The Discrete Surface Kernel: Framework and Applications

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***Abstract*—This paper presents a framework for the computation of a discrete surface kernel, defined as the set of points from which the whole surface is visible. The first part of the paper exposes the theoretical background related to the kernel computation. In this part we also demonstrate the invariance of the surface kernel to rigid geometric transformations. The second part describes two exploitations of the surface kernel concept in the computer vision area, namely spherical surface shape representation and surface registration. This framework has been tested and experimented on real 3D face surfaces[1].**

I. Introduction

The last decade has witnessed a proliferation of the 3D digitizers, scanners and a substantial developments in techniques for modelling, and digitizing 3D object shapes. This has led to an explosion in the number of available 3D models in the world wide web and in domain-specific databases. This progress fuelled the development of 3D shape analysis, description, modelling and classification methods to bridge the gap between the 3D shape digitizing technology and the potential applications. In this context, we propose a methodology for the computation of particular surface attribute, namely the surface kernel. Basically the kernel of a given surface is the space from which the whole surface is visible. This methodology is built upon the concept of 'starshapeness' defined in the continuous space [1]. In this paper, we project the concept of 'starshapeness' on the class of discrete manifolds (triangular-meshed surface) to derive a theocratical framework that allows the determination of the discrete set of points satisfying the visibility condition. This framework covers both closed and open surfaces.

The rest of the paper is organized as follows: Section 2 describes the theatrical foundations of the kernel computation and its implementation. Section 3 handles the invariance of the kernel with respect to rigid geometric transformations. Section 4 exposes two applications of the kernel concept, namely the spherical parametrization of the surface, and the registration of the 3D surfaces. Section 5 terminates with concluding remarks and future work.

[1]The face surface data are from the public BU-3DFEFB database of the Department of Computing Sciences, State University of New York at Binghamton.

II. The discrete surface kernel

Notation
$\mathcal{S}$ a 2-D manifold surface.

$Int(\mathcal{S})$ The interior of $\mathcal{S}$.

$\mathcal{S}_n$: a triangle mesh surface composed of n triangles facets $t_i,\ i = 1..n$.

$\mathcal{P}_i$: Plane associated to the triangle facet t_i, having a normal $\mathbf{n_i}$, passing by the point x_i and defined by $\mathcal{P}_i = \{x \in R^3 \setminus (x - x_i).\mathbf{n_i} = 0\})$.

$\mathcal{H}_i$: negative half space associated to the facet t_i, defined by $\mathcal{H}_i = \{x \in R^3|, \setminus (x - x_i).\mathbf{n_i} \leq 0\})$. $\mathcal{H}_i$ is adopted as the interior of the facet t_i.

Definition 1: Let y a point $\in Int(\mathcal{S})$, Let $x \in \mathcal{S}$, we say that y sees x via $Int(\mathcal{S})$ if the segment $[x\,y] \subset Int(\mathcal{S})$.

Definition 2: Let y a point $\in \mathcal{S}$, the star of y with respect to $\mathcal{S}$, noted $Star(y, \mathcal{S})$, is the set of all points of $\mathcal{S}$ that can be seen from y via $Int(\mathcal{S})$.

Definition 3:A Star-center of S is a point $y \in Int(\mathcal{S})$ such that $Star(y, \mathcal{S}) = S$. We Say that $\mathcal{S}$ is a star-shaped with respect to y.

Definition 4: The kernel of $\mathcal{S}$, noted $Ker(\mathcal{S})$, is the set of all the Star-centers of $\mathcal{S}$.

Definition 5: $\mathcal{S}$ is Star-shaped if $Ker(\mathcal{S}) \neq \emptyset$.

Lemma 1 The kernel of a single triangle facet t_i is the half space $\mathcal{H}_i$

Proof
$\mathcal{H}_i$ is the kernel of the facet t_i if any point in $\mathcal{H}_i$ can see the all the points of the facet t_i. This can be translated to the following statement: $\forall x \in t_i, \forall y \in \mathcal{H}_i,$ the segment $[xy] \in \mathcal{H}_i$. To prove this statement we need to demonstrate that any point of the segment $[xy]$ belongs to $\mathcal{H}_i$. A point p of the segment $[x\ y]$ can be expressed by $p = y + \lambda(x - y)$, $\lambda \in [0\ 1]$. To belong to $\mathcal{H}_i$, p must satisfy

$$(p - x_i).\mathbf{n_i} \leq 0$$

$(p - x_i).\mathbf{n_i} = (y + \lambda(x - y) - x_i).\mathbf{n_i}$,
$x \in t_i$, then it can be expressed by

$$x = x_i + \mathbf{u}$$

,

with $\mathbf{u}.\mathbf{n_i} = 0$.

We get then

$$\begin{aligned}(p - x_i).\mathbf{n_i} &= (y + \lambda(x_i + \mathbf{u} - y) - x_i).\mathbf{n_i} \\ &= ((y - x_i) - \lambda(y - x_i)).\mathbf{n_i} + \lambda\mathbf{u}.\mathbf{n_i} \\ &= (1 - \lambda)(y - x_i).\mathbf{n_i}\end{aligned}$$

$(1 - \lambda) \geq 0$ and $y \in \mathcal{H}_i$, then

$(1 - \lambda)(y - x_i).\mathbf{n_i} \leq 0$.

Thus $p \in \mathcal{H}_i$.

Theorem 1: Let $\mathcal{S}_n$ a triangle mesh surface, then $Ker(\mathcal{S}_n)$ is the intersection of the half spaces $\mathcal{H}_i, i = 1..n$

Proof: By induction

The theorem holds for a single facet surface $\mathcal{S}_1$ as using lemma 1 we have $\mathcal{H}_1$ is the kernel of the facet t_1.

Let assume that the theorem holds for $\mathcal{S}_n$, let prove that it does also for $\mathcal{S}_{n+1}$.

We have $Ker(\mathcal{S}_n) = \bigcap_{i=1}^{n} \mathcal{H}_i$. In another hand, Let $x \in Ker(\mathcal{S}_{n+1})$, then x can see $t_1, t_2, ..., t_{n+1}$, then x can see $\mathcal{S}_n$ and can see t_{n+1}, then $x \in Ker(\mathcal{S}_n)$ and $x \in Ker(t_{n+1})$, then $x \in \bigcap_{i=1}^{n} \mathcal{H}_i$ and $x \in \mathcal{H}_{n+1}$, then $x \in \bigcap_{i=1}^{n} \mathcal{H}_i \bigcap \mathcal{H}_{n+1}$, then $x \in \bigcap_{i=1}^{n+1} \mathcal{H}_i$. Therefore $Ker(\mathcal{S}_{n+1})$ is the intersection of the half spaces $\mathcal{H}_i, i = 1..n+1$

A. Computing the Kernel

The Kernel of the surface $\mathcal{S}_n$ is determined with the following simple algorithm

Generate a 3-D Grid of points $\mathcal{G}$ that contain the surface $\mathcal{S}_n$ (via its convex hull).

$Ker(\mathcal{S}_n) \longleftarrow \mathcal{G}$

For each facet t_i

Find, in $Ker(\mathcal{S}_n)$, the set of points Y_i that lie in the half-space $\mathcal{H}_i$

$Ker(\mathcal{S}_n) \longleftarrow Y_i$

End For

In the worst case, this algorithm has a complexity of $O(N \times M)$, where N and M are the number of facets and the number of points in the initial grid respectively. However since the number is expected to decrease across the surface facets, the average complexity can be reasonably estimated to $O(L)$,where L is a fraction of $N \times M$.

III. Invariance of the kernel with respect to geometric transformations

In this section we will demonstrate that the discrete surface kernel is invariant with respect to rigid geometric transformation.

Lemma 2: Let $\mathcal{T}$ a geometric transformation defined by $\mathcal{T}(X) = R(X) + T$, where R and T are a $3D$ rotation and translation respectively, let t_i a triangle facet, let $t_i^{'} = \mathcal{T}(t_i)$ then $Ker(t_i^{'}) = \mathcal{T}(Ker(t_i))$

Proof

Let y_i a point of $\mathcal{H}_i$ the kernel of t_i, then we have

$$\begin{aligned}y_i^{'} &= \mathcal{T}(y_i) \\ &= R(y_i) + T\end{aligned}$$

Let $t_i{}' = \mathcal{T}(t_i)$, then we have

$$\begin{aligned}x_i{}' &= \mathcal{T}(x_i) = R(x_i) + T \\ n_i^{'} &= R(n_i)\end{aligned}$$

We need to demonstrate that $y_i^{'} \in Ker(t_i^{'})$, i.e.

$$(x_i^{'} - y_i^{'}).n_i^{'} \leq 0$$

$$\begin{aligned}(x_i^{'} - y_i^{'}).n_i^{'} &= (R(x_i) + T - R(y_i) - T).R(n_i) \\ &= R(x_i - y_i).R(n_i)\end{aligned}$$

The invariance of the dot product with respect to rotation, yields to

$$R(x_i - y_i).R(n_i) = (x_i - y_i).n_i$$

In another hand, $y_i \in \mathcal{H}_i$ implies

$$(x_i - y_i).n_i \leq 0$$

Then $R(x_i - y_i).R(n_i) \leq 0$, and thus

$$(x_i^{'} - y_i^{'}).n_i^{'} \leq 0$$

Therefore $y_i^{'} \in ker(t_i^{'})$

Theorem 2: Let $\mathcal{S}_n$ a triangle mesh surface, Let $\mathcal{T}$ a geometric transformation defined by $\mathcal{T}(X) = R(X) + T$, where R and T are a $3D$ rotation and translation respectively, and let $\mathcal{S}_n^{'} = \mathcal{T}(\mathcal{S}_n)$, then we have $Ker(\mathcal{S}_n^{'}) = \mathcal{T}(Ker(\mathcal{S}_n))$

*Proof:*By Induction

The theorem holds for a single facet surface $\mathcal{S}_1$. This can be proven using Lemma 2.

Assuming that the theorem holds for $\mathcal{S}_n$, let prove that it also does for $\mathcal{S}_{n+1}$.

According to Theorem 1, we have

$$Ker(\mathcal{S}_{n+1}) = \bigcap_{i=1}^{n+1} \mathcal{H}_i$$

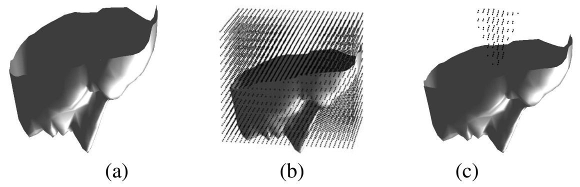

Fig. 1. (a) The 3D face surface, (b) the 3D grid of points, (c) the discrete kernel of the face surface.

then we can write

$$\begin{aligned}\mathcal{T}(Ker(\mathcal{S}_{n+1})) &= \mathcal{T}(\bigcap_{i=1}^{n+1}\mathcal{H}_i) \\ &= \mathcal{T}(\bigcap_{i=1}^{n}\mathcal{H}_i\bigcap\mathcal{H}_{n+1}) \\ &= \mathcal{T}(\bigcap_{i=1}^{n}\mathcal{H}_i)\bigcap\mathcal{T}(\mathcal{H}_{n+1}) \\ &= \mathcal{T}(Ker(S_n))\bigcap\mathcal{T}(Ker(t_{n+1}))\end{aligned}$$

The theorem holds for S_n then we have

$$\begin{aligned}\mathcal{T}(Ker(S_n)) &= Ker(\mathcal{T}(S_n)) \\ &= Ker(\mathcal{S}_n^{'})\end{aligned}$$

then using Theorem 1, we can write

$$\mathcal{T}(Ker(S_n)) = \bigcap_{i=1}^{n}\mathcal{H}_i^{'}$$

Using lemma 2, we have

$$\begin{aligned}\mathcal{T}(Ker(t_{n+1})) &= Ker(\mathcal{T}(t_{n+1})) \\ &= Ker(t_{n+1}^{'}) \\ &= \mathcal{H}_{n+1}^{'}\end{aligned}$$

Therefore we can write

$$\begin{aligned}\mathcal{T}(Ker(\mathcal{S}_{n+1})) &= \bigcap_{i=1}^{n}\mathcal{H}_i^{'}\bigcap\mathcal{H}_{n+1}^{'} \\ &= \bigcap_{i=1}^{n+1}\mathcal{H}_i^{'} \\ &= Ker(\mathcal{S}_{n+1}^{'}) \\ &= Ker(\mathcal{T}(\mathcal{S}_{n+1}))\end{aligned}$$

IV. Applications

In this section we shade some light on two particular applications of the surface kernel, namely spherical parametrization and surface registration

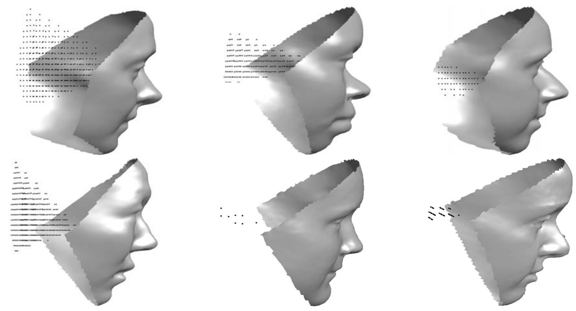

Fig. 2. Samples of 3D face surface kernels.

A. Spherical parametrization

Spherical representation of surface shapes is of great usefulness in many applications related to 3D shape modelling and analysis retrieval. For instance when meshing free-from shapes that are topologically equivalent to a sphere, it is best to parameterize the mesh over a domain which is topologically equivalent to it [2]. A spherical parametrization would permit the use of harmonic functions on the sphere, such as spherical harmonics [3], double Fourier series, [4], spherical diffusion, [5] and spherical wavelets [6], for the purpose of 3D shape description, analysis, and retrieval.

The spherical parametrization projects the surface on the unit sphere around its center. The surface can then be described using standard spherical coordinates $r = r(\phi, \theta)$, where r is the distance form the origin to the point on the surface, r here is a radial function with two parameters ϕ and θ representing the latitude and the longitude respectively.

However, to cover the whole surface, the sphere's center must satisfy the visibility requirement, i.e. the whole surface must be visible from the center, or else the radial function will exhibit gaps and discontinuities. The point derived from the center of mass of the surface, and which usually adopted as the sphere's center do not necessarily satisfy that requirement, particularly when the surface shape is not convex.

Determining the surface kernel has three benefits 1) It indicates whether or not the a surface can be spherically parameterized, by checking whether or not the kernel is not empty. 2) It allows choosing a sphere's center which is guaranteed to satisfy the visibility requirement. 3) it allows to determine the sphere's center that ensure a maximum extend of the radial function $r = r(\phi, \theta)$ over the unit sphere.

Figure 3 shows radial functions of three surface face samples and their corresponding surface reconstructions. Figure 4 depicts radial functions and surface reconstructions, of a same face, corresponding to three different sphere centers. The three centers belong to the surface kernel. The first and the third are respectively the closest and the distant point to the face surface along the face orientation, the third is the midway point of the two extremes. It is interesting to note that the closer is the sphere's center to the face surface the larger is the radial function domain, and the better is the surface reconstruction.

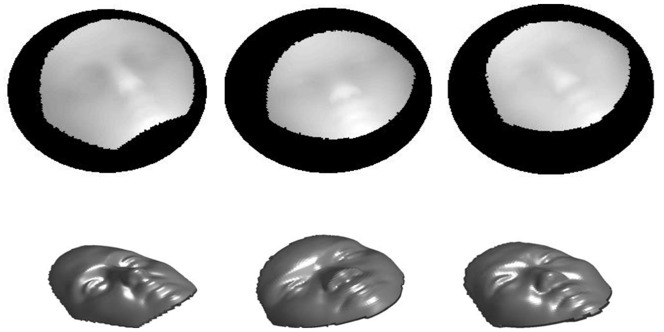

Fig. 3. First row: radial function on the sphere of some face surface samples. Second row: related face surface reconstruction.

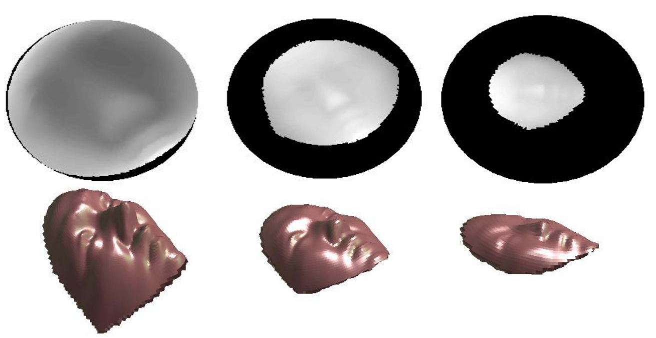

Fig. 4. First row: radial function on the sphere of the same face, corresponding to three different spheres centers. Second row: the corresponding face surface reconstruction.

B. Surface registration

3 Surface registration used in a variety of applications that span building terrain maps in the context of autonomous vehicle [7], [8], recognizing and retrieving objects from 3D object model databases[9], and reconciling various imaging modalities in biomedical imaging [10].

The registration of two surfaces consists in estimating the mapping between coordinate systems associated with each surface. I.e. estimating the geometric transformation (Rotation and translation) that maps the two surfaces. The registration has been one of the most intriguing problems in computer vision. In effect, it raises hard issues, particularly the elaboration of the surface representation and the matching (corresponding) between features derived from the surface representation [11].

The invariance of the kernel with respect to rigid transformation, allows to use surfaces' kernels in the registration rather than the surfaces themselves. There are two reasons that would favor this alternative 1) The simplicity of the kernel shape. In effect the kernel has a convex shape that exhibits attractive properties, e.g. well defined orientation, smoothness. 2) the area of kernel is quite small relatively to the surface's area, this would reduce considerably the space of correspondences. These interesting aspects suggest that employing the surface in the registration would alleviate the complexity of the aforementioned issues.

We applied kernel based registration on a set of 3D face surfaces (samples are depicted in Figure 5). In this experiment we took group of face surface pairs, each pair contains two face surface samples at different positions and orientations (Figure 5.(a,b)). The kernels of the face surfaces in the pair are determined (Figure 5.c), then used to estimate the geometric transformation that maps them (Figure 5.d). The geometric transformation maps also their corresponding face surfaces as shown in (Figure 5.e). We mention here, that we have used a simple and naive registration technique, as our objective here is to demonstrate the feasibility of the approach rather than obtaining an accurate registration.

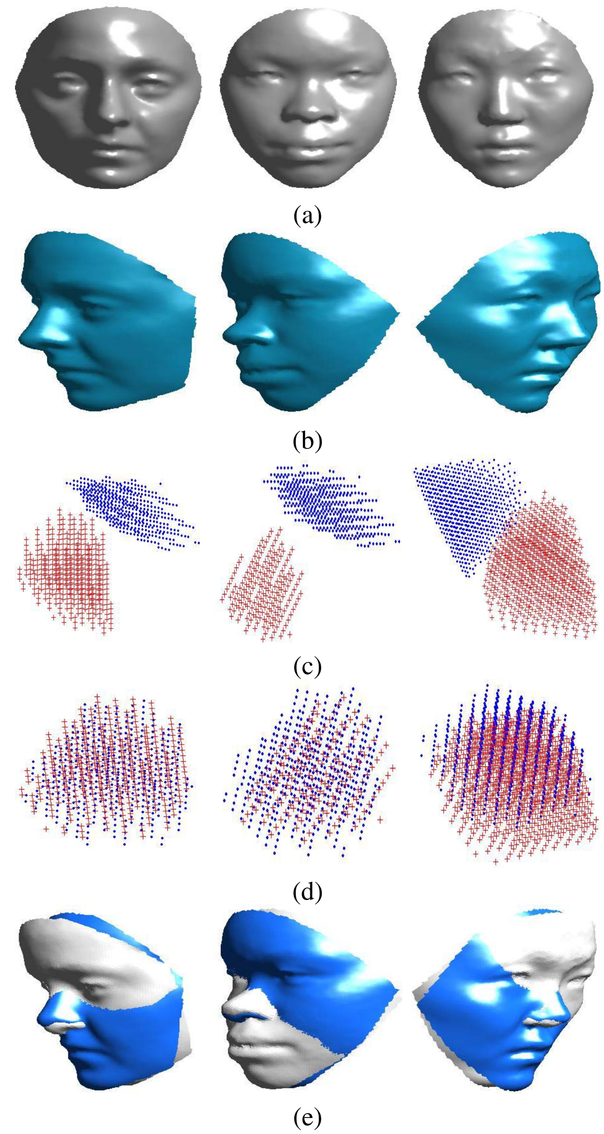

Fig. 5. Three examples of face surfaces' registration using the surface kernel. (a) samples of face surfaces. (b) Transformed Surfaces. (c) Kernel's locations before registration. (d) registered kernels. (e) registered face surfaces.

V. CONCLUSION

In this paper, we elaborated a theoretical framework for determining a surface kernel for the class of triangular meshed

2D manifolds. We established a simple algorithm for estimating the kernel, and we proved the invariance of the kernel with respect to rigid geometric transformations. We illustrated the usefulness of the surface kernel on two applications: spherical parametrization and registration of 3D surfaces. The surface examples treated in this paper are open surfaces, yet the framework holds also close surfaces. The surface kernel can also be exploited in classification tasks, for example categorizing surfaces into starshaped and non-starshaped ones, depending whether or not the corresponding kernels are empty or not.

This framework can also be used to establish a metric for measuring the starshapeness of a surface, for example by using the size of the kernel's volume. We plan to extensively explore this aspect in the future. We plan also to investigate how we can automatically determine the optimal point from the discrete kernel to be used as the sphere'center, guided by the observations of Figure 4.

ACKNOWLEDGMENT

The author would like to thank the Department of Computing Sciences in State University of New York at Binghamton for providing the 3D faces database.

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