An Improved Adaptive CUSUM Control Chart for Monitoring Process Mean

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***Abstract*—Adaptive CUSUM (namely ACUSUM) charts have received much attention recently. It is found that, by adjusting the reference parameter *k* dynamically, an ACUSUM chart can achieve better performance over a range of mean shifts than the conventional CUSUM chart that is designed to have maximal detection effectiveness at a particular level of shift. This article studies a new feature of the ACUSUM chart regarding an additional charting parameter, *w* (the exponential of the sample mean shift in $(x_t - \mu_0)^w$), which is also adapted according to the on-line estimated value of the mean shift. The testing cases reveal that this new adaptive CUSUM chart outperforms the earlier ACUSUM chart to a substantial degree.**

***Keywords*—adaptive control chart, CUSUM chart, loss function, quality control, statistical process control**

I. Introduction

The CUSUM control charts have been well recognized across industries owing to the fact that on-line measurement and distributed computing systems become a norm in today's Statistical Process Control (SPC) applications [1]. Usually, two symmetrical CUSUM charts are used together for detecting two-sided mean shifts. In a high-sided CUSUM chart for detecting increasing mean shifts, the statistic C_t to be updated for the (t)th sample is

$$\begin{aligned} &C_0 = 0, \\ &C_t = \max(0,\ C_{t-1} + (x_t - \mu_0) - k), \end{aligned} \tag{1}$$

where x_t is the (t)th sample value of a quality characteristic x following an independent and identical normal distribution, $N(\mu, \sigma^2)$; μ_0 is the in-control mean of x; and k is the reference parameter. The difference $(x_t - \mu_0)$ is a sample value of the mean shift $\delta\sigma$. In this article, it is assumed that the in-control mean μ_0 and standard deviation σ_0 of x are known a priori (for example they can be estimated from the field test records or historical data). Moreover, the standard deviation σ is assumed to be unchanged, i.e., $\sigma \equiv \sigma_0$.

The quality characteristic x can be converted to z that has a standard normal distribution when process is in control.

$$z = (x - \mu_0)/\sigma . \tag{2}$$

Correspondingly, Equation (1) becomes

$$\begin{aligned} &C_0 = 0, \\ &C_t = \max(0,\ C_{t-1} + z_t - k), \end{aligned} \tag{3}$$

and the mean shift is δ.

A conventional CUSUM chart often determines the reference parameter k with reference to a special mean shift δ_s. As a result, it performs optimally when the mean shift δ is equal to δ_s. However, since it is quite difficult to predict the actual magnitude of δ for most of the applications, there is no guarantee that the conventional CUSUM chart always performs well during the operation.

Recently, Sparks (2000) proposed the concept of adaptive CUSUM (namely ACUSUM) chart which adjusts the reference parameter k according to the on-line estimated value $\hat{\delta}$ of the mean shift [2]. Compared with the conventional CUSUM charts, the adaptive feature makes the ACUSUM chart more efficient in signaling a range of future expected but unknown mean shifts from a holistic viewpoint. Shu and Jiang (2006) simplified the design and implementation of the ACUSUM chart [3].

Recently, some researchers have found that an exponential w will influence the sensitivity of the CUSUM chart with respect to mean shifts δ if the variable z_t in (3) is replaced by z_t^w, or the term $(x_t - \mu_0)$ in (1) is replaced by $(x_t - \mu_0)^w$ [4]. Usually, a larger w makes the CUSUM chart more effective for detecting larger δ; whilst a smaller w makes it more sensitive to smaller δ.

In this article, we propose a new ACUSUM chart in which both k and w are adapted dynamically in accordance with the current estimated value $\hat{\delta}$ of the mean shift. For the purpose of detecting increasing mean shifts, the statistic C_t in the new ACUSUM chart will be updated by

$$\begin{aligned} &C_0 = 0 \\ &C_t = \max\left(0,\ C_{t-1} + q - k\right), \end{aligned}$$

$$q=\begin{cases} z_t^w, & if \quad z_t \ge 0, \\ -(-z_t)^w, & if \quad z_t < 0. \end{cases} \tag{4}$$

where, the parameters k, w and q all depend on the current sample value z_t. The statistic C_t may increase or decrease depending on whether the sample value z_t (or $x_t - \mu_0$) is larger or smaller than zero. However, C_t is always shrunk toward zero by the reference parameter k. When an increasing mean shift occurs, C_t is likely to become larger and larger. Sooner or later, a subsequent sample point will exceed the control limit H of the ACUSUM chart, and thereby produce an out-of-control signal.

In order to differentiate the two versions of ACUSUM charts, the first ACUSUM chart studied by Sparks (2000) and Shu and Jiang (2006) is called as the ACUSUM I chart, whilst the one proposed in this article as the ACUSUM II chart. It is noted that, if w is equal to one, Equation (4) is exactly the same as (3). This means that the ACUSUM I chart is just a special case of the ACUSUM II chart per se. As revealed by the performance studies in the subsequent sections, the new adaptive feature of the ACUSUM II chart is able to increase the detection effectiveness by about 20%, on average, compared with the ACUSUM I chart in which only the reference parameter k is adapted.

This article focuses on the study of the high-sided ACUSUM II chart. However, a symmetrical low-sided counterpart can be built straightforwardly.

$$C_0^- = 0,$$
$$C_t^- = \min\left(0,\ C_{t-1}^- + q + k\right),$$

$$q=\begin{cases} z_t^w, & if \quad z_t \ge 0, \\ -(-z_t)^w, & if \quad z_t < 0. \end{cases} \tag{5}$$

The performance of a control chart can be measured by the Average Run Length (ARL), meaning the average number of samples required to signal an out-of-control case or produce a false alarm. The out-of-control ARL_1 is commonly used as an indicator of the power (or effectiveness) of the control chart, whereas the in-control ARL_0 for the false alarm rate. In this article, the out-of-control ARL_1 will be computed under the steady-state mode. It assumes that the process has reached its stationary distribution at the time when the process shift occurs. Since production processes often operate in in-control condition for most or relatively long periods of time [5], the steady-state mode is therefore more realistic than the zero-state mode.

II. Design and Implementation of an ACUSUM II Chart

In this section, the idea of sub-cusum chart is first introduced and is used to discretize the ACUSUM II chart. Then an optimization model is presented for the design of this chart. It is followed by the selection of the objective function for the optimization design. The implementation of the ACUSUM II chart is outlined at last.

A. Sub-cusum Charts

It seems desirable to adjust the parameters k and w (Equation (4)) of an ACUSUM II chart continuously in accordance with the current estimated value $\hat{\delta}_t$ of the mean shift. However, studies on VSSI (Variable Sample Sizes and Sampling Intervals) CUSUM charts discover that using n (n = 2 or 3) individual cusum charts (called the *sub-cusum* charts) may gain most of the benefits that can be reached by a VSSI CUSUM chart [6], and are relatively easier to implement. It suggests that, in the implementation of an ACUSUM II chart, one may only use n sub-cusum charts, each of which has different values of k_i and w_i (i = 1, 2, …, n) and each is best for detecting a particular discrete value of δ_i ($\delta_{\min} < \delta_1 < \ldots < \delta_n < \delta_{\max}$). Jointly, the n sub-cusum charts will optimize the holistic performance of the ACUSUM II chart over the entire mean shift range. The ACUSUM II chart keeps on switching among the n sub-cusum charts depending on which δ_i is closest to the current estimated $\hat{\delta}_t$. Any moment one, and only one, sub-cusum chart that is best for detecting $\hat{\delta}_t$ is in use (or active). Suppose, in a moment, if the (i)th sub-cusum chart is active, the parameters k and w in (4) will take the values of k_i and w_i, respectively. In this article, (n = 2) is always used because of its ease for design and implementation. The results of numerical studies show that, no matter (n = 2) or (n = 3) is employed, the performance of the ACUSUM II charts is nearly the same.

Each of the n discrete δ_i is set at the center of one of the n equal intervals between $\delta_{\min}$ and $\delta_{\max}$, that is,

$$\delta_i = \delta_{\min} + (i-0.5)\cdot D, \quad i = 1,\ 2,\ \ldots,\ n,$$
$$D = (\delta_{\max} - \delta_{\min})/n, \tag{6}$$

where D is the distance between two discrete values δ_i and δ_{i+1}. Each of the n sub-cusum charts has different values of k_i and w_i. When the (i)th sub-cusum chart is active, Equation (4) is discretized to:

$$C_0 = 0,$$
$$C_t = \max(0,\ C_{t-1} + q - k_i),$$

$$q=\begin{cases} z_t^{w_i}, if\ z_t \ge 0, \\ -(-z_t)^{w_i}, if\ z_t < 0, \end{cases} \quad i = 1,2,\ldots,n \tag{7}$$

Like in an ACUSUM I chart, $\hat{\delta}_t$ is updated by a EWMA procedure in an ACUSUM II chart.

$$\hat{\delta}_0 = \delta_1,$$
$$\hat{\delta}_t = \left\|(1-\lambda)\hat{\delta}_{t-1} + \lambda z_t\right\|. \tag{8}$$

The operator $\|\ \|$ makes $\hat{\delta}_t$ equal to one of the n discrete δ_i whichever is closest to $((1-\lambda)\hat{\delta}_{t-1} + \lambda z_t)$. Then the corresponding sub-cusum chart is selected to update C_t using

(7). When the smoothing parameter λ equals one, $\hat{\delta}_t$ is completely determined by z_t; otherwise $\hat{\delta}_t$ also depends on the information in the sequence of the sample points.

For example, suppose $n = 2$, $\lambda = 0.4$, $\delta_{min} = 1$ and $\delta_{max} = 3$, then, Equation (6) gives $D = 1$, $\delta_1 = 1.5$ and $\delta_2 = 2.5$. Now, suppose $\hat{\delta}_{t-1} = \delta_1$ and $z_t = 3.0$,

$$\begin{aligned}\hat{\delta}_t &= \|(1-0.4)\times 1.5 + 0.4\times 3.0\| \\ &= \|2.1\| = \delta_2.\end{aligned} \quad (9)$$

Thus, the 2nd sub-cusum chart will be activated to update C_t, or

$$C_t = \max\left(0,\ C_{t-1} + z_t^{w_2} - k_2\right) \quad (10)$$

B. Design Model

To design an ACUSUM II chart, three parameters need to be specified: (1) τ, the minimum allowable in-control ARL_0 for a one-sided ACUSUM II chart; (2) δ_{min}, the lower bound of mean shift; and (3) δ_{max}, the upper bound of mean shift.

Based on the specifications, the charting parameters of an ACUSUM II chart will be determined in an integrative and optimal manner using the following design model:

$$\text{Objective function: } U = \text{minimum.} \quad (11)$$

$$\text{Constraint function: } ARL_0 = \tau. \quad (12)$$

$$\text{Design variables: } k_i,\ w_i\ (i = 1, 2, \ldots, n),\ \lambda,\ H. \quad (13)$$

where H is the control limit of the ACUSUM II chart. When $n = 2$, there are in total six design variables, among which k_1, w_1, k_2, w_2 and λ are independent. The control limit H is adjusted to ensure that the ARL_0 of the ACUSUM II chart is equal to τ. The optimization aims to find the optimal values of these design variables so that the objective function U is minimized or both small and large mean shifts can be detected quickly. The selection of U will be discussed shortly.

Any nonlinear optimization program may be used to search the optimal solution. In our study, the simple, yet reliable, Hooke-Jeeves procedure is employed [7]. It can complete the design of an ACUSUM II chart in a few CPU seconds with a personal computer.

C. Design Objective

Since our goal is to make control charts efficient at signalling a range of mean shifts, the objective function should measure the holistic performance of the charts across the range rather than the effectiveness at a particular point.

Furthermore, since it is assumed that all mean shifts within a range are equally important [2], a uniform distribution for δ is implied. The comparison of the overall performance of two charts may be formulated as follows:

$$RARL = \frac{1}{\delta_{max} - \delta_{min}} \cdot \int_{\delta_{min}}^{\delta_{max}} \frac{ARL(\delta)}{ARL_{benchmark}(\delta)} d\delta\ , \quad (14)$$

where, $ARL(\delta)$ is produced by a particular chart at δ and $ARL_{benchmark}(\delta)$ is generated by a chart that acts as the benchmark. Obviously, if the *RARL* value of a chart is larger than one, this chart is generally less effective than the benchmark, and *vice versa*.

An alternative is to use the *Extra Quadratic Loss* (*EQL*) to measure and compare the performance of the charts. When σ is a constant, the quadratic loss incurred by a mean shift δ is simply equal to $(\delta\sigma_0)^2$ [4, (Wu *et al.* 2004), because

$$\begin{aligned}&\left[\sigma^2 + (\mu - \mu_0)^2\right] - \sigma_0^2 \\ &= \left[\sigma_0^2 + (\delta\sigma_0)^2\right] - \sigma_0^2 = (\delta\sigma_0)^2\end{aligned}. \quad (15)$$

Moreover, since the quality cost is proportional to $ARL(\delta)$, the overall *EQL* can be calculated as follows:

$$EQL = \frac{\sigma_0^2}{\delta_{max} - \delta_{min}} \cdot \int_{\delta_{min}}^{\delta_{max}} \delta^2 \cdot ARL(\delta) \cdot d\delta\ . \quad (16)$$

Both *EQL* and *RARL* are acquired by integration across the whole shift range. The integration can be computed by a numerical method and the $ARL(\delta)$ of the ACUSUM II chart is calculated by the formulae derived in Appendix A. The index *EQL* based on loss function has two advantages compared with *RARL*. Firstly, the loss function is a more comprehensive measure of the charting performance than *ARL*, because it considers all the contributors to the quality cost including the time to signal and the magnitude of δ. Secondly, the evaluation of *EQL* does not require a predetermined benchmark chart. In view of this, *EQL* will be used as the objective function U in (11) for the design of the ACUSUM II charts. The minimization of *EQL* will reduce the loss in quality (or the cost, or the damage) incurred in the out-of-control cases. Like *RARL*, the ratio between the *EQL* values of two control charts serves as a measure of the relative effectiveness of the two charts.

D. Implementation

After an ACUSUM II chart has been designed, it can be implemented as follows:

1) Initialize $\hat{\delta}_0$ as δ_1 (note, $\hat{\delta}_0$ means the estimated mean shift at the beginning when $t = 0$, and δ_1 is the designated value of the shift for the first sub-cusum chart), and the statistic C_0 in (7) as zero.

2) Take a sample value x_t of the quality characteristic.

3) Covert x_t to z_t using (2).

4) Update $\hat{\delta}_t$ by (8).

5) If $\hat{\delta}_t = \delta_i$, use the charting parameters k_i, w_i of the (i)th sub-cusum chart to update C_t, that is,

$$C_t = \max\left(0,\ C_{t-1} + q - k_i\right),$$
$$q = \begin{cases} z_t^{w_i} & if \quad z_t \geq 0, \\ -(-z_t)^{w_i} & if \quad z_t < 0. \end{cases} \quad (17)$$

6) If $C_t \leq H$, the process is thought in control, and go back to step 2) for the next sample.

7) Otherwise (i.e., $C_t > H$), the ACUSUM II chart produces an out-of-control signal, and the process is stopped immediately for investigation.

III. Comparative Studies

In this section, the performance of four control charts is compared. For the convenience, all charts are studied as one-sided charts with an upper control limit for detecting increasing mean shifts. Furthermore, in-control μ_0 and σ_0 are assumed as zero and one, respectively.

1) The conventional CUSUM chart: The design of a conventional CUSUM chart aims at minimizing the out-of-control ARL_1 at a specified mean shift level of δ_s. Usually, δ_s is set as δ_{min}, because δ_s should be the smallest shift such that any shift δ larger than δ_s is considered important enough to be detected quickly [8]. Consequently, the reference parameter k is set as 0.5 δ_{min}.

2) The optimal CUSUM chart: This chart is very similar to a conventional CUSUM chart in aspects of the updating of the cumulative statistic C_t (as in (1)), the fixed parameter k, and the operational rules. However, the optimal CUSUM chart is designed by a new optimization algorithm in which the parameters k and H are optimized by using *EQL* (i.e., the performance over the whole shift range) as the objective function and ($ARL_0 = \tau$) as the constraint function.

3) The ACUSUM I chart: This is the adaptive CUSUM chart with k being adjusted during the operation. Here, the model developed by Shu and Jiang (2006) is adopted, because this model is easier to be designed than the model proposed by Sparks (2000). However the performance of both models is almost the same. In this article, the two charting parameters λ and Q_0 of an ACUSUM I chart are optimized so that *EQL* is minimized subject to ($ARL_0 = \tau$).

4) The ACUSUM II chart proposed in this article.

The first comparison is carried out under a general condition with (τ = 740, δ_{min} = 0.5, δ_{max} = 4). The specification of (τ = 740) ensures that the resultant false alarm rate is identical to that of a typical 3-sigma $\bar{X}$ chart when two symmetrical CUSUM charts are used simultaneously to detect the two-sided mean shifts.

With these specifications, the four control charts are designed and the resultant charting parameters and the *ARL* values, are summarized in Table I. There are several interesting findings.

TABLE I. ARL Comparison among CUSUM and ACUSUM Charts

(δ_{min} = 0.5, δ_{max} = 4.0)

δ	***con CUSUM***	***opt CUSUM***	***ACUSUM I***	***ACUSUM II***
	k=0.250 H=8.009	k=0.825 H=3.048	H=1.706 λ=0.400 Q_0=3.417 L=4.000	H=6.898 λ=0.456 k_1=0.594 k_2=1.154 w_1=1.435 w_2=1.750
0.00	739.39	739.48	743.18	739.16
0.50	25.51	54.59	26.48	40.15
1.00	9.79	11.13	10.55	10.14
1.50	6.08	4.98	6.09	5.22
2.00	4.46	3.17	4.09	3.38
2.50	3.56	2.36	3.03	2.42
3.00	2.99	1.92	2.37	1.85
3.50	2.60	1.62	1.97	1.48
4.00	2.31	1.40	1.68	1.24
EQL	20.709	15.375	17.343	14.398
$\frac{EQL}{EQL_{ACUSUM\ II}}$	1.438	1.068	1.205	1.000
RARL	1.349	1.074	1.164	1.000

1) Both the ACUSUM I and II charts are more effective than the conventional CUSUM chart almost across the entire shift range except for $\delta \leq 1$. The ACUSUM II chart outperforms the conventional CUSUM chart to a more significant degree than the ACUSUM I chart does.

2) The ACUSUM II chart also outperforms the ACUSUM I chart for most of the cases. It is only less sensitive than the latter to very small δ (i.e., when $\delta \leq 0.5$). It is noted that, δ_{min} is specified as 0.5 in this case. Then, a low *ARL* value for ($\delta \leq \delta_{min}$) will be considered as a drawback, because it may result in over-correction and introduce extra variability [8].

3) The optimal CUSUM chart has achieved significant improvement in detection effectiveness compared with the conventional CUSUM chart. The optimal CUSUM chart has a larger ARL_1 only for minor mean shifts. As long as $\delta > 1$, it becomes much more effective than the conventional CUSUM chart. The optimal CUSUM chart may even outperforms the ACUSUM I chart. But it is generally less effective than the ACUSUM II chart.

As aforementioned, for most of the cases, no chart will give better performance than other charts for all shifts [9]. Consequently, in order to make an accurate and objective decision about the relative effectiveness of the charts, it is necessary to evaluate the values of the following three holistic measures of the charts.

1) *EQL* (Equation (16));
2) EQL / $EQL_{ACUSUM\ II}$, the ratio between the *EQL* of a chart and the *EQL* of the ACUSUM II chart; and
3) *RARL* (Equation (14)), the ratio between the *ARL* of a chart and the *ARL* of the ACUSUM II chart, i.e., using the ACUSUM II chart as the benchmark.

The results are enumerated at the bottom of Table I. It is interesting to find that the values of EQL / $EQL_{ACUSUM\ II}$ and *RARL* of a chart are often fairly close to each other. They reveal that:

1) When considering the whole shift range, both the ACUSUM I and ACUSUM II charts obviously outperform the conventional CUSUM chart. If measured by *EQL*, the ACUSUM I and ACUSUM II charts are more effective than the conventional CUSUM chart by 19.4% and 43.8%, respectively.

2) Between the ACUSUM I and ACUSUM II charts, the latter outperforms the former by more than 20%, on average, across the entire shift range.

3) The optimal CUSUM chart uses a fixed k as the conventional CUSUM chart, but its overall performance has been significantly improved. However, the optimal CUSUM chart is less effective than the ACUSUM II chart by about 7% measured by either *EQL* or *RARL*. The optimal CUSUM chart seems simpler than the ACUSUM II chart. But in a computerized environment, the operation of both charts is equally easy.

Next, the effectiveness of the four charts is further compared in a factorial experimental design with four different cases (combinations) of δ_{min} and δ_{max}: (1) δ_{min} = 0.25, δ_{max} = 3.0; (2) δ_{min} = 0.25, δ_{max} = 5.0; (3) δ_{min} = 0.75, δ_{max} = 3.0; (4) δ_{min} = 0.75, δ_{max} = 5.0.

In all four cases, the values of *ARL*, *EQL* and *RARL* reveal the performance characteristics similar to those shown in Table I.

IV. Conclusions and Discussions

This article has proposed an improved adaptive CUSUM chart, the ACUSUM II chart, for detecting process shifts in mean. This chart further enhance the performance of the earlier ACUSUM I chart to a promising extent. The improvement is attributable to the on-line adaption of an additional charting parameter, *w*, which is the exponential of the sample mean shifts. The ACUSUM II chart is also much easier to design in terms of the required CPU time as its *ARL* can be evaluated by a Markov procedure.

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Appendix
Calculation of the ARL of the ACUSUM II Chart

The ACUSUM II chart can be described by a two-dimensional Markov chain. Suppose that the statistic C_t in (7) experiences *m* different transitional states before being absorbed into the out-of-control state. States 0 to (*m*–1) are in-control states and state *m* is an out-of-control state. The width of the interval of each state is given as

$$d = H / (m - 0.5). \quad (A.1)$$

The center, O_j, of state *j* is given as

$$O_j = j \cdot d \qquad j = 0, 1, \ldots, m. \quad (A.2)$$

Meanwhile, *n* sub-cusum charts are employed corresponding to *n* discrete mean shifts $\delta_1, \delta_2, \ldots, \delta_n$.

$$\delta_i = \delta_{min} + (i - 0.5)\, D \quad i = 1, 2, \ldots, n \quad (A.3)$$

where,

$$D = (\delta_{max} - \delta_{min}) / n \quad (A.4)$$

is the distance between these discrete mean shift values.

In a two-dimensional Markov chain, a point (*i*, *j*) represents a status in which the (*i*)th sub-cusum chart is in use and the statistic C_t is equal to O_j. Let $p_{ij\text{-}uv}$ be the transition probability from point (*i*, *j*) to point (*u*, *v*).

$$\begin{aligned} p_{ij-uv} &= \int_{\Omega} f(z_t) \cdot dz_t \\ &= \int_{\Omega} \frac{1}{\sqrt{2\pi}} \cdot \exp\left[-0.5 \cdot (z_t - \mu)^2\right] \cdot dz_t \end{aligned}, \quad (A.5)$$

where, $f\,(z_t)$ is the density function of z_t, and Ω is the intersection of the following two domains Ω_1 and Ω_2:

1) Domain Ω_1

$$\Omega_1\text{:}\;\; LB \le z_t \le UB. \quad (A.6)$$

It is the region for which the ACUSUM II chart will use the (*u*)th sub-cusum chart for the (*t*)th sample, given that the (*i*)th cub-cusum chart is employed for the (*t*-1)th sample; or the estimated $\hat{\delta}$ is closest to δ_u for the (*t*)th sample, given that it is equal to δ_i for the (*t*-1)th sample. Referring to (A.3), the lower bound *LB* and upper bound *UB* of the region Ω_1 can be determined as follows:

$$LB = \begin{cases} -\infty, if\;\; \mu = 1, \\ \left[\delta_u - 0.5D - (1-\lambda)\delta_i\right] / \lambda, if\;\; \mu > 1. \end{cases} \quad (A.7)$$

$$UB = \begin{cases} \left[\delta_u + 0.5D - (1-\lambda)\delta_i\right] / \lambda, if\;\; \mu < n, \\ +\infty, if\;\; \mu = n. \end{cases} \quad (A.8)$$

2) Domain Ω_2

$$\Omega_2\text{:}\;\; lb \le z_t \le ub. \quad (A.9)$$

It is the region for which the statistic C_t will be closest to O_v, given that C_{t-1} is equal to O_j. To make this transition (see (7)),

$$Q_L < q < Q_U, \quad if\;\; v > 0, \quad (A.10)$$

or

$$-\infty < q < Q_U, \quad if\;\; v = 0, \quad (A.11)$$

where,

$$Q_L = (O_v - 0.5d - O_j) + k_u,$$

$$Q_U = (O_v + 0.5d - O_j) + k_u. \quad \text{(A.12)}$$

Then, since (from (7))

$$z_t = \begin{cases} q^{-1/w_u}, if\ q \geq 0, \\ -(-q)^{-1/w_u}, if\ q < 0, \end{cases} \quad \text{(A.13)}$$

therefore the lower bound lb and upper bound ub of z_t in the region Ω_2 are determined as follows:

$$lb = \begin{cases} -\infty, if\ v = 0, \\ Q_L^{1/w_u}, if\ v > 0\ and\ Q_L \geq 0, \\ -(-Q_L)^{1/w_u}, if\ v > 0\ and\ Q_L < 0, \end{cases} \quad \text{(A.14)}$$

and

$$ub = \begin{cases} Q_U^{1/w_u}, if\ Q_U \geq 0, \\ -(-Q_U)^{1/w_u}, if\ Q_U < 0. \end{cases} \quad \text{(A.15)}$$

The transition probability p_{ij-uv} in (A.5) can be actually computed by

$$p_{ij-uv} = \Phi(\min(ub, UB) - u) - \Phi(\max(lb, LB) - u), \quad \text{(A.16)}$$

if min(ub, UB)>max(lb, LB); otherwise, $p_{ij-uv} = 0$. And Φ() is the cumulative probability function of the standard normal distribution.

When computing the in-control ARL_0, the transition probability p_{ij-uv} is calculated with $\mu = 0$. Based on p_{ij-uv}, the in-control transition matrix $\boldsymbol{R_0}$ can be established. It is a (n m) × (n m) matrix excluding the elements associated with the absorbing (or out-of-control) state. The zero-state ARL_0 is equal to the first element of the vector $\boldsymbol{V}$ given by the following expression:

$$\boldsymbol{V} = (\boldsymbol{I} - \boldsymbol{R_0})^{-1}\ \mathbf{1}, \quad \text{(A.17)}$$

where $\boldsymbol{I}$ is an identity matrix and $\mathbf{1}$ is a vector with all elements equal to one.

The transition matrix $\boldsymbol{R_1}$ for calculating the out-of-control ARL_1 can be established similarly to $\boldsymbol{R_0}$ except that the transition probability p_{ij-uv} in $\boldsymbol{R_1}$ must be evaluated using the out-of-control μ (= δ). The out-of-control ARL_1 under the steady-state mode is calculated as the following:

$$ARL_1 = \boldsymbol{B}^T (\boldsymbol{I} - \boldsymbol{R_1})^{-1}\ \mathbf{1}, \quad \text{(A.18)}$$

where, $\boldsymbol{B}$ is the steady-state probability vector with ($\mu = 0$). It is obtained by first normalizing $\boldsymbol{R_0}$ and then solving the following equation:

$$B = R_0^T B, \quad \text{(A.19)}$$

subject to

$$1^T B = 1. \quad \text{(A.20)}$$

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