbf{B_{1346}}$ for (17). Since $[\mathbf{123}]=0$, then

$$\begin{aligned}
&\mathbf{B_{1379}}-\mathbf{B_{1278}}\\
=\;&\frac{\mu[\mathbf{457}]\mathbf{13}}{[\mathbf{137}]}-\frac{\mu[\mathbf{457}]\mathbf{12}}{[\mathbf{127}]}\\
=\;&\frac{\mu[\mathbf{457}]}{[\mathbf{137}][\mathbf{127}]}(\underline{[\mathbf{127}]\mathbf{13}}-[\mathbf{137}]\mathbf{12})\\
=\;&\frac{\mu[\mathbf{457}][\mathbf{123}]}{[\mathbf{137}][\mathbf{127}]}\mathbf{17}\\
=\;&0,
\end{aligned}$$

where the fifth formula in Table II is used to expand the term $\underline{[\mathbf{127}]\mathbf{13}}$. And that,

$$\begin{aligned}
&\mathbf{B_{1379}}-\mathbf{B_{2389}}\\
=\;&\frac{\mu[\mathbf{457}]\mathbf{13}}{[\mathbf{137}]}-\frac{\mu[\mathbf{458}]\mathbf{23}}{[\mathbf{238}]}\\
=\;&\frac{\mu}{[\mathbf{137}][\mathbf{238}]}([\mathbf{457}]\underline{[\mathbf{238}]\mathbf{13}}-[\mathbf{458}][\mathbf{137}]\mathbf{23})\\
=\;&\frac{\mu}{[\mathbf{137}][\mathbf{238}]}([\underline{\mathbf{45}}\mathbf{7}][\mathbf{8}\underline{\mathbf{13}}]-[\underline{\mathbf{45}}\mathbf{8}][\underline{\mathbf{13}}\mathbf{7}])\mathbf{23}\\
=\;&\frac{\mu}{[\mathbf{137}][\mathbf{238}]}(\mathbf{78}\wedge\mathbf{45}\wedge\underline{\mathbf{13}})\mathbf{23}\\
=\;&\frac{\mu}{[\mathbf{137}][\mathbf{238}]}(\mathbf{78}\wedge\mathbf{45}\wedge(k\mathbf{12}))\mathbf{23}\\
=\;&0,
\end{aligned}$$

where the fifth formula in Table II is used to expand the term $\underline{[\mathbf{238}]\mathbf{13}}$ and the last one in Table II is used to contract the term $[\underline{\mathbf{45}}\mathbf{7}][\mathbf{8}\underline{\mathbf{13}}]-[\underline{\mathbf{45}}\mathbf{8}][\underline{\mathbf{13}}\mathbf{7}]$ respectively. Notice that $\mathbf{13}=k\mathbf{12}$ because $[\mathbf{123}]=0$ which equal to the three point $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ are collinear. Similarly, $\mathbf{B_{4578}}=\mathbf{B_{4679}}=\mathbf{B_{5689}}$. On the other hand, when $\mu\neq 0$, easy to verify that $\mathbf{B_{1245}}\neq\mathbf{B_{1379}}$, $\mathbf{B_{1245}}\neq\mathbf{B_{4579}}$, and $\mathbf{B_{1379}}\neq\mathbf{B_{4578}}$. Finally, 3 distinct planes are upgraded.

We see that there are many possible reconstructions for the line drawing of Sugihara-torus. Corresponding these reconstructions, their complete classifications shown in the above list are incapable of being obtained by traditional methods.

VI. Experimental results

On a 1.70GHz CPU and 248MB RAM PC with the operating system of Windows 2000, Our parametric propagation algorithm implemented in Maple 8 runs fast enough and some tested examples are shown in Fig. 3. These examples include both spherical and non-spherical polyhedral scenes. It takes less than 0.1 second to obtain the solution spaces of reconstruction for every line drawing. With such encouraging results, reconstructing more complex objects is a challenge for future research.

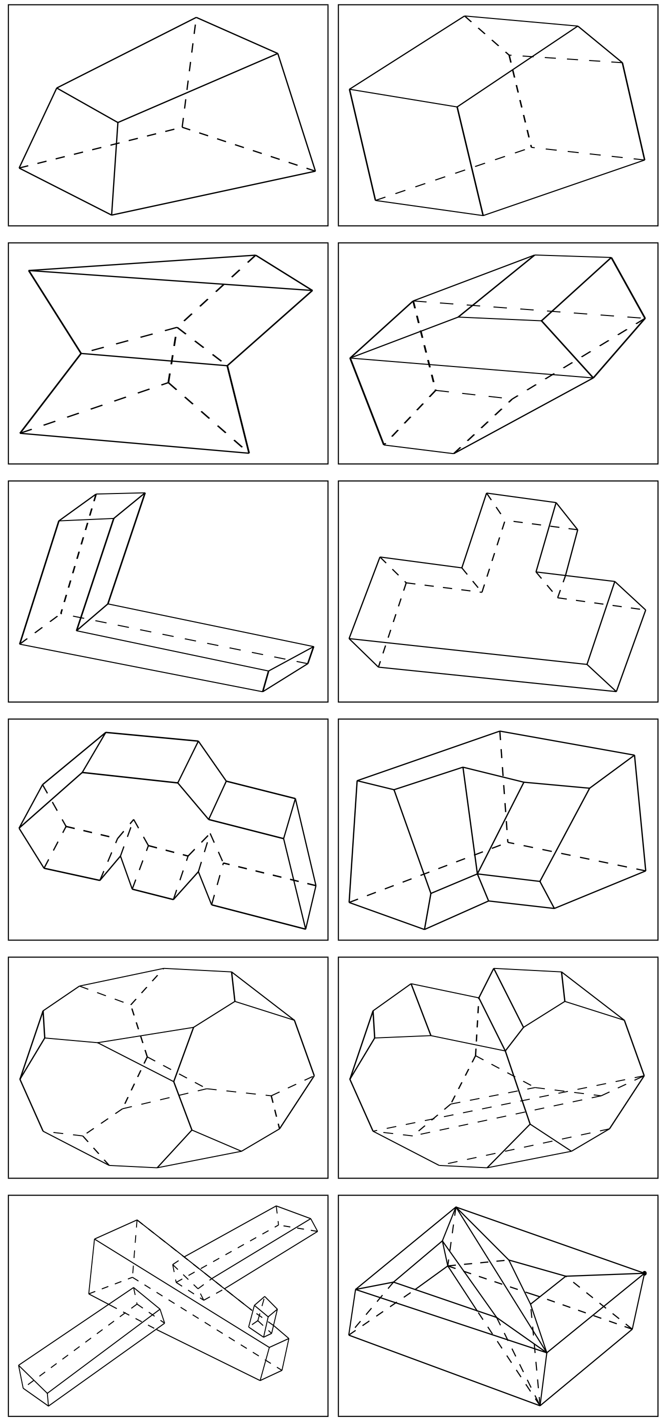

Fig. 3. Some line drawings which have been reconstructed.

VII. Conclusions

In this paper we present a novel approach, called parametric propagation, to solve the reconstruction problem from a single 2D line drawing in polyhedra scene analysis. In parametric propagation, 2D realizability conditions and 3D coordinates parameterization are carried out at the same time. By solving the $\mathbf{B}$-system equations, the complete set of possible reconstructions are obtained. Our future work will consider a wider range of scenes with additional properties such as concave or convex polyhedra, spatial symmetries, curved models, and so on.

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