An Analysis of Stability of Systems with Impulse Effects: Application to Biped Robots

Li mei Liu[1,2], Yan tao Tian[1] , Pei jie Zhang[1] , Zhen ze Liu[1]

1. College of Communication Engineering, Ji lin University, Chang chun, 130025, China
2. Department of Mathematics of Chang chun Taxation College, Chang chun, 130117, China
daliu137@sina.com.cn, tianyt@mail.jlu.edu.cn

***Abstract*—The approximate Jacobian matrix of the Poincaré return map at the fixed point is presented for the nonlinear system with impulse effects. And the sufficient condition to the existence of this approximate Jacobian matrix is given with the disturbance theory and linearization method. Since this approximate expression depends only on the configuration of the system with impulse effects, then the uniqueness of this approximate expression can be guaranteed and this approximate expression can be obtained precisely. In addition, this approximate Jacobian matrix can be used as a tool to study the asymptotical stability of the system with impulse effects. Since the biped robot gaits can be described by the nonlinear system with impulse effects, then the stability of the biped robot walking cycle can be studied with this tool. In order to study the stability, this approximate Jacobian matrix is applied to the compass-like passive biped robot gaits. It is shown that the approximate Jacobian matrix proposed in this paper is as useful as the ones proposed in the numerical methods. In the end this result is confirmed by simulations.**

***Keywords*—stability, Poincaré map, Jacobian matrix, limit cycle, nonlinear system with impulse effects, biped robot**

Ⅰ. INTRODUCTION

The nonlinear system with impulse effects is a hybrid dynamic system consisting of the ordinary differential equations and the algebraic mapping dynamic systems. If the nonlinear system with impulse effects is stable, then the system has a stable limit cycle. The stability of the nonlinear system with impulse effects has been studied using the theory of the Lyapunov function method. But the most primary method is using the theory of the Poincaré return map. Since an asymptotically stable fixed point of Poincaré return map corresponds to an asymptotically stable periodic trajectory of the system. It is well known how to use the numerical methods to compute the Poincaré return map. McGeer, Goswami and Garcia have studied the stability of passive gait via the methods of the Poincaré return map [1, 2, 3, 4, 5, 6], but they all use the numerical methods. Since Jacobian matrix of the Poincaré return map at the fixed point has uncertainty through the numerical method, so the result of the stability on biped robot gait is rough. Grizzle has proposed the restricted Poincaré section theory to study the system with the impulse effects, and applied it to the biped locomotion[7,8,9]. But themethod proposed by Grizzle is not convenient to the hyperspace.

In this paper, an approximate expression of the Jacobian matrix of the Poincaré return map at the fixed point is proposed for the system with impulse effects. And this expression is obtained on the basis of the disturbance theory and the linearization method. Furthermore, the sufficient condition to the existence of this expression is proposed. This approximate expression bases only on the configuration of the system; however the Jacobian matrix in the numerical method bases on the data that are chosen. So the result of the approximate expression proposed in this paper is unique, but the results in numerical methods are various. Since the stability of the system with impulse effects can be determined on the basis of Jacobian matrix of the Poincaré return map, then using the approximate expression proposed in this paper can achieve the stability of the nonlinear system with impulse effects exactly. Specially, either planar biped robot gaits or non-planar biped robot gaits can be all expressed by the system with impulse effects, so the stability of biped robot gaits can be achieved by the approximate expression proposed in this paper. In this paper, the stability of a compass-like passive biped robot gaits is argued with the approximate Jacobian matrix both proposed in this paper and in numerical method. It is demonstrated that the approximate Jacobian matrix proposed in this paper has uniqueness and accuracy of judgment on the stability of the system with impulse effects.

Ⅱ. SYSTEM WITH IMPULSE EFFECTS

Consider an ordinary differential equation

$$\dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right) \cdot u(t) \qquad (1)$$

where $x(t) \in \mathrm{X}$, and X is a connected open subset of $\mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and f and the columns of g are continuous vector fields on X. Let $\mathbb{S} := \{x(t) \in \mathrm{X} \,|\, r\left(x(t)\right) = 0\}$, where $r : \mathrm{X} \to \mathbb{R}$, and r is called the impulse effect detection function. Let $H : \mathbb{S} \to \mathrm{X}$ is called the impulse effect transition function. A system with impulse effects is presented as follows:

$$\sum : \begin{cases} \dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right) \cdot u(t) & x^-(t) \notin \mathbb{S} \\ x^+(t) = H\left(x^-(t)\right) & x^-(t) \in \mathbb{S} \end{cases} \qquad (2)$$

The corresponding author: tianyt@mail.jlu.edu.cn

This work was supported by the National High Technology Research and Development Program of China (863Program) (Grant No: 2006AA04Z251)

and $x^-(t) := \lim_{\tau \to t^-} x(\tau)$ and $x^+(t) := \lim_{\tau \to t^+} x(\tau)$ denote respectively the left and the right limits of the trajectory $x(\tau)$ at $\tau = t$. In simple words, a trajectory of the above model (2) is specified by the differential equation (1) until its state "impacts" the hyper surface $\mathbb{S}$. At this point, the impulse mode H compresses the impact event into an instantaneous moment of time, resulting in a discontinuity in the state trajectory. The ultimate result of the impact model is a new initial condition, from which the differential equation model evolves until the next impact with $\mathbb{S}$. In order to avoid the state having to take on two values at the "impact time", the impact event is described in terms of the values of the state "just prior to impact" at time "t^-" and "just after impact" at time "t^+". These values are represented by the left limit x^- and the right limit x^+ of equation (1) respectively. Using the theory of the Poincaré return map is the primary method to study the stability of the system with impulse effects. Let $x_{k+1} = P(x_k)$ be the Poincaré return map of the model (2). The instant section after impact is chosen as the Poincaré section. Let $\varphi(t, 0, x_k)$ be a solution of equation (1), so $P(x_k) := H \circ \varphi(t, 0, x_k)$. The relationship between modules of the eigenvalues λ of the Poincaré return map P and the stability of the model (2) is as follows:

$$\begin{cases} |\lambda| < 1 & \textit{asymptotically stable} \\ |\lambda| \le 1 & \textit{stable} \\ |\lambda| > 1 & \textit{unstable} \end{cases}$$

Ⅲ. MAIN RESULT

The Poincaré return map is denoted as $x_{k+1} = P(x_k)$, where x_k represents the state after the $k-th$ impulse effect. The instant after the impulse effect just happened is chosen as the Poincaré section. Since x^* is a fixed point of the mapping, that is $P(x^*) = x^*$, the linearization of the mapping P at the fixed point is

$$P(x^* + \Delta x) \approx P(x^*) + \frac{\partial P(x^*)}{\partial x} \cdot \Delta x = x^* + \frac{\partial P(x^*)}{\partial x} \cdot \Delta x .$$

Then $P(x^* + \Delta x) - x^* \approx \frac{\partial P(x^*)}{\partial x} \cdot \Delta x$, where $J := \frac{\partial P(x^*)}{\partial x}$ is the Jacobian matrix of P at the fixed point x^*. In particular,

$$x_{i+1} - x^* = P(x_i) - P(x^*) \approx J \cdot (x_i - x^*),$$

so $\Delta x_{i+1} \approx J \cdot \Delta x_i$ [6,10,11].

In this section, the motion between impulse effects is expressed as an ordinary differential equation $\dot{x}(t) = f(x(t))$. Let $x^*(t)$ be a limit cycle trajectory with the initial point x^* at $t = 0$ of $\dot{x}(t) = f(x(t))$ and let τ^* denote the limit cycle time between the impulse effects. $x^+ = H(x^-)$ denotes the impulse effect transition function, such that $x^* = H[x^*(\tau^*)]$. The impulse effect detection function is defined as $r(x^-)$. Therefore an impulse effect occurs when $r(x^-) = 0$, that is, $r[x^*(\tau^*)] = 0$. Under the above hypotheses, the system with impulse effects in this section is denoted as

$$\Sigma: \begin{cases} \dot{x}(t) = f(x(t)) & x^-(t) \notin \mathbb{S} \\ x^+(t) = H(x^-(t)) & x^-(t) \in \mathbb{S} \end{cases}, \tag{3}$$

where $\mathbb{S} := \{ x(t) \in \mathrm{X} \subseteq \mathbb{R}^n \mid r(x(t)) = 0 \}$.

Lemma 1: Consider the system with impulse effects (3) satisfying the following hypotheses:

(1) $f(x(t))$ is continuous and differentiable on X;

(2) a solution of $\dot{x}(t) = f(x(t))$ from a given initial condition is unique and depends continuously on the initial condition;

(3) the model (3) exits a limit cycle trajectory $x^*(t)$ with the initial point x^* at $t = 0$;

(4) a perturbation $\hat{x}(0)$ is given at the initial point x^*.

Then the perturbation trajectory $\hat{x}(t)$ of the limit cycle trajectory $x^*(t)$ can be obtained as

$$\hat{x}(t) \approx e^{Df(x^*(t)) \cdot t} \cdot \hat{x}(0), \text{ where } Df(x^*(t)) := \frac{\partial f[x^*(t)]}{\partial x}.$$

Proof: If a perturbation $\hat{x}(0)$ is given at the initial point x^*, the limit cycle trajectory has changed immediately.
Since $\hat{x}(t)$ denotes the perturbation trajectory of the limit cycle trajectory $x^*(t)$, so $x^*(t) + \hat{x}(t)$ denotes the limit cycle of model (3) with the initial point $x^* + \hat{x}(0)$ at $t = 0$.

For ε is sufficiently small, then

$$(\dot{x}^*(t) + \varepsilon \dot{\hat{x}}(t)) = f(x^*(t) + \varepsilon \hat{x}(t)).$$

And $\dot{x}^*(t) + \varepsilon \dot{\hat{x}}(t) \approx f(x^*(t)) + \frac{\partial f[x^*(t)]}{\partial x} \cdot \varepsilon \hat{x}(t)$.

Since $\dot{x}^*(t) = f(x^*(t))$, so $\dot{\hat{x}}(t) \approx \frac{\partial f[x^*(t)]}{\partial x} \cdot \hat{x}(t)$,

where $Df(x^*(t)) := \frac{\partial f[x^*(t)]}{\partial x}$, so $\hat{x}(t) \approx e^{Df(x^*(t)) \cdot t} \cdot \hat{x}(0)$. □

Lemma 2: Supppose that $r(x^-)$ is a continuous and differentiable function on X, then the second derivative of $r(x^-)$ is also continuous.

So under the hypotheses of lemma 1, the perturbation $\hat{\tau}$ of the limit cycle time between impulse effects τ^* almost is

$$\hat{\tau} \approx -\frac{Dr\left[x^*(\tau^*)\right]\cdot\hat{x}(\tau^*)}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}, \text{ where } Dr\left[x^*(\tau^*)\right] := \frac{\partial r\left[x^*(\tau^*)\right]}{\partial x}.$$

Proof: From Lemma 1, it is necessary to change the time between impulse effects. The perturbation of the limit cycle time between impulse effects τ^* is expressed by $\hat{\tau}$.

For ε is sufficiently small,

$$0 = r\left[x^*(\tau^* + \varepsilon\hat{\tau}) + \varepsilon\hat{x}(\tau^* + \varepsilon\hat{\tau})\right]$$

$$\approx r\left[x^*(\tau^* + \varepsilon\hat{\tau})\right] + \frac{\partial r\left[x^*(\tau^* + \varepsilon\hat{\tau})\right]}{\partial x}$$

$$\cdot\varepsilon\hat{x}(\tau^* + \varepsilon\hat{\tau}) + o(\varepsilon^2)$$

$$\approx r\left[x^*(\tau^*)\right] + \frac{\partial r\left[x^*(\tau^*)\right]}{\partial x}\cdot\dot{x}^*(\tau^*)\cdot\varepsilon\hat{\tau}$$

$$+\left[\frac{\partial r\left[x^*(\tau^*)\right]}{\partial x} + \frac{\partial^2 r\left[x^*(\tau^*)\right]}{\partial x^2}\cdot\dot{x}^*(\tau^*)\cdot\varepsilon\hat{\tau}\right]$$

$$\cdot\varepsilon\left[\hat{x}(\tau^*) + \frac{\partial\hat{x}(\tau^*)}{\partial t}\cdot\varepsilon\hat{\tau}\right] + o\left(\varepsilon^2\right)$$

$$\approx r\left[x^*(\tau^*)\right] + \frac{\partial r\left[x^*(\tau^*)\right]}{\partial x}\cdot\dot{x}^*(\tau^*)\cdot\varepsilon\hat{\tau}$$

$$+\frac{\partial r\left[x^*(\tau^*)\right]}{\partial x}\cdot\hat{x}(\tau^*)\cdot\varepsilon + o\left(\varepsilon^2\right)$$

$$\approx \frac{\partial r\left[x^*(\tau^*)\right]}{\partial x}\cdot\dot{x}^*(\tau^*)\cdot\varepsilon\hat{\tau} + \frac{\partial r\left[x^*(\tau^*)\right]}{\partial x}\cdot\hat{x}(\tau^*)\cdot\varepsilon$$

So $\hat{\tau} \approx -\dfrac{Dr\left[x^*(\tau^*)\right]\cdot\hat{x}(\tau^*)}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}$,

where $Dr\left[x^*(\tau^*)\right] := \dfrac{\partial r\left[x^*(\tau^*)\right]}{\partial x}$. □

Lemma 3: It is assumed that $H\left(x^-(t)\right)$ is continuous and differentiable on X. Moreover under the hypotheses of lemma 1 and lemma 2, the perturbation $\hat{x}^+$ of the result of the impulse effect is expressed as

$$\hat{x}^+ \approx DH\left[x^*(\tau^*)\right]\cdot\left[I - \frac{\dot{x}^*(\tau^*)\cdot Dr\left[x^*(\tau^*)\right]}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}\right]\cdot\hat{x}\left(\tau^*\right),$$

where $DH\left[x^*(\tau^*)\right] := \dfrac{\partial H\left[x^*(\tau^*)\right]}{\partial x}$; $Dr\left[x^*(\tau^*)\right] := \dfrac{\partial r\left[x^*(\tau^*)\right]}{\partial x}$.

Proof:Since a perturbation is added to the initial point x^* of the limit cycle trajectory $x^*(t)$, in terms of lemma1 and lemma 2, the result of the impulse effect is changed too. Let $\hat{x}^+$ be the perturbation of the result of the impulse effect. Then $x^* + \varepsilon\hat{x}^+ = H\left[x^*(\tau^* + \varepsilon\hat{\tau}) + \varepsilon\hat{x}(\tau^* + \varepsilon\hat{\tau})\right]$.

For ε is sufficiently small,

$$x^* + \varepsilon\hat{x}^+ = H\left[x^*(\tau^* + \varepsilon\hat{\tau}) + \varepsilon\hat{x}(\tau^* + \varepsilon\hat{\tau})\right]$$

$$\approx H\left[x^*(\tau^* + \varepsilon\hat{\tau})\right] + \frac{\partial H\left[x^*(\tau^* + \varepsilon\hat{\tau})\right]}{\partial x}$$

$$\cdot\varepsilon\hat{x}(\tau^* + \varepsilon\hat{\tau}) + o(\varepsilon^2)$$

$$\approx H\left[x^*(\tau^*)\right] + \frac{\partial H\left[x^*(\tau^*)\right]}{\partial x}\cdot\dot{x}^*(\tau^*)\cdot\varepsilon\hat{\tau}$$

$$+\frac{\partial H\left[x^*(\tau^* + \varepsilon\hat{\tau})\right]}{\partial x}\cdot\varepsilon\left[\hat{x}(\tau^*) + \dot{\hat{x}}(\tau^*)\cdot\varepsilon\hat{\tau}\right]$$

$$+o(\varepsilon^2)$$

$$\approx x^* + \frac{\partial H\left[x^*(\tau^*)\right]}{\partial x}\cdot\dot{x}^*(\tau^*)\cdot\varepsilon\hat{\tau}$$

$$+\frac{\partial H\left[x^*(\tau^*)\right]}{\partial x}\cdot\varepsilon\hat{x}(\tau^*) + o(\varepsilon^2)$$

So $\hat{x}^+ \approx \dfrac{\partial H\left[x^*(\tau^*)\right]}{\partial x}\cdot\left[I - \dfrac{\dot{x}^*(\tau^*)\cdot Dr\left[x^*(\tau^*)\right]}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}\right]\cdot\hat{x}\left(\tau^*\right)$,

where I denotes the identity matrix and $Dr\left[x^*(\tau^*)\right] :=$ $\dfrac{\partial r\left[x^*(\tau^*)\right]}{\partial x}$. □

Theorem 1: Consider the nonlinear system with impulse effects

$$\Sigma:\begin{cases}\dot{x}(t) = f\left(x(t)\right) & x^-(t) \notin \mathbb{S} \\ x^+(t) = H\left(x^-(t)\right) & x^-(t) \in \mathbb{S}\end{cases}$$

where $\mathbb{S} = \left\{x(t) \in \mathrm{X} \subseteq \mathbb{R}^n \,\middle|\, r\left(x(t)\right) = 0\right\}$, which satisfies the following conditions:

(1) $f\left(x(t)\right)$ is continuous and differentiable on X;

(2) a solution of $\dot{x}(t) = f\left(x(t)\right)$ from a given initial condition is unique and depends continuously on the initial condition;

(3) the system exits a limit cycle trajectory $x^*(t)$ with the initial point x^* at $t = 0$, and the limit cycle time

between the impulse effects τ^* is known ;

(4) x^* is the fixed point of the Poincaré return map P that corresponds to this system;

(5) $r(x^-)$ is a continuous and differentiable function on X ;

(6) $H\left(x^-(t)\right)$ is continuous and differentiable on X .

Then Jacobian matrix J of the Poincaré return map P at the fixed point x^* for this system with impulse effects is expressed approximately as

$$DH\left[x^*(\tau^*)\right]\cdot\left[I-\frac{\dot{x}^*(\tau^*)\cdot Dr\left[x^*(\tau^*)\right]}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}\right]\cdot e^{Df\left(x^*(\tau^*)\right)\cdot\tau^*},$$

where $DH\left[x^*(\tau^*)\right]:=\dfrac{\partial H\left[x^*(\tau^*)\right]}{\partial x}$;

$$Dr\left[x^*(\tau^*)\right]:=\frac{\partial r\left[x^*(\tau^*)\right]}{\partial x}.$$

Proof: With the whole hypotheses,

$$\hat{x}^+\approx DH\left[x^*(\tau^*)\right]\cdot\left[I-\frac{\dot{x}^*(\tau^*)\cdot Dr\left[x^*(\tau^*)\right]}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}\right]\cdot e^{Df\left(x^*(\tau^*)\right)\cdot\tau^*}\cdot\hat{x}(0)$$

where $DH\left[x^*(\tau^*)\right]:=\dfrac{\partial H\left[x^*(\tau^*)\right]}{\partial x}$ is obtained immediately from lemma 1 and lemma 3. In terms of lemma 3, if $\hat{x}_i$ denotes $\hat{x}(0)$, and $\hat{x}_{i+1}$ denotes $\hat{x}^+$, then

$$\hat{x}_{i+1}\approx\frac{\partial H\left[x^*(\tau^*)\right]}{\partial x}\cdot\left[I-\frac{\dot{x}^*(\tau^*)\cdot Dr\left[x^*(\tau^*)\right]}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}\right].\cdot e^{Df\left(x^*(\tau^*)\right)\cdot\tau^*}\cdot\hat{x}_i$$

In the end,

$$J:=\frac{\partial f(x^*)}{\partial x}\approx DH\left[x^*(\tau^*)\right]\cdot\left[I-\frac{\dot{x}^*(\tau^*)\cdot Dr\left[x^*(\tau^*)\right]}{Dr\left[x^*(\tau^*)\right]\cdot\dot{x}^*(\tau^*)}\right]\cdot e^{Df\left(x^*(\tau^*)\right)\cdot\tau^*}.\ \square$$

This expression only needs to find a limit cycle trajectory $x^*(t)$ and the limit cycle time between the impulse effects τ^*. And the limit cycle and the limit cycle time between impulse effects can be obtained through the numerical method. So the stability of the nonlinear system with impulse effects can be studied by the Jacobian matrix of the Poincaré return map at the fixed point.

IV. BIPED ROBOT AND STABILITY

In this section, the goal is to study the stability of a walking cycle for the compass-like passive biped robot using the Jacobian matrix of the Poincaré return map at the fixed point. So the simplest passive biped robot is introduced. The simplest passive robot consists of a hip and two legs of equal length with no ankles and no knees [1]. Figure 1 presents the sketch of the simplest passive robot. The walking cycle is composed of the swing phase of motion (the single support-only one leg is touching the ground) and the collision stage (the double support-two legs are all touching the ground). Since the swing phase of motion can be described by the nonlinear differential equation, and the double support stage can be described by the nonlinear algebraic mapping dynamic system. So the model of the walking cycle of this biped robot can be described by the nonlinear system with impulse effects. The dynamic model of the robot between successive impacts is derived from the theory of Lagrange, and the expression in a standard second order system in terms of the normalized parameters μ and β is

$$M_n(\theta)\ddot{\theta}+N_n(\theta,\dot{\theta})\dot{\theta}+\frac{1}{a}g_n(\theta)=0,$$

where $\theta=(\theta_{ns},\theta_s)^T$ are the angles made respectively by the swing and the support leg with the vertical(counterclockwise positive).

$M_n(\theta), N_n(\theta,\dot{\theta}), g_n(\theta)$ depend on μ and β, and they are deduced from the Lagrange, where

$$M_n(\theta)=\begin{pmatrix}\beta^2 & -(1+\beta)\beta\cos 2\alpha\\ -(1+\beta)\beta\cos 2\alpha & (1+\beta)^2(\mu+1)+1\end{pmatrix};$$

$$N_n(\theta,\dot{\theta})=\begin{pmatrix}0 & (1+\beta)\beta\dot{\theta}_s\sin(\theta_s-\theta_{ns})\\ -(1+\beta)\beta\dot{\theta}_{ns}\sin(\theta_s-\theta_{ns}) & 0\end{pmatrix};$$

$$g_n(\theta)=\begin{pmatrix}g\beta\sin(\theta_{ns})\\ -\left((1+\mu)(1+\beta)+1\right)g\sin(\theta_s)\end{pmatrix}.$$

And $\beta=\dfrac{b}{a};\mu=\dfrac{m_H}{m}$.

So the nonlinear differential equation of the single support is expressed by

$$\dot{x}=\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix}=M_n^{-1}(\theta)\left[-N_n(\theta,\omega)\omega-\frac{1}{a}g_n(\theta)\right]:=f(x).$$

The state space for the system is taken as

$$X:=\left\{x:=(\theta',\omega')'\,\middle|\,\theta\in(-\pi,\pi)^2;\omega\in\mathbb{R}^2\right\}.$$

The collision stage is described as $x^+=W_n(\alpha)x^-$, and

$W_n(\alpha)$ denotes the collision transition function, where

$$W_n(\alpha)=\begin{pmatrix} K & 0 \\ 0 & H_n(\alpha) \end{pmatrix},$$

that is, $x^+ = H(x^-) = W(\alpha)\cdot x^-$, where

$$K=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\ H_n(\alpha)=\left(Q_n^+(\alpha)\right)^{-1} Q_n^-(\alpha);$$

$$Q_n^+(\alpha)=\begin{pmatrix} -\beta & -\beta+\left(\mu(1+\beta)^2+2(1+\beta)\right)\cos 2\alpha \\ 0 & -\beta \end{pmatrix};$$

$$Q_n^-(\alpha)=\begin{pmatrix} \beta(\beta-(1+\beta)\cos 2\alpha) & (1+\beta)((1+\beta)-\beta\cos 2\alpha)+1 \\ & +\mu(1+\beta)^2 \\ \beta^2 & -\beta(1+\beta)\cos 2\alpha \end{pmatrix}.$$

The collision surface is

$$\mathbb{S}=\left\{(\theta',\omega')' \middle| \theta_{ns}+\theta_s=-2\phi;\theta\in(-\pi,\pi)^2;\omega\in\mathbb{R}^2\right\}.$$

The impact detection function is $r(x^-)=\theta_{ns}^-+\theta_s^-+2\phi=0$.

The premises underlying this model are that: (1) the impact takes place instantaneously; (2) the impact of the swing leg with the ground is assumed to be inelastic and without sliding; (3) the tip of the support leg is assumed not to be slip, and the robot behaves as a ballistic double-pendulum; (4) a knee-less robot with a rigid leg can not clear the ground.

We choose the parameters as follows:

$$\beta=1;\mu=2;\alpha=0.2710;\phi=3^\circ;a=b=0.5$$

Through the numerical method, a fixed point is

$$x^*(0)=[-0.3234, 0.2186, -0.3779, -1.0922]^T.$$

The limit cycle time between the collisions is $\tau^* = 0.7343$.

$$x^*(\tau^*)=[0.2186, -0.3234, -1.8024, -1.4936]^T$$

By means of calculation with theorem 1, the Jacobian matrix J of the Poincaré return map at fixed point $x^*(0)$ is

$$J=\begin{bmatrix} -0.0188 & 2.6073 & -0.0391 & 0.8251 \\ 0.0188 & -2.6073 & 0.0391 & -0.8251 \\ -1.1487 & 7.5955 & -0.0050 & 2.0366 \\ -0.4951 & 8.9029 & -0.1649 & 2.8578 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix J of the Poincaré return map at fixed point $x^*(0)$ are

$$-0.0345+0.5115i, -0.0345-0.5115i, 0, 0.2958.$$

The modules of these eigenvalues are 0.5127, 0.5127,0, and 0.2958. Since all of the modules of eigenvalues are strictly less than one, the biped gaits are asymptotically stable. The existence and the characteristics of steady passive compass gaits are confirmed by simulations (fig.2).

Under the same parameters conditions, the Jacobian matrix of Poincaré return map at the fixed point $x^*(0)$ obtained by the numerical method which is introduced in[12] is

$$J_1=\begin{bmatrix} -0.7574 & -2.0589 & 0.0204 & -0.7249 \\ 0.7574 & 2.0589 & -0.0204 & 0.7249 \\ 2.4692 & -0.2433 & -0.0141 & 0.2172 \\ -0.7151 & -4.4003 & 0.0705 & -1.5411 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix J_1 of Poincaré return map at the fixed point $x^*(0)$ obtained by the numerical method are

$$-0.0699+0.7599i, -0.0699-0.7599i, 0, -0.1139.$$

The modules of the eigenvalues of J_1 are 0.7631, 0.7631, 0 and 0.1139. They are all less than 1.

However under the same hypotheses, the Jacobian matrix of the Poincaré return map at the fixed point $x^*(0)$ obtained by the same numerical method[2] is

$$J_2=\begin{bmatrix} -0.439 & -0.500 & -0.003 & -0.169 \\ 0.439 & 0.500 & 0.003 & 0.169 \\ 0.147 & -2.933 & 0.082 & -1.065 \\ -0.877 & -1.846 & 0.011 & -0.633 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix J_2 of the Poincaré return map at fixed point $x^*(0)$ are

$$-0.252+0.215i, -0.252-0.215i, 0, 0.014.$$

The modules of the eigenvalues of J_2 are 0.332, 0.332, 0.014 and 2.554×10^{-9}. They are less than 1, too.

Since J is obtained on basis of the configuration of the model, then J is unique. However $J_1;J_2$ are achieved on the data that are chosen in the numerical method, so $J_1;J_2$ are different.

In the following, the linearity errors of J and J_1 are compared in the same conditions. The results are shown in Table1. It is shown that the linearity error of J is close to the linearity error of J_1. So the accuracy of judgment on the stability of the system with impulse effects is guaranteed in using the approximate Jacobian matrix proposed in this paper.

In a word, the Jacobian matrix of the Poincaré return map at the fixed point in the numerical method has uncertainty and the judgment on the stability is consistent in the judgment through the Jacobian matrix proposed in this paper. So the method in this paper is prior to the numerical method, when the stability of the system with impulse effects is studied.

TABLE1. RESULTS OF COMPARISON BETWEEN J AND J_1

initial point of limit cycle	linearity error of J	linearity error of J_1
$[-0.3224, 0.2186, -0.3779, -1.0922]^T$	0.13%	0.27%
$[-0.3234, 0.2196, -0.3779, -1.0922]^T$	1.03%	0.41%
$[-0.3234, 0.2186, -0.3779, -1.0912]^T$	0.25%	0.22%
$[-0.3233, 0.2187, -0.3778, -1.0921]^T$	0.11%	8.1657e-004
$[-0.3234, 0.2186, -0.3779, -1.0912]^T$	0.32%	0.26%
$[-0.3234, 0.2196, -0.3779, -1.0932]^T$	0.92%	0.37%
$[-0.3234, 0.2186, -0.3879, -1.0922]^T$	0.12%	0.14%
$[-0.3234, 0.2186, -0.2779, -1.0922]^T$	1.25%	1.58%
$[-0.3334, 0.2186, -0.3779, -1.0922]^T$	1.09%	4.18%

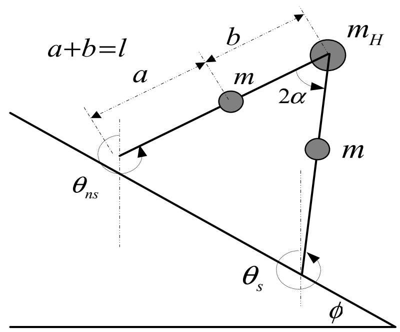

Figure1. Model of a compass-like passive biped robot walking down a slope. (Swing stage)

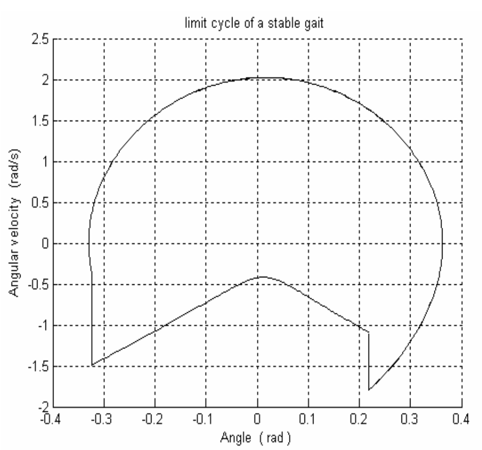

Figure 2. A limit cycle of stable biped gaits

V. CONCLUSION

With the disturbance theory and the linearization method, an approximate Jacobian matrix of the Poincaré return map at the fixed point is presented for the nonlinear system with impulse effects. The sufficient condition to the existence of this approximate Jacobian matrix is given. And this approximate Jacobian matrix is compared with the ones obtained in numerical method. In conclusions, firstly this approximate Jacobian matrix depends only on the configuration of the model; secondly this approximate Jacobian matrix has uniqueness that does not exist in the numerical method; thirdly the stability of the system with impulse effects is the same in using this approximate Jacobian matrix and the ones obtained in numerical method. So the approximate Jacobian matrix in this paper can be used to study the stability of the system accurately and conveniently. Since the biped robot gaits can be presented by a system with impulse effects, so the method proposed in this paper can be used to study the stabilities of the biped robot gaits. But this approximate expression needs to find the limit cycle $x^*(t)$ and the limit cycle time between impulse effects τ^*. So the next job is to find the fixed points of the Poincaré map with the theory of mathematics.

REFERENCES

[1] T.McGeer, "Passive dynamic walking," Internationl Journal of Robotics Research, vol.9, No. 2, pp. 62-82, April, 1990.

[2] A. Goswami, B. Thuilot, and B.Espiau,"Compass-like biped robot Part1: Stability and bifurcation of passive gaits," Rapport de recherche INRIA, RhoneALPES, 2996, unite de Rapport st.Martin France, october 1996.

[3] B.Thuilot, A.Goswami, and B.Espiau,"Bifurcation and chaos in a simple passive bipedal gait,"IEEE International Conference on Robotics and Automation ,vol.1, Issue , 20-25 , pp.792 – 798, April,1997.

[4] A. Goswami, B. Thuilot, and B. Espiau, "A study of the passive gait of a compass-like biped robot:symmetry and chaos," International Journal of Robotics Research,vol.17, No.12, pp. 1282–1301, December,1998.

[5] A.Goswami, B.Espiau, and A.Keramane,"Limit cycles in a passive compass gait biped and passivity-mimicking control laws," Journal of Autonomous Robots, vol. 4 , No.3, pp. 273-286, 1997.

[6] M.Garcia, A.Ruina, A.Chatterjee, and M.Coleman,"The Simplest Walking Model: Stability, Complexity and Scaling," ASME Journal of Biomechanical Engineering, vol. 120, No. 2, pp. 281-288, April, 1998.

[7] J. W.Grizzle, F. Plestan, and G.Abba, "Poincaré maps's method for systems with impulse effects: application tomechanical biped locomotion," Decision and Control, 1999. Proceedings of the 38th IEEE Conference ,vol. 4, pp. 3869 – 3876, 1999.

[8] J.W.Gizzle, G.Abba,and F.Plestan, " Asymptotically Stable Walking for Biped Robots: Analysis via Systems with Impulse Effects,"IEEE Trans. Autom. Control, vol.46, No.1, pp.51-64, January, 2001.

[9] E.Plestan, J.W.Gizzle , E.R.Westervelt, and G.abba, "Stable walking of a 7-DOF biped robot,"IEEE Transactions on Robotics and Automation,vol.19, No.4, pp.653-668, August, 2003.

[10] M.Coleman, A.Chatterjee, and A.Ruia, "Motion of a rimless spoked wheel: A simple 3D system with impact," Dynamic and Stability of Systems, vol.12, No. 3, pp. 139-169, Spetember, 1997.

[11] M.W.Spong and F.Bullo, "Controlled symmetries and passive walking," IEEE Transactions on Automation and Control,vol.50, No.7, pp. 1025-1031, July, 2005.

[12] Zhenze Liu, "On Passive-Dynamic Model of Human Gait and Some Useful Control Strategies," Doctor thesis, October 2007, unpublished.