Extension of the Second Paden-Kahan Sub-problem and its’ Application in the Inverse Kinematics of a Manipulator

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Abstract—Three basic sub-problems of screw theory are acceptable for some particular configuration manipulators’ inverse kinematics, which can not solve the inverse kinematics of all configuration manipulators. A new sub-problem is extended based on sub-problem 2, that is, rotation about two disjoint axes, and the inverse kinematics thereof is solved in this paper. Based on the extended sub-problem, a manipulator, the inverse kinematics of which can not be solved by the three sub-problems without the participation of the new sub-problem, is constructed. The inverse kinematics of the manipulator is solved with the help of the extended sub-problem, therefore a close-form solution expressed by motion screw is gained.

Keywords—motion screw, Paden-Kahan sub-problem, inverse kinematics

I. INTRODUCTION

In recent years, screw theory is widely used in kinematics, dynamics and control of the robot and gets a fine effect. For a general configuration manipulator, Denavit-Hartenberg parameter method is employed to make mathematical model for kinematics and dynamics of the robot, and then a number of value iterative methods [1,2] or neural network methods [3,4,5] or genetic algorithm [6] are taken to solve the inverse kinematics or dynamics of the robot, which not only cost much calculating time for its’ high computational complexity, but also the integrality and convergency of the solution can not be warranted. On the other hand, the model based on D-H parameter method requires that each joint and the end-effector has to be connected with a local coordinate in order to represent four parameters of each joint, and different configurations correspond to different solution methods, thus it is not a general method [9]. There are two main advantages to using screw theory. The first is that they allow a global description of rigid body motion which does not suffer from singularities due to the use of local coordinates. Such singularities are inevitable when one chooses to represent rotation via Euler angles, for example. The second advantage is that screw theory provides a very geometric description of rigid motion which greatly simplifies the analysis of mechanisms [7]. The method to solve the inverse kinematics problem based on screw theory generally reduces the full inverse kinematics problem into appropriate sub-problems whose solutions are known. For any configuration manipulator, the inverse kinematics of the manipulator is rather simple if the sub-problems employed are not out of the existing ones. Some scholars such as Chen and Gao [8], Zhao Jie [9], etc, have solved the inverse kinematics for some manipulators having a special configuration basing on the following three sub-problems and exponential equation. For a manipulator only including rotary joints, the typical sub-problems includes

Sub-problem 1: Rotation about a single axis;
Sub-problem 2: Rotation about two subsequent axes;
Sub-problem 3: Rotation to a given distance.

The above-mentioned three sub-problems are basic ones, which have limitations in application, that is, they can not solve the inverse kinematics of some manipulator in the case all the axes of the adjacent joints are not crossed. In order to solve this problem, a new sub-problem based on the sub-problem 2 is developed, that is, rotation about two disjoint subsequent axes. The solution for the new sub-problem is solved in the paper, and then a special manipulator whose solution has to employ the new sub-problem is configured, finally the inverse kinematics is made to the manipulator based on the screw theory.

II. EXTENSION OF SUBPROBLEM 2: ROTATION ABOUT TWO DISJOINT SUBSEQUENT AXES

Where $\xi_1$ and $\xi_2$ are two zero-pitch, unit magnitude twists with non-intersecting axes and $p, q \in \mathbb{R}^3$ two points, find $\theta_1$ and $\theta_2$ such that

$$e^{\xi_1 \theta_1} e^{\xi_2 \theta_2} p = q$$

The problem is similar as sub-problem 2 described in reference [1] except that the two axes here are non-intersecting.

In Fig.1, $c$ is a point such that

$$e^{\tilde{\xi} \theta} p = c = e^{-\tilde{\xi} \theta} q$$

(1)
Fig. 1 an extended sub-problem of the 2th Paden-Kahan sub-problem
and as shown in Fig. 1, \( d \) is a common normal between axes \( \xi_1 \) and \( \xi_2 \), and \( d = \| r_2 - r_1 \| (\omega_1 \times \omega_2) \), wherein \( r_1 \) and \( r_2 \) is the point of intersection between the common normal and axes \( \xi_1 \) and \( \xi_2 \) respectively, hence we can get
\[
e^{\hat{\xi}_1 \theta_1} (p - r_2) = c - r_2 = z_2 = e^{-\hat{\xi}_1 \theta_1} q - r_2
\]
(2)

And
\[
c - r_1 = z_1 = e^{\hat{\xi}_1 \theta_1} q - r_1, \quad d = r_2 - r_1 = z_1 - z_2
\]
so that equation (2) becomes
\[
e^{\hat{\xi}_1 \theta_1} (p - r_2) = e^{-\hat{\xi}_1 \theta_1} q - d - r_1 = e^{-\hat{\xi}_1 \theta_1} (q - r_1) - d
\]
(3)

Let \( u = p - r_2, \quad v = q - r_1, \quad z_1 = c - r_1, \quad z_2 = c - r_2 \).
Substituting these expressions into equation (3) give
\[
e^{\hat{\xi}_1 \theta_1} u = z_2 = z_1 - d = e^{-\hat{\xi}_1 \theta_1} v - d
\]
(4)

From equation (4) we can get
\[
a^\alpha v = a^\alpha z_1 \quad \text{and} \quad a^\alpha u = a^\alpha z_2 = a^\alpha (z_1 - d), \quad \text{and}
\]
\[
\| u \| = \| z_1 \| = \| z_1 - d \|, \quad \| v \| = \| z_2 \|
\]
(5)

Since \( \omega_1, \omega_2 \) and \( \omega_1 \times \omega_2 \) are linearly independent, we can write
\[
z_2 = a \omega_1 + \beta \omega_2 + \gamma (\omega_1 \times \omega_2)
\]
(6)
\[
z_1 = a \omega_1 + \beta \omega_2 + \gamma (\omega_1 \times \omega_2) + d = a \omega_1 + \beta \omega_2 + (\gamma + \| d \|) (\omega_1 \times \omega_2)
\]
(7)

Substituting equation (6) and (7) into equation (5) gives
\[
a^\alpha v = a + \beta \omega_1 \omega_3 + \gamma (\omega_1 \times \omega_2)
\]
(8)

Substituting equation (6) and (7) into equation (5) gives
\[
a^\alpha v = a + \beta \omega_1 \omega_3 + \gamma (\omega_1 \times \omega_2)
\]
(9)

\[
\alpha^\beta u = \beta + \alpha \omega_1 \omega_3
\]
(10)

From equation (9) and (10) we can get
\[
\alpha = \frac{(\alpha^\beta \omega_1) \omega_3 u - \alpha^\beta v}{(\alpha^\beta \omega_1)^2 - 1}
\]
(11)
\[
\beta = \frac{(\alpha^\beta \omega_1) \omega_3 v - \alpha^\beta u}{(\alpha^\beta \omega_1)^2 - 1}
\]
(12)

Substituting equation (11) and (12) into equation (8) we can get
\[
\gamma = \frac{\| u \|^2 - \alpha^2 - \beta^2 - 2 \alpha \beta \omega_1 \omega_3}{\| \omega_1 \times \omega_2 \|^2}
\]
(13)

Substituting \( \alpha, \beta \) and \( \gamma \) into equation (6) and (7) we can find \( z_1, z_2 \), and hence \( c \) in the case that a solution exists. Obviously the equation (1) has a same form as that of the sub-problem 1, so we can solve the equation (1) to find \( \theta_1 \) and \( \theta_2 \)
\[
\begin{align*}
\theta_1 &= a \tan 2(\omega_1 (u' \times z_2'), u' z_2') \\
\theta_2 &= b \tan 2(\omega_1 (z_1' \times v'), z_1' v')
\end{align*}
\]
(14)

Where \( u' \) and \( z_2' \) are the projections of \( u \) and \( z_2 \) on \( \omega_2 \), and \( z_1', v' \) are the projections of \( z_1, v \) on \( \omega_1 \). If there are multiple solutions for \( c \), each of these solutions gives a value for \( \theta_1 \) and \( \theta_2 \). Two solutions exist in the case where the circles in Figure 1 intersect at two points, one solution when the circles are tangential, and none when the circles fail to intersect.

III. EXPERIMENTS

To verify the correctness of the solution the extended sub-problem, we configure a 5-DOF manipulator for which the extended sub-problem must be employed, as shown in Figure 2.
In figure 2, The axes of joint 1 and joint 2 are perpendicular to each other and not intersecting, and the axes of joint 4, 5, 6 intersect at a point \( q_w \). Where, \( l_1 = 355 \), \( l_2 = 245 \), \( l_3 = 90 \), \( l_4 = 300 \), \( l_5 = 180 \).

If \( \theta = 0 \), the configuration of the fixed coordinate relative to the end-effector coordinator is given by

\[
g_w(0) = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 1 & l_1 + l_4 + l_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

and we can choose axis points

\[
q_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} l_2 \\ l_2 \\ l_2 + l_4 \end{bmatrix}, \quad q_w = \begin{bmatrix} l_3 \\ l_5 \\ l_5 \\ l_5 \end{bmatrix}
\]

Apply both sides of equation (15) to a point \( p_w \) which is the common point of intersection for the wrist axes 4, 5 and 6, this yields

\[
e^{\gamma_1 \theta_1} e^{\gamma_2 \theta_2} e^{\gamma_3 \theta_3} e^{\gamma_4 \theta_4} e^{\gamma_5 \theta_5} q_w = q_1 q_w \Rightarrow e^{\gamma_1 \theta_1} e^{\gamma_2 \theta_2} q_w = q_1 q_w
\]

(16)

Obviously the equation (16) is in the form of the extended subproblem. Hence applying the extended subproblem, we solve for \( \theta_1 \) and \( \theta_2 \).

(2) Solve for \( \theta_3, \theta_4 \) and \( \theta_5 \)

Since \( \theta_1 \) and \( \theta_2 \) are known, so

\[
e^{\gamma_4 \theta_4} e^{\gamma_5 \theta_5} e^{\gamma_4 \theta_4} e^{\gamma_5 \theta_5} g_1 = e^{-\gamma_4 \theta_4} e^{-\gamma_5 \theta_5} g_1
\]

Let \( e^{-\gamma_4 \theta_4} e^{-\gamma_5 \theta_5} g_1 = g_2 \), we get

\[
e^{\gamma_4 \theta_4} e^{\gamma_5 \theta_5} e^{\gamma_4 \theta_4} e^{\gamma_5 \theta_5} g_2 = g_2
\]

(17)

in this case, we can apply subproblem 2 to solve for \( \theta_3 \) and \( \theta_4 \), and get \( \theta_5 \) finally with subproblem 1, the concrete steps can refer to reference [8].

To verify the correctness of the algorithm, we give arbitrary groups of joint angles, and get the positions and orientations of the end-effector by applying the forward kinematics. Taking the computed positions and orientations of the end-effector as a known condition, we solve for the joint angles of the manipulator by applying the inverse kinematics. Comparing the given joint angles with the computed joint angle, we found that they are identical, with which the correctness of the algorithm is verified.

**IV. CONCLUSION**

Based on sub-problem 2 of Paden-Kahan, a new sub-problem is developed and its solving method is provided herein, and its solution is given, which can be applied directly in the inverse kinematics of a manipulator, providing a new approach for the inverse kinematics of a general configuration manipulator. The inverse kinematics process based on the screw theory is more explicit and direct than that of D-H parameters method.

**REFERENCES**


