Representations of Continuous Attractors of Discrete-Time Cellular Neural Networks

Jiali Yu, Zhang Yi and Lei Zhang
Computational Intelligence Laboratory
School of Computer Science and Engineering
University of Electronic Science and Technology of China
Chengdu 610054, P. R. China.
{yujiali, zhangyi, leilazhang}@uestc.edu.cn

***Abstract*—To describe the encoding of continuous stimuli in neural networks, continuous attractors have been recognized as promising models. A continuous attractor is a set of connected stable equilibrium points. It exhibits interesting dynamical properties in many recurrent neural networks. This paper studies the continuous attractors of discrete-time Cellular Neural Networks (DCNNs). The main contribution is that the representations of continuous attractors for DCNNs are obtained under some conditions. Such important results provide clear and complete descriptions to the continuous attractors of DCNNs.**

***Index Terms*—Continuous Attractors, Discrete-time Cellular Neural Networks, Multistability.**

I. Introduction

The model of Cellular Neural Networks (CNNs) was first proposed in [1], [2]. It has been studied by many authors in recent years. CNNs have been successfully applied to pattern recognition, signal processing, associative memories [3], [4], communication problems [5], Euclidean distance transformation [6], and especially in image processing, see for examples, [7]–[11], etc.

The transfer function of CNNs is defined by piecewise linear function. The piecewise linear function is essentially nonlinear, many important dynamic properties and applications of CNNs are crucially dependent on its properties. Many interesting properties of the transfer functions of CNNs can be found in [12]. Base on the properties of transfer functions, many dynamical properties of CNNs are established, see for examples [10], [11]. So far, most of the results on the dynamics of CNNs are focused on global stability, complete stability, chaotical properties, see for examples [13]–[20], etc.

Recently, dynamics of continuous attractors have been recognized as important properties of recurrent neural networks. In biological neural networks, it is believed that external inputs are encoded in neural activity patterns in the brain. The brain can reliably retrieve the stored information even when external stimuli are incomplete or noisy, achieving the associative memory or invariant object recognition [28]. Recurrent neural networks are often used for invariant object recognition as an associative memory. In most models of associative memory, memories are stored as attractive fixed points at discrete locations in state space. However, discrete attractors may not be appropriate for patterns with continuous variability, like the images of a three-dimensional object from different viewpoints. So the other way of representing each object by a continuous manifold of fixed points is natural [27].

Continuous attractor has been studied by many authors [21]–[32]. In [22], it shows that the memory of eye position is stored in a neural network with an approximate line attractor dynamics. If synaptic strengths and other parameters are precisely tuned by learning mechanisms, a linear network can exactly realize a line attractor dynamics [23]. Because the stable states are arranged in a continuous dynamical attractor, the network can store a memory of eye position with analog neural encoding. Moreover, it was given in [23] that the condition for the existence of line attractors of linear networks is that the largest eigenvalue of connection weight matrix is precisely unit and the rest of the eigenvalues have real parts that are less than unity. In [24], nonlinear network models of the oculomotor integrator are discussed. The synaptic weight matrix is tuned by minimizing the mean squared drift velocity of the eyes over a range of eye positions, leading to an approximate line attractor dynamics. In [28], two important issues when applying continuous attractors in neural systems are discussed. One is the computational robustness of continuous attractors with respect to input noises and the other is the implementation of Bayesian online coding. In [29], the dynamics and the computational properties of continuous attractors are investigated.

In this paper, we study the continuous attractors of discrete-time Cellular Neural Networks (DCNNs). DCNNs belong to an important class of recurrent neural networks and have been studied by many authors, see for examples [33]–[38]. We will address the important problem of continuous attractors: the explicit representations of continuous attractors. By explicit representations of continuous attractors, clear and complete descriptions of continuous attractors can be given. Using the properties of the transfer function of the DCNNs and by rigorous mathematical analysis, the explicit representations of continuous attractors of DCNNs are obtained. In [27], [28], one-dimensional continuous attractors are studied. However, a continuous attractor could be in multidimensional. The continuous attractors in this paper are not restricted to be in one dimensional, they can be in multi-dimensional.

The rest of the paper is organized as follows. Preliminaries are presented in Section II. The main results on the representation of continuous attractors of DCNNs are given in Section III. Simulations are given in Section IV to illustrate the theory. Finally, conclusions are drawn in Section V.

II. Preliminaries

The discrete-time Cellular Neural Networks (DCNNs) will be studied can be described by

$$x(k+1) = f\left(Wx(k) + b\right) \tag{1}$$

for $k \geq 0$, where $x = (x_1, \cdots, x_n)^T \in R^n$ is the state vector, $W = (W_{ij})_{n\times n}$ is the connection weight matrix which is symmetric, $b = (b_1, \cdots, b_n)^T$ denotes the external input. For any $x \in R^n$, $f(x) = (f(x_1), f(x_2), \cdot, f(x_n))^T$, and the function f is defined as follows:

$$f(s) = \frac{|s+1| - |s-1|}{2}, \quad s \in R.$$

The transfer function $f(\cdot)$ is a piecewise linear function, which is continuous but non-differentiable. Figure 1 shows this function. The transfer function $f(\cdot)$ is an important characteristic of the DCNNS. In fact, many dynamic properties and applications of DCNNS are crucially dependent on its properties. The transfer function f is composed of three parts with explicit boundaries, i.e.,

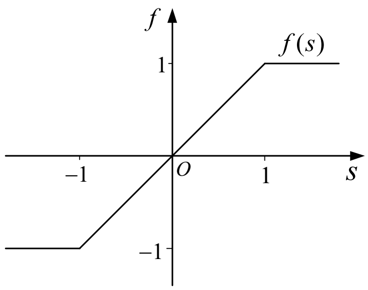

Fig. 1. The transfer function of network (1).

$$f(s) = \begin{cases} -1, & s \leq -1, \\ s, & -1 < s < 1, \\ 1, & s \geq 1. \end{cases}$$

Definition 1: A vector $x^* \in R^n$ is called an equilibrium point of (1), if it satisfies

$$x^* = f\left(Wx^* + b\right).$$

Lemma 1: The set

$$B = \{x \,||x_i| \leq 1, (i = 1, \cdots, n)\}$$

is an invariant set of the network (1), i.e., each trajectory starting in B remains in B for ever.

Proof: Given any $x(0) \in B$, we have

$$x_i(k+1) = f\left(Wx_i(k) + b\right) \tag{2}$$

for $k \geq 0$ and $i = 1, \cdots, n$. Because

$$|f(\cdot)| \leq 1$$

then, from (2),

$$|x_i(k)| \leq 1, (i = 1, \cdots, n)$$

for $k \geq 0$. The result now follows and the proof is completed.

Let Ω be an nonempty set, denote the distant from a point $x \in R^n$ to the set Ω by

$$dist\,(x, \Omega) = \min_{x^\dagger \in \Omega} \|x - x^\dagger\|.$$

Definition 2: A set of equilibrium points C is said to be stable, if given any constant $\epsilon > 0$, there exists a constant $\delta > 0$ such that

$$dist\,(x(0), C) \leq \delta$$

implies that

$$dist\,(x(k), C) \leq \epsilon$$

for all $k \geq 0$.

Definition 3: A set of equilibrium points C is called a continuous attractor if it is connected and stable.

The problem of representation of continuous attractors is to derive the representations of the continuous attractors explicitly.

III. Continuous Attractors

In this section, we study the continuous attractors of the DCNNs. We will derive explicit representations of continuous attractors of DCNNs.

By assumption, the synaptic matrix W is a symmetric matrix. Let $\lambda_i (i = 1, \cdots, n)$ be all the eigenvalues of W ordered by $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$. Suppose that $S_i (i = 1, \cdots, n)$ is an orthonormal basis in R^n such that each S_i is an eigenvector of W corresponding to the eigenvalue λ_i. Let σ be the largest eigenvalue of W with multiplicity m, clearly, $\sigma = \lambda_1$.

Suppose that

$$b = \sum_{i=1}^{n} \tilde{b}_i \cdot S_i. \tag{3}$$

Theorem 1: Suppose that $\sigma = 1$, $b \perp V_\sigma$, and $|\lambda_i| < 1 (i = m+1, \cdots, n)$. If

$$\sum_{j=m+1}^{n} \frac{|\tilde{b}_j|}{1 - \lambda_j} < 1,$$

then, the network (1) has a continuous attractor and it can be represented by

$$C = \left\{ x \middle| x = \sum_{i=1}^{m} c_i S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1-\lambda_j} S_j, \right. \left. \sqrt{\sum_{i=1}^{m} c_i^2 + \sum_{j=m+1}^{n} \left(\frac{|\tilde{b}_j|}{1-\lambda_j}\right)^2} \leq 1, c_i \in R \right\}.$$

Proof: Since $b \perp V_\sigma$, from (3), then $\tilde{b_1} = \cdots = \tilde{b_m} = 0$.

Firstly, we prove that $C \subset B$. Given any $x^* \in C$, then there exist constants $c_i^* \in R (i = 1, \cdots, m)$ with

$$\sqrt{\sum_{i=1}^{m} {c_i^*}^2 + \sum_{j=m+1}^{n} \left(\frac{|\tilde{b}_j|}{1-\lambda_j}\right)^2} \leq 1$$

such that

$$x^* = \sum_{i=1}^{m} c_i^* S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1-\lambda_j} S_j.$$

Then

$$\begin{aligned} \|x^*\| &= \sqrt{\sum_{i=1}^{m} c_i^{*2} + \sum_{j=m+1}^{n} \left(\frac{|\tilde{b}_j|}{1-\lambda_j}\right)^2} \\ &\leq 1. \end{aligned}$$

That is,

$$x^* \in B.$$

This shows that $C \subset B$.

Next, we prove that each point of C is an equilibrium point. Given any $x^* \in C$, it follows that

$$\begin{aligned} & f\left(Wx^* + b\right) \\ =\ & f\left(\sum_{i=1}^{m} c_i^* W S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1-\lambda_j} W S_j + \sum_{i=1}^{n} \tilde{b}_i \cdot S_i\right) \\ =\ & f\left(\sum_{i=1}^{m} c_i^* S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1-\lambda_j} \lambda_j S_j + \sum_{j=m+1}^{n} \tilde{b}_j \cdot S_j\right) \\ =\ & f\left(\sum_{i=1}^{m} c_i^* S_i + \sum_{j=m+1}^{n} \frac{\tilde{b}_j}{1-\lambda_j} S_j\right) \\ =\ & f\left(x^*\right) \\ =\ & x^*. \end{aligned}$$

This shows clearly that x^* is an equilibrium point of (1).

Next, we prove that C is stable. Given any $x(0) \in B$, let $x(k)$ be the trajectory starting from $x(0)$. Since $S_i (i = 1, \cdots, n)$ compose an orthonormal basis of R^n, then $x(k)$ can be represented as

$$x(k) = \sum_{i=1}^{n} z_i(k) S_i \tag{4}$$

for $k \geq 0$, where $z_i(k)$ are some functions. By Lemma 1,

$$x(k) = \sum_{i=1}^{n} z_i(k) S_i \in B, k \geq 0.$$

Then,

$$Wx(k) = \sum_{i=1}^{n} z_i(k) W S_i = \sum_{i=1}^{n} z_i(k) \lambda_i S_i,$$

for $k \geq 0$.

Since

$$|\lambda_i| \leq 1, (1 \leq i \leq n)$$

and

$$\sum_{j=m+1}^{n} \frac{|\tilde{b}_j|}{1-\lambda_j} < 1,$$

clearly, $Wx(k) + b \in B$ for all $k \geq 0$.

Then, it follows from (1) that

$$z_i(k+1) = z_i(k), \quad (i = 1, \cdots, m) \tag{5}$$

and

$$z_i(k+1) = \lambda_i z_i(k) + \tilde{b}_i, \quad (i = m+1, \cdots, n) \tag{6}$$

for $k \geq 0$. Solving equations (5) and (6), it gives that

$$z_i(k) = \begin{cases} z_i(0), & 1 \leq i \leq m \\ \lambda_i^k z_i(0) + \sum_{r=1}^{k} \lambda_i^{r-1} \tilde{b_i}, & m+1 \leq i \leq n \end{cases}$$

for $t \geq 0$. Then, from (4), it follows that

$$\begin{aligned} x(k) &= \sum_{i=1}^{m} z_i(0) S_i \\ &\quad + \sum_{j=m+1}^{n} \left(\lambda_j^k z_j(0) + \sum_{r=1}^{k} \lambda_j^{r-1} \tilde{b_j}\right) S_j \end{aligned} \tag{7}$$

for $k \geq 0$.

Given any $\epsilon > 0$, choose a constant $\delta = \epsilon$, we have

$$\begin{aligned} dist\left(x(0), C\right) &= \min_{x^* \in C} \left\{\|x(0) - x^*\|\right\} \\ &= \min_{c_i \in R (1 \leq i \leq m)} \left\{\left\| \sum_{i=1}^{m} \left(z_i(0) - c_i\right) \cdot S_i \right.\right. \\ &\qquad \left.\left. + \sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j}\right) \cdot S_j \right\|\right\} \\ &= \min_{c_i \in R (1 \leq i \leq m)} \left\{P\right\} \end{aligned}$$

where

$$P = \sqrt{\sum_{i=1}^{m} \left(z_i(0) - c_i\right)^2 + \sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j}\right)^2}.$$

Clearly,

$$\min_{c_i \in R (1 \leq i \leq m)} \left\{P\right\} = \sqrt{\sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j}\right)^2}.$$

Then,

$$dist\left(x(0), C\right) = \sqrt{\sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j}\right)^2}.$$

If

$$dist\left(x(0), C\right) \leq \delta,$$

then, from (7) that

$$\begin{aligned} & dist\,(x(k), C) \\ = & \min_{x^* \in C} \{\|x(k) - x^*\|\} \\ = & \min_{c_i \in R(1 \le i \le m)} \left\{ \left\| \sum_{i=1}^{m} (z_i(0) - c_i) \cdot S_i \right. \right. \\ & + \sum_{j=m+1}^{n} \lambda_j^k z_j(0) \cdot S_j \\ & \left. \left. + \sum_{j=m+1}^{n} \left(\sum_{t=1}^{k} \lambda_j^{t-1} \tilde{b_j} - \frac{\tilde{b}_j}{1-\lambda_j} \right) \cdot S_j \right\| \right\} \\ = & \min_{c_i \in R(1 \le i \le m)} \{Q\} \end{aligned}$$

where

$$Q = \sqrt{\sum_{i=1}^{m} (z_i(0) - c_i)^2 + \sum_{j=m+1}^{n} \left(\lambda_j^k z_j(0) - \frac{\lambda_j^k \tilde{b}_j}{1-\lambda_j} \right)^2}.$$

It is easy to check that

$$\min_{c_i \in R(1 \le i \le m)} \{Q\} \le \sqrt{\sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j} \right)^2}.$$

Then,

$$\begin{aligned} & dist\,(x(k), C) \\ \le & \sqrt{\sum_{j=m+1}^{n} \left(z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j} \right)^2} \\ = & dist\,(x(0), C) \\ \le & \delta \\ = & \epsilon \end{aligned}$$

for all $k \ge 0$. By Definition 3, the set C is a continuous attractor of the network (1). The proof is completed.

Theorem 1 gives a representation for the continuous attractor of the network (1). By this representation, one can completely master the properties of the continuous attractors.

IV. SIMULATIONS

In this section, we will give some simulations to illustrate the continuous attractors theory established in last section. Let us first consider a three dimensional DCNN network.

Consider the three dimensional DCNN

$$x(k+1) = f\left(\begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix} x(k) \right) \qquad (8)$$

for $k \ge 0$. Clearly,

$$W = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$$

and

$$b = [0, 0, 0]^T.$$

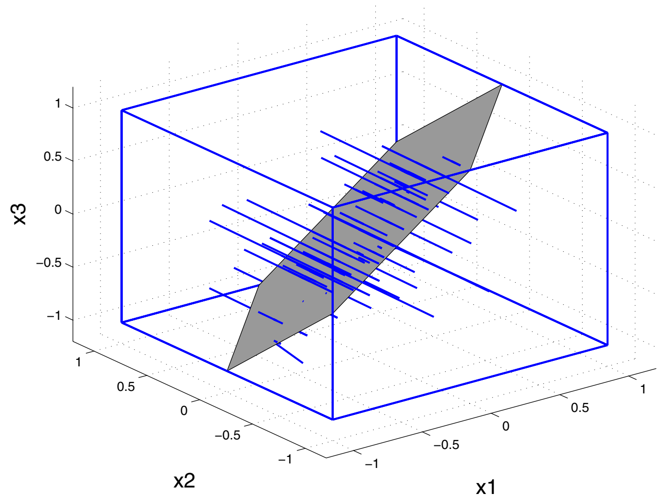

Fig. 2. Two dimensional continuous attractor of the network (8). It divides the three dimensional box into two parts. Trajectories are attracted to the continuous attractor.

It can be checked that W has the largest eigenvalue $\sigma = 1$ with multiplicity 2, and W has another eigenvalue -0.5. Moreover, $b \perp V_\sigma$. By Theorem 1, the network exists a continuous attractor. The continuous attractor is in two dimensional. Figure 2 shows this two dimensional continuous attractor of the network (8). The polygon in the figure represents the continuous attractor. It divides the three dimensional box into two parts. The figure shows that forty trajectories starting from randomly selected initial point are attracted to the attractor.

Next, let us consider a two dimensional DCNN

$$x(k+1) = f\left(\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \right) \qquad (9)$$

for $k \ge 0$. Clearly,

$$W = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}.$$

It can be checked that W has the largest eigenvalue $\sigma = 1$ with multiplicity 1, and it has another eigenvalue 0. Moreover, $b \perp V_\sigma$. By Theorem 1, the network (9) exists one dimensional continuous attractor. Figure 3 shows the continuous attractor of the network. The line in the figure is the continuous attractor. It divides the two dimensional square into two parts. The figure shows that fifty trajectories starting from randomly selected initial point are attracted to the attractor.

V. CONCLUSIONS

There are two ways of representing objects as attractors of a recurrent neural network dynamics: discrete attractors and continuous attractors. In this paper, we have studied the continuous attractor of DCNNs. Representations of continuous attractors are obtained. Such important results provide clear and complete descriptions for mastering continuous attractors. The methods presented in this paper could be further developed to study related problems of other class of neural networks. More research on this direction are required.

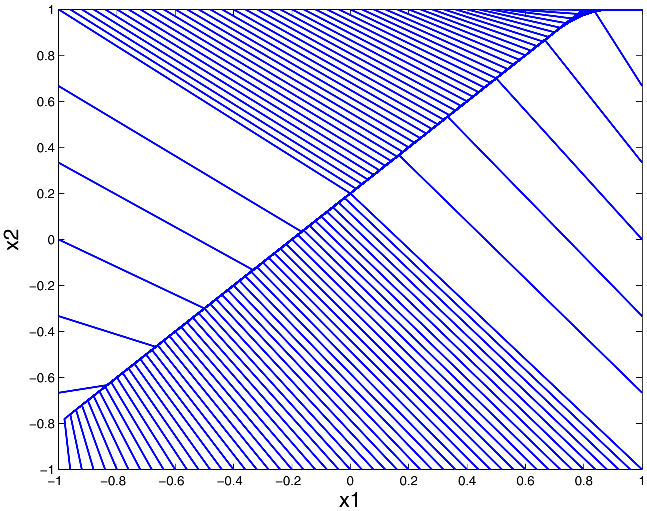

Fig. 3. One dimensional continuous attractor of the network (9). It divides the two dimensional square into two parts. Trajectories are attracted to the continuous attractor.

ACKNOWLEDGMENT

This work was supported by Chinese 863 High-Tech Program under Grant 2007AA01Z321.

REFERENCES

[1] L. O. Chua and L. Yang, "Cellular neural networks: theory", *IEEE Trans. Circuits Syst. I*, vol. 35, pp.1257-1272, 1988.

[2] L. O. Chua and L. Yang, "Cellular neural networks: application", *IEEE Trans. Circuits Syst. I*, vol. 35, pp.1273-1290, 1988.

[3] G. Grassi, "A new approach to design cellular neural networks for associative memories", *IEEE Trans. Circuits Syst. I*, vol. 44, no. 9, pp.835-838, 1998.

[4] G. Grassi, On discrete-time cellular neural networks for associative memories, *IEEE Trans. Circuits Syst. I*, vol. 48, no. 1, pp.107-111, 2001.

[5] R. Fantacci, M. Forti, M. Marini and L. Pancani, "Cellular neural networks approach to a class of comunication problems," *IEEE Trans. Circuits Syst. I*, vol. 46, no. 12, pp. 1457-1467, 1999.

[6] H. Yang and L. B. Yang, "Application of fuzzy cellular neural networks to Euclidean distance transformation," *IEEE Trans. Circuits Syst. I*, vol. 44, no. 3, pp. 242-246, 1997.

[7] P. Julian, R. Dogaru and L. O. Chua, "A piecewise-linear simplicial coupling cell for CNN gray-level image processing," *IEEE Trans. Circuits Syst. I*, vol. 49, no. 7, pp. 904-913, 2002.

[8] A. Zarandy, L. Orzo, E. Grawes and F. Werblin, "CNN-based models for color vision and visual illusions" *IEEE Trans. Circuits Syst. I*, vol. 46, no. 2, pp. 229-238, 1999.

[9] T. Sziranyi and J. Zerubia, "Markov random field image segmentation using cellular neural networks," *IEEE Trans. Circuits Syst. I*, vol. 44, no. 1, pp. 86-89, 1997.

[10] L. O. Chua and T. Roska, "The CNN paradigm," *IEEE Trans. Circuits Syst. I*, vol. 40, no. 3, pp. 147-156, 1993.

[11] K. R. Crounse and L. O. Chua, "Methods for image processing and pattern formation in Cellular Neural Networks: a tutorial," *IEEE Trans. Circuits Syst. I*, vol. 42, no. 10, pp. 583-601, 1995.

[12] Z. Yi and K. K. Tan, Convergence analysis of recurrent neural networks, *Kluwer Academic Publishers*, Boston, 2004.

[13] C. W. Wu and L. O. Chua, "Amore rigorous proof of complete stability of cellular neural networks," *IEEE Trans. Circuits Syst. I*, vol. 44, pp. 370-371, 1998.

[14] N. Takahashi and L. O. Chua, "On the complete stability of nonsymmetric cellular neural networks," *IEEE Trans. Circuits Syst. I*, vol. 45, pp. 754-758, 1998.

[15] S. Lin and C. Shih, "Complete stability for standard cellular neural networks," *Int. J. Bifurcation and Choas*, vol. 9, pp. 909-918, 1999.

[16] G. D. Sandre, "Stability of 1-D-CNNs with Dirichlet boundary conditions and global propagation dynamics," *IEEE Tran. Circuits Syst. I*, vol. 47, pp. 785-792, 2000.

[17] M. Forti and A. Tesi, "A new method to analyze complete stability of PWL cellular neural networks," *Int. J. Bifurcation and Choas*, vol. 11, pp. 655-676, 2001.

[18] M. Forti, "Some extensions of a new method to analyze complete stability of neural networks," *IEEE Trans. Neural Networks*, vol. 13, pp. 1230-1238, 2002.

[19] M. Di Marco, M. Forti and A. Tesi, "Complex dynamics in nearly symmetric three-cell cellular neural networks," *Int. J. Bifurcation and Choas*, vol. 12, pp. 1357-1362, 2002.

[20] X. Li, C. Ma and L. Huang, "Invariance principle and complete stability for cellular neural networks," *IEEE Trans. Circuits Syst. II*, vol. 53, pp. 202-206, 2006.

[21] D. J. Amit, "Modeling Brain Function: The World of Attractor Neural Networks," *Cambridge UP*, New York, 1989.

[22] H. Sebastian Seung, "How the brain keeps the eyes still," *Proc. Natl. Acad. Sci. USA*, vol. 93, pp.13339-13344, 1996.

[23] H. Sebastian Seung, "Continouous attractors and oculomotor control," *Neural Networks*, vol. 11, pp.1253-1258, 1998.

[24] D. D. Lee, B. Y. Reis, H. S. Seung, and D. W. Tank. "Nonlinear network models of the oculomotor integrator," *In Computational Neuroscience: Trends in Research*. New York, Plenum Press. 1997.

[25] H. Sebastian Seung, "Pattern analysis and synthesis in attractor neural networks," *IEEE Trans. Syst, Man, Cybern.*, vol. 15, no. 1, pp.116-132, 1985.

[26] H. Sebastian Seung and D. D. Lee, "The manifold ways of perception," *Science*, vol. 290, pp.2268-2269, 2000.

[27] H. Sebastian Seung, "Learning continuous attractors in recurrent networks," *Adv. Neural Info. Proc. Syst.*, vol. 10, pp.654-660, 1998.

[28] S. Wu and S. Amari, "Computing with continuous attractors: stability and online aspects," *Neural Computation*, vol. 17, pp.2215-2239, 2005.

[29] S. Wu and S. Amari, "Dynamics and Computation of continuous attractors," *Neural Computation*, in press.

[30] Paul Miller, "Analysis of Spike Statistics in Neuronal Systems with Continuous Attractors or Multiple, Discrete Attractor States," *Neural Computation*, vol. 18, pp.1268-1317, 2006.

[31] F.P. Battaglia and A. Treves, "Attractor neural networks storing multiple space representations: A model for hippocampal place fields," *Physical Review E* vol.58, pp. 7738-7753,1998.

[32] A. Samsonovich, B.L. McNaughton, "Path integration and cognitive mapping in a continuous attractor neural network model," *J. Neurosci*, vol.17, pp. 5900-5920,1997.

[33] V. M. Brea, D. L. Vilarino, A. Paasio and D. Cobello, "Design of the processing core of a mixed-signal CMOS DTCNN chip for pixellevel snakes," *IEEE Trans. Circuits Syst. I*, vol. 51, pp.997-1013, 2004.

[34] M. Brucoli, L. Carnimeo and G. Grassi, "A global approach to the design of discrete-time cellular neural networks for associative memories," *Int. J. Circuit Theory Appl.*, vol. 24, pp.489-510, 1996.

[35] M. Brucoli, L. Carnimeo and G. Grassi, "Discrete-time cellular neural networks for associative memories with learning and forgetting capabilities," *IEEE Trans. Circuits Syst. I*, vol. 42, pp.396-399, 1995.

[36] S. Chen and C. Shih, "Dynamics for discrete-time cellular neural networks," *Int. J. Bifurcation and Chaos* vol. 14, pp.2667-2687, 2004.

[37] H. Harrer and J. A. Nossek, "Discrete-time cellular neural networks," *Int. J. Circuit Theory Appl.* vol. 20, pp.453-467, 1992.

[38] N. Takahashi, T. Otake and M. Tanaka, "The template optimization of discrete-time CNN for image compression and reconstruction," *IEEE Int. Symposium Circuits and Syst.* vol. 1, pp.237-240, 2002.