Operational Separability Criteria and Matrix Reorderings

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***Abstract*—So far the positive partial transpose (PPT) criterion and the computable cross norm (CCN) criterion are two strongest operational separability criteria of bipartite mixed quantum states. In this paper, we exactly connect the PPT criterion and the CCN criterion with the matrix reorderings found by Poluikis and Hill [J. A. Poluikis and R. D. Hill, Linear Algebra and Its Applications 35: 1-10 (1981)] and Oxenrider and Hill [C. J. Oxenrider and R. D. Hill, Linear Algebra and Its Applications 69: 205-212 (1985)]. Motivated by work by Chen etc. [K. Chen, S. Albeverio, and S.-M. Fei, Physical Review Letters 95, 040504 (2005), Physical Review Letters 95, 210501 (2005)], we compute the trace norm of several bound entangled states and illuminate the equivalence among the elements of group generated by the eight matrix reorderings.**

***Keywords*—quantum computing, quantum entanglement, separability, positive partial transpose, computable cross norm, matrix reorderings, trace norm**

I. INTRODUCTION

Since the famous Einstein, Podolsky, and Rosen [1] and Schrödinger [2] papers, quantum entanglement still remains one of the most striking quantum phenomena. In recent years, great effort has been made to understand the role of entanglement in quantum information processing [3-13]. However, some fundamental problems about quantum entanglement such as entanglement criteria and entanglement measures are still far from being solved completely, especially in the context of multipartite systems.

Despite considerable effort, the operational characterization of separable states is still an open problem. There are simple, efficiently computable, tests that can establish the entanglement of a large subset of states. The most famous of these criteria is the positive partial transpose (PPT) criterion [14, 15]. This simply requires making an appropriate rearrangement of the matrix elements, corresponding to transposing one of the parties, and checking that the resulting matrix is positive. Nevertheless, since 1996 when Peres designed the PPT criterion, no better computable separability criterion has been provided for a long time. Only recently a new operational separability criterion, i.e., the computable cross norm (CCN) criterion or the realignment criterion, was found by Rudolph in [16] and Chen and Wu in [17]. This method has recently been developed further in [18-31]. It is independent of the PPT criterion. As the PPT criterion, the CCN criterion also requires making an appropriate rearrangement of the matrix elements, and checking whether the trace norm of the resulting matrix is larger than one or not. Both the PPT criterion and the CCN criterion require making an appropriate rearrangement of the matrix elements. Naturally, one question was raised: what is the relationship between the rearrangement of the matrix elements of the PPT criterion and that of the CCN criterion?

From the point of view of mathematics, in fact, the similar rearrangements of the matrix elements, i.e., the matrix reorderings, were previously studied in the context of general matrix algebra by Poluikis and Hill in [32] and Oxenrider and Hill in [33]. In [33], Oxenrider and Hill discussed eight natural matrix reorderings and explored some of their algebraic properties and their properties as linear transformations. In the case of the PPT and CCN criteria are concerned, as stated in section 5 below, they just corresponding with two elements of the group generated by the eight matrix reorderings. In section 4 below, we shall know that except for the elements corresponding the PPT and CCN criteria, there are still twenty-two different matrix reorderings in the group generated by the eight natural matrix reorderings. Then another question was raised: Whether other elements of the group are independent of the two elements corresponding to the PPT and CCN criteria or not? If yes, then we shall obtain new operational separability criteria. Unfortunately, all elements of the group are equivalent to either the elements corresponding to the PPT or CCN criteria or identity. Therefore, there is no new operational separability criteria in this framework.

From the point of view of physics, as Nielsen stated in [34], "…what we really want is deep theorems connecting measures of entanglement in surprising ways to other problems in quantum information sciences.…At the present time I believe it is fair to say that few deep results connecting measures of entanglement to other problems are known." In [25] and [26], Chen, Albeverio, and Fei found an essential quantitative relation between entanglement measures, i.e., concurrence and entanglement of formation, and available strong separability criteria, i.e., the PPT criterion and the CCN criterion, respectively. Motivated by Ref. [25-26], we compute trace norms of four examples of bound entangled states [35-38] to illuminate equivalence in $C^3 \otimes C^3$ composite quantum systems.

Although the Horodeckis family [27] and Życzkowski and Bengtsson [39] once noted that the PPT and CCN criteria are part of a large group of matrix reorderings studied in the 1980s, the present paper makes this comparison very explicit from the point of view of general matrix algebra.

The remainder of the paper is organized as follows. In sections 2 and 3, the PPT and CCN criteria are presented, respectively. Section 4 is devoted to the matrix reorderings. In section 5, the relationship between the PPT and CCN criteria and the matrix reorderings are obtained. In section 6, we give four known bound entangled states and compute the trace norm of all matrix reorderings to illuminate the equivalence in $C^3 \otimes C^3$ composite quantum systems. Finally, in section 7, we leave one open problem for further work.

II. THE PPT CRITERION IN $C^M \otimes C^N$ COMPOSITE QUANTUM SYSTEMS

In the case of $C^M \otimes C^N$ composite quantum systems, Peres and the Horodeckis family gave the connection between the partial transposes of density matrices with separability of bipartite mixed quantum states in [14] and [15]. In the following, we recall the main result.

If the $M \times N$ state ρ can be written as

$$\rho = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{21} & A_{22} & \cdots & A_{2M} \\ \vdots & \vdots & \vdots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MM} \end{pmatrix},$$

with $N \times N$ matrices A_{mn} acting on the second (C^N) space. They are defined by their matrix elements as $\{A_{mn}\}_{\mu\nu} \equiv \rho_{m\nu,n\mu}$. Then the partial transpositions ρ^{T_1} and ρ^{T_2} will be realized simply by commutation and transposition of all of these matrices A_{mn}, respectively, namely,

$$\rho^{T_1} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{M1} \\ A_{12} & A_{22} & \cdots & A_{M2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1M} & A_{2M} & \cdots & A_{MM} \end{pmatrix}, \quad \rho^{T_2} = \begin{pmatrix} A_{11}^T & A_{12}^T & \cdots & A_{1M}^T \\ A_{21}^T & A_{22}^T & \cdots & A_{2M}^T \\ \vdots & \vdots & \vdots & \vdots \\ A_{M1}^T & A_{M2}^T & \cdots & A_{MM}^T \end{pmatrix}.$$

The PPT Criterion [14, 15]

If an $m \times n$ bipartite density matrix ρ_{AB} is separable, then

$$\|\rho^{T_1}\|_1 \le 1 \text{ (or } \|\rho^{T_2}\|_1 \le 1).$$

In the following we give an example in $C^M \otimes C^N$ composite quantum systems to display the concise and explicit expression of the partial transposes of density matrices.

Example 2.1. The Horodecki 3 ⊗ 3 bound entangled state [35]

The density matrix ρ_a is

$$\rho_a = \frac{1}{8a+1} \times \left[\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+a) & 0 & \frac{1}{2}\sqrt{1-a^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{1}{2}\sqrt{1-a^2} & 0 & \frac{1}{2}(1+a) \end{array}\right],$$

where $0 \le a \le 1$. ρ_a can be written as $\rho_a = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$, where A_{ij} are 3×3 matrices. Then we have

$$\rho_a^{T_1} = \rho_a^{T_2} = \frac{1}{8a+1} \times \left[\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & a & 0 & 0 \\ \hline 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ \hline 0 & 0 & a & 0 & 0 & 0 & \frac{1}{2}(1+a) & 0 & \frac{1}{2}\sqrt{1-a^2} \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{1-a^2} & 0 & \frac{1}{2}(1+a) \end{array}\right].$$

III. THE CCN CRITERION IN $C^M \otimes C^N$ COMPOSITE QUANTUM SYSTEMS

In the case of $C^M \otimes C^N$ composite quantum systems, Rudolph and Chen and Wu obtain the computable cross norm (CCN) criterion or the realignment criterion in [16] and [17]. We recall the main result below.

If the $M \times N$ state ρ can be written as

$$\rho = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{21} & A_{22} & \cdots & A_{2M} \\ \vdots & \vdots & \vdots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MM} \end{pmatrix},$$

with $N \times N$ matrices A_{mn} acting on the second (C^N) space. They are defined by their matrix elements as $\{A_{mn}\}_{\mu\nu} \equiv \rho_{m\nu,n\mu}$. Then the realignment matrix $\tilde{\rho}$ will be realized simply by commutation and transposition of all of these matrices A_{mn}, namely,

$$\tilde{\rho} = \begin{pmatrix} vec(A_{11})^T \\ \vdots \\ vec(A_{M1})^T \\ \vdots \\ vec(A_{1M})^T \\ \vdots \\ vec(A_{MM})^T \end{pmatrix}.$$

The CCN Criterion [16, 17]

If an $m \times n$ bipartite density matrix ρ_{AB} is separable, then

$$\|\tilde{\rho}\|_1 \le 1.$$

In the following we give an example in $C^M \otimes C^N$ composite quantum systems to display the concise and explicit expression of the realignment of density matrices.

Example 3.1. The Horodecki 3 ⊗ 3 bound entangled state [35]

The density matrix ρ_a is same as **Example 2.1**.

Then we have

$$\tilde{\rho}_a = \frac{1}{8a+1} \times \left[\begin{array}{ccccccccc} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ \hline 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ \hline a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ \hline 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ \hline \frac{1}{2}(1+a) & 0 & \frac{1}{2}\sqrt{1-a^2} & 0 & a & 0 & \frac{1}{2}\sqrt{1-a^2} & 0 & \frac{1}{2}(1+a) \end{array}\right].$$

IV. THE MATRIX REORDERINGS IN $C^M \otimes C^N$ COMPOSITE QUANTUM SYSTEMS

In [33], Oxenrider and Hill discussed eight natural matrix reorderings. First, Let us recall the notation of the eight natural matrix reorderings as follows:

Given positive integers t, u, let $S = \{(i, j): i = 1,\ldots, t; j = 1, \ldots, u\}$; endow S with two orderings, viz., the lexicographical ordering

$$(i, j) < (r, s) \quad \text{iff} \quad i < r \text{ or } (i = r \text{ and } j < s)$$

and the antilexicographical ordering

$$(i, j) < (r, s) \quad \text{iff} \quad j < s \text{ or } (j = s \text{ and } i < r).$$

Corresponding to each of these orderings we have bijections from the set of ordered pairs S onto $\{1, \ldots, tu\}$ defined by $[i, j] = (i-1)\, u + j$ and $<i, j> = (j-1)\, t + i$.

Eight reorderings naturally appear:

$$\Gamma(T)_{rs}^{ij} = t_{[i,j],[r,s]}, \ \Upsilon(T)_{rs}^{ij} = t_{<i,j>,[r,s]},$$

$$\Xi(T)_{rs}^{ij} = t_{[i,j],<r,s>}, \ \Theta(T)_{rs}^{ij} = t_{<i,j>,<r,s>},$$

where $i, j = 1,\ldots, q$; $r, s = 1, \ldots, n$, and

$$\Psi(T)_{rs}^{ij} = t_{<r,s>,<i,j>}, \ \Lambda(T)_{rs}^{ij} = t_{<r,s>,[i,j]},$$

$$\Delta(T)_{rs}^{ij} = t_{[r,s],<i,j>}, \ \Omega(T)_{rs}^{ij} = t_{[r,s],[i,j]},$$

where $i, j = 1,\ldots, n$; $r, s = 1, \ldots, q$.

Now we will present these results for density matrices in $C^3 \otimes C^3$ composite quantum systems to display the concise and explicit expression of the matrix reorderings as follows:

Example 4.1. Let ρ be a $3 \otimes 3$ bipartite density matrix as follows:

$$\rho = \left[\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} & \rho_{17} & \rho_{18} & \rho_{19} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} & \rho_{27} & \rho_{28} & \rho_{29} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{36} & \rho_{37} & \rho_{38} & \rho_{39} \\ \hline \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & \rho_{46} & \rho_{47} & \rho_{48} & \rho_{49} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & \rho_{56} & \rho_{57} & \rho_{58} & \rho_{59} \\ \rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & \rho_{66} & \rho_{67} & \rho_{68} & \rho_{69} \\ \hline \rho_{71} & \rho_{72} & \rho_{73} & \rho_{74} & \rho_{75} & \rho_{76} & \rho_{77} & \rho_{78} & \rho_{79} \\ \rho_{81} & \rho_{82} & \rho_{83} & \rho_{84} & \rho_{85} & \rho_{86} & \rho_{87} & \rho_{88} & \rho_{89} \\ \rho_{91} & \rho_{92} & \rho_{93} & \rho_{94} & \rho_{95} & \rho_{96} & \rho_{97} & \rho_{98} & \rho_{99} \end{array}\right].$$

Then the eight natural matrix reorderings are

$$\Gamma(\rho) = \left[\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{21} & \rho_{22} & \rho_{23} & \rho_{31} & \rho_{32} & \rho_{33} \\ \rho_{14} & \rho_{15} & \rho_{16} & \rho_{24} & \rho_{25} & \rho_{26} & \rho_{34} & \rho_{35} & \rho_{36} \\ \rho_{17} & \rho_{18} & \rho_{19} & \rho_{27} & \rho_{28} & \rho_{29} & \rho_{37} & \rho_{38} & \rho_{39} \\ \hline \rho_{41} & \rho_{42} & \rho_{43} & \rho_{51} & \rho_{52} & \rho_{53} & \rho_{61} & \rho_{62} & \rho_{63} \\ \rho_{44} & \rho_{45} & \rho_{46} & \rho_{54} & \rho_{55} & \rho_{56} & \rho_{64} & \rho_{65} & \rho_{66} \\ \rho_{47} & \rho_{48} & \rho_{49} & \rho_{57} & \rho_{58} & \rho_{59} & \rho_{67} & \rho_{68} & \rho_{69} \\ \hline \rho_{71} & \rho_{72} & \rho_{73} & \rho_{81} & \rho_{82} & \rho_{83} & \rho_{91} & \rho_{92} & \rho_{93} \\ \rho_{74} & \rho_{75} & \rho_{76} & \rho_{84} & \rho_{85} & \rho_{86} & \rho_{94} & \rho_{95} & \rho_{96} \\ \rho_{77} & \rho_{78} & \rho_{79} & \rho_{87} & \rho_{88} & \rho_{89} & \rho_{97} & \rho_{98} & \rho_{99} \end{array}\right];$$

$$\Theta(\rho) = \left[\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{14} & \rho_{17} & \rho_{41} & \rho_{44} & \rho_{47} & \rho_{71} & \rho_{74} & \rho_{77} \\ \rho_{12} & \rho_{15} & \rho_{18} & \rho_{42} & \rho_{45} & \rho_{48} & \rho_{72} & \rho_{75} & \rho_{78} \\ \rho_{13} & \rho_{16} & \rho_{19} & \rho_{43} & \rho_{46} & \rho_{49} & \rho_{73} & \rho_{76} & \rho_{79} \\ \hline \rho_{21} & \rho_{24} & \rho_{27} & \rho_{51} & \rho_{54} & \rho_{57} & \rho_{81} & \rho_{84} & \rho_{87} \\ \rho_{22} & \rho_{25} & \rho_{28} & \rho_{52} & \rho_{55} & \rho_{58} & \rho_{82} & \rho_{85} & \rho_{88} \\ \rho_{23} & \rho_{26} & \rho_{29} & \rho_{53} & \rho_{56} & \rho_{59} & \rho_{83} & \rho_{86} & \rho_{89} \\ \hline \rho_{31} & \rho_{34} & \rho_{37} & \rho_{61} & \rho_{64} & \rho_{67} & \rho_{91} & \rho_{94} & \rho_{97} \\ \rho_{32} & \rho_{35} & \rho_{38} & \rho_{62} & \rho_{65} & \rho_{68} & \rho_{92} & \rho_{95} & \rho_{98} \\ \rho_{33} & \rho_{36} & \rho_{39} & \rho_{63} & \rho_{66} & \rho_{69} & \rho_{93} & \rho_{96} & \rho_{99} \end{array}\right];$$

$$\Omega(\rho) = \left[\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{21} & \rho_{31} & \rho_{12} & \rho_{22} & \rho_{32} & \rho_{13} & \rho_{23} & \rho_{33} \\ \rho_{41} & \rho_{51} & \rho_{61} & \rho_{42} & \rho_{52} & \rho_{62} & \rho_{43} & \rho_{53} & \rho_{63} \\ \rho_{71} & \rho_{81} & \rho_{91} & \rho_{72} & \rho_{82} & \rho_{92} & \rho_{73} & \rho_{83} & \rho_{93} \\ \hline \rho_{14} & \rho_{24} & \rho_{34} & \rho_{15} & \rho_{25} & \rho_{35} & \rho_{16} & \rho_{26} & \rho_{36} \\ \rho_{44} & \rho_{54} & \rho_{64} & \rho_{45} & \rho_{55} & \rho_{65} & \rho_{46} & \rho_{56} & \rho_{66} \\ \rho_{74} & \rho_{84} & \rho_{94} & \rho_{75} & \rho_{85} & \rho_{95} & \rho_{76} & \rho_{86} & \rho_{96} \\ \hline \rho_{17} & \rho_{27} & \rho_{37} & \rho_{18} & \rho_{28} & \rho_{38} & \rho_{19} & \rho_{29} & \rho_{39} \\ \rho_{47} & \rho_{57} & \rho_{67} & \rho_{48} & \rho_{58} & \rho_{68} & \rho_{49} & \rho_{59} & \rho_{69} \\ \rho_{77} & \rho_{87} & \rho_{97} & \rho_{78} & \rho_{88} & \rho_{98} & \rho_{79} & \rho_{89} & \rho_{99} \end{array}\right];$$

$$\Psi(\rho) = \left[\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{41} & \rho_{71} & \rho_{14} & \rho_{44} & \rho_{74} & \rho_{17} & \rho_{47} & \rho_{77} \\ \rho_{21} & \rho_{51} & \rho_{81} & \rho_{24} & \rho_{54} & \rho_{84} & \rho_{27} & \rho_{57} & \rho_{87} \\ \rho_{31} & \rho_{61} & \rho_{91} & \rho_{34} & \rho_{64} & \rho_{94} & \rho_{37} & \rho_{67} & \rho_{97} \\ \hline \rho_{12} & \rho_{42} & \rho_{72} & \rho_{15} & \rho_{45} & \rho_{75} & \rho_{18} & \rho_{48} & \rho_{78} \\ \rho_{22} & \rho_{52} & \rho_{82} & \rho_{25} & \rho_{55} & \rho_{85} & \rho_{28} & \rho_{58} & \rho_{88} \\ \rho_{32} & \rho_{62} & \rho_{92} & \rho_{35} & \rho_{65} & \rho_{95} & \rho_{38} & \rho_{68} & \rho_{98} \\ \hline \rho_{13} & \rho_{43} & \rho_{73} & \rho_{16} & \rho_{46} & \rho_{76} & \rho_{19} & \rho_{49} & \rho_{79} \\ \rho_{23} & \rho_{53} & \rho_{83} & \rho_{26} & \rho_{56} & \rho_{86} & \rho_{29} & \rho_{59} & \rho_{89} \\ \rho_{33} & \rho_{63} & \rho_{93} & \rho_{36} & \rho_{66} & \rho_{96} & \rho_{39} & \rho_{69} & \rho_{99} \end{array}\right];$$

$$\Upsilon(\rho) = \left[\begin{array}{ccc|ccc|ccc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{41} & \rho_{42} & \rho_{43} & \rho_{71} & \rho_{72} & \rho_{73} \\ \rho_{14} & \rho_{15} & \rho_{16} & \rho_{44} & \rho_{45} & \rho_{46} & \rho_{74} & \rho_{75} & \rho_{76} \\ \rho_{17} & \rho_{18} & \rho_{19} & \rho_{47} & \rho_{48} & \rho_{49} & \rho_{77} & \rho_{78} & \rho_{79} \\ \hline \rho_{21} & \rho_{22} & \rho_{23} & \rho_{51} & \rho_{52} & \rho_{53} & \rho_{81} & \rho_{82} & \rho_{83} \\ \rho_{24} & \rho_{25} & \rho_{26} & \rho_{54} & \rho_{55} & \rho_{56} & \rho_{84} & \rho_{85} & \rho_{86} \\ \rho_{27} & \rho_{28} & \rho_{29} & \rho_{57} & \rho_{58} & \rho_{59} & \rho_{87} & \rho_{88} & \rho_{89} \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{61} & \rho_{62} & \rho_{63} & \rho_{91} & \rho_{92} & \rho_{93} \\ \rho_{34} & \rho_{35} & \rho_{36} & \rho_{64} & \rho_{65} & \rho_{66} & \rho_{94} & \rho_{95} & \rho_{96} \\ \rho_{37} & \rho_{38} & \rho_{39} & \rho_{67} & \rho_{68} & \rho_{69} & \rho_{97} & \rho_{98} & \rho_{99} \end{array}\right];$$

$$\Xi(\rho)=\left[\begin{array}{ccc|ccc|ccc}
\rho_{11} & \rho_{14} & \rho_{17} & \rho_{21} & \rho_{24} & \rho_{27} & \rho_{31} & \rho_{34} & \rho_{37} \\
\rho_{12} & \rho_{15} & \rho_{18} & \rho_{22} & \rho_{25} & \rho_{28} & \rho_{32} & \rho_{35} & \rho_{38} \\
\rho_{13} & \rho_{16} & \rho_{19} & \rho_{23} & \rho_{26} & \rho_{29} & \rho_{33} & \rho_{36} & \rho_{39} \\ \hline
\rho_{41} & \rho_{44} & \rho_{47} & \rho_{51} & \rho_{54} & \rho_{57} & \rho_{61} & \rho_{64} & \rho_{67} \\
\rho_{42} & \rho_{45} & \rho_{48} & \rho_{52} & \rho_{55} & \rho_{58} & \rho_{62} & \rho_{65} & \rho_{68} \\
\rho_{43} & \rho_{46} & \rho_{49} & \rho_{53} & \rho_{56} & \rho_{59} & \rho_{63} & \rho_{66} & \rho_{69} \\ \hline
\rho_{71} & \rho_{74} & \rho_{77} & \rho_{81} & \rho_{84} & \rho_{87} & \rho_{91} & \rho_{94} & \rho_{97} \\
\rho_{72} & \rho_{75} & \rho_{78} & \rho_{82} & \rho_{85} & \rho_{88} & \rho_{92} & \rho_{95} & \rho_{98} \\
\rho_{73} & \rho_{76} & \rho_{79} & \rho_{83} & \rho_{86} & \rho_{89} & \rho_{93} & \rho_{96} & \rho_{99}
\end{array}\right];$$

$$\Delta(\rho)=\left[\begin{array}{ccc|ccc|ccc}
\rho_{11} & \rho_{21} & \rho_{31} & \rho_{14} & \rho_{24} & \rho_{34} & \rho_{17} & \rho_{27} & \rho_{37} \\
\rho_{41} & \rho_{51} & \rho_{61} & \rho_{44} & \rho_{54} & \rho_{64} & \rho_{47} & \rho_{57} & \rho_{67} \\
\rho_{71} & \rho_{81} & \rho_{91} & \rho_{74} & \rho_{84} & \rho_{94} & \rho_{77} & \rho_{87} & \rho_{97} \\ \hline
\rho_{12} & \rho_{22} & \rho_{32} & \rho_{15} & \rho_{25} & \rho_{35} & \rho_{18} & \rho_{28} & \rho_{38} \\
\rho_{42} & \rho_{52} & \rho_{62} & \rho_{45} & \rho_{55} & \rho_{65} & \rho_{48} & \rho_{58} & \rho_{68} \\
\rho_{72} & \rho_{82} & \rho_{92} & \rho_{75} & \rho_{85} & \rho_{95} & \rho_{78} & \rho_{88} & \rho_{98} \\ \hline
\rho_{13} & \rho_{23} & \rho_{33} & \rho_{16} & \rho_{26} & \rho_{36} & \rho_{19} & \rho_{29} & \rho_{39} \\
\rho_{43} & \rho_{53} & \rho_{63} & \rho_{46} & \rho_{56} & \rho_{66} & \rho_{49} & \rho_{59} & \rho_{69} \\
\rho_{73} & \rho_{83} & \rho_{93} & \rho_{76} & \rho_{86} & \rho_{96} & \rho_{79} & \rho_{89} & \rho_{99}
\end{array}\right];$$

$$\Lambda(\rho)=\left[\begin{array}{ccc|ccc|ccc}
\rho_{11} & \rho_{41} & \rho_{71} & \rho_{12} & \rho_{42} & \rho_{72} & \rho_{13} & \rho_{43} & \rho_{73} \\
\rho_{21} & \rho_{51} & \rho_{81} & \rho_{22} & \rho_{52} & \rho_{82} & \rho_{23} & \rho_{53} & \rho_{83} \\
\rho_{31} & \rho_{61} & \rho_{91} & \rho_{32} & \rho_{62} & \rho_{92} & \rho_{33} & \rho_{63} & \rho_{93} \\ \hline
\rho_{14} & \rho_{44} & \rho_{74} & \rho_{15} & \rho_{45} & \rho_{75} & \rho_{16} & \rho_{46} & \rho_{76} \\
\rho_{24} & \rho_{54} & \rho_{84} & \rho_{25} & \rho_{55} & \rho_{85} & \rho_{26} & \rho_{56} & \rho_{86} \\
\rho_{34} & \rho_{64} & \rho_{94} & \rho_{35} & \rho_{65} & \rho_{95} & \rho_{36} & \rho_{66} & \rho_{96} \\ \hline
\rho_{17} & \rho_{47} & \rho_{77} & \rho_{18} & \rho_{48} & \rho_{78} & \rho_{19} & \rho_{49} & \rho_{79} \\
\rho_{27} & \rho_{57} & \rho_{87} & \rho_{28} & \rho_{58} & \rho_{88} & \rho_{29} & \rho_{59} & \rho_{89} \\
\rho_{37} & \rho_{67} & \rho_{97} & \rho_{38} & \rho_{68} & \rho_{98} & \rho_{39} & \rho_{69} & \rho_{99}
\end{array}\right].$$

In [33], Oxenrider and Hill observed that Γ, Θ, Ω, Ψ, Υ, Ξ, Δ, and Λ do not preserve the matrix properties of determinant, rank, and trace. However, the eight natural matrix reorderings do generate the transpose operator, which preserves all of the above properties and types of matrices. Moreover, under composition they generate a 24-element group, all members of which preserve the inner product and thus the corresponding metric properties.

Observation 4.1. For any $m \otimes n$ bipartite density matrix ρ_{AB}, if both dimensions *m* and *n* are squares of nature numbers, and $\rho_{AB}^{T}=\rho_{AB}$, then the eight natural matrix reorderings, i.e., $\Gamma(\rho_{AB})$, $\Theta(\rho_{AB})$, $\Omega(\rho_{AB})$, $\Psi(\rho_{AB})$, $\Upsilon(\rho_{AB})$, $\Xi(\rho_{AB})$, $\Delta(\rho_{AB})$, and $\Lambda(\rho_{AB})$, generate a 12-element group under composition, and all members of which preserve the inner product and thus the corresponding metric properties.

V. The Relationship Among the PPT Criterion, the CCN Criterion, and the Matrix Reorderings

In the case of the above $C^3 \otimes C^3$ composite quantum systems, we can rewrite the partial transpositions ρ^{T_1}, ρ^{T_2} and the realignment matrix $\widetilde{\rho}$ as follows:

$$\rho^{T_1}=\left[\begin{array}{ccc|ccc|ccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{41} & \rho_{42} & \rho_{43} & \rho_{71} & \rho_{72} & \rho_{73} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{51} & \rho_{52} & \rho_{53} & \rho_{81} & \rho_{82} & \rho_{83} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{61} & \rho_{62} & \rho_{63} & \rho_{91} & \rho_{92} & \rho_{93} \\ \hline
\rho_{14} & \rho_{15} & \rho_{16} & \rho_{44} & \rho_{45} & \rho_{46} & \rho_{74} & \rho_{75} & \rho_{76} \\
\rho_{24} & \rho_{25} & \rho_{26} & \rho_{54} & \rho_{55} & \rho_{56} & \rho_{84} & \rho_{85} & \rho_{86} \\
\rho_{34} & \rho_{35} & \rho_{36} & \rho_{64} & \rho_{65} & \rho_{66} & \rho_{94} & \rho_{95} & \rho_{96} \\ \hline
\rho_{17} & \rho_{18} & \rho_{19} & \rho_{47} & \rho_{48} & \rho_{49} & \rho_{77} & \rho_{78} & \rho_{79} \\
\rho_{27} & \rho_{28} & \rho_{29} & \rho_{57} & \rho_{58} & \rho_{59} & \rho_{87} & \rho_{88} & \rho_{89} \\
\rho_{37} & \rho_{38} & \rho_{39} & \rho_{67} & \rho_{68} & \rho_{69} & \rho_{97} & \rho_{98} & \rho_{99}
\end{array}\right];$$

$$\rho^{T_2}=\left[\begin{array}{ccc|ccc|ccc}
\rho_{11} & \rho_{21} & \rho_{31} & \rho_{14} & \rho_{24} & \rho_{34} & \rho_{17} & \rho_{27} & \rho_{37} \\
\rho_{12} & \rho_{22} & \rho_{32} & \rho_{15} & \rho_{25} & \rho_{35} & \rho_{18} & \rho_{28} & \rho_{38} \\
\rho_{13} & \rho_{23} & \rho_{33} & \rho_{16} & \rho_{26} & \rho_{36} & \rho_{19} & \rho_{29} & \rho_{39} \\ \hline
\rho_{41} & \rho_{51} & \rho_{61} & \rho_{44} & \rho_{54} & \rho_{64} & \rho_{47} & \rho_{57} & \rho_{67} \\
\rho_{42} & \rho_{52} & \rho_{62} & \rho_{45} & \rho_{55} & \rho_{65} & \rho_{48} & \rho_{58} & \rho_{68} \\
\rho_{43} & \rho_{53} & \rho_{63} & \rho_{46} & \rho_{56} & \rho_{66} & \rho_{49} & \rho_{59} & \rho_{69} \\ \hline
\rho_{71} & \rho_{81} & \rho_{91} & \rho_{74} & \rho_{84} & \rho_{94} & \rho_{77} & \rho_{87} & \rho_{97} \\
\rho_{72} & \rho_{82} & \rho_{92} & \rho_{75} & \rho_{85} & \rho_{95} & \rho_{78} & \rho_{88} & \rho_{98} \\
\rho_{73} & \rho_{83} & \rho_{93} & \rho_{76} & \rho_{86} & \rho_{96} & \rho_{79} & \rho_{89} & \rho_{99}
\end{array}\right];$$

$$\widetilde{\rho}=\left[\begin{array}{ccc|ccc|ccc}
\rho_{11} & \rho_{21} & \rho_{31} & \rho_{12} & \rho_{22} & \rho_{32} & \rho_{13} & \rho_{23} & \rho_{33} \\
\rho_{41} & \rho_{51} & \rho_{61} & \rho_{42} & \rho_{52} & \rho_{62} & \rho_{43} & \rho_{53} & \rho_{63} \\
\rho_{71} & \rho_{81} & \rho_{91} & \rho_{72} & \rho_{82} & \rho_{92} & \rho_{73} & \rho_{83} & \rho_{93} \\ \hline
\rho_{14} & \rho_{24} & \rho_{34} & \rho_{15} & \rho_{25} & \rho_{35} & \rho_{16} & \rho_{26} & \rho_{36} \\
\rho_{44} & \rho_{54} & \rho_{64} & \rho_{45} & \rho_{55} & \rho_{65} & \rho_{46} & \rho_{56} & \rho_{66} \\
\rho_{74} & \rho_{84} & \rho_{94} & \rho_{75} & \rho_{85} & \rho_{95} & \rho_{76} & \rho_{86} & \rho_{96} \\ \hline
\rho_{17} & \rho_{27} & \rho_{37} & \rho_{18} & \rho_{28} & \rho_{38} & \rho_{19} & \rho_{29} & \rho_{39} \\
\rho_{47} & \rho_{57} & \rho_{67} & \rho_{48} & \rho_{58} & \rho_{68} & \rho_{49} & \rho_{59} & \rho_{69} \\
\rho_{77} & \rho_{87} & \rho_{97} & \rho_{78} & \rho_{88} & \rho_{98} & \rho_{79} & \rho_{89} & \rho_{99}
\end{array}\right].$$

In the following **Observations 5.1 - 5.5**, we shall give the relationship among the PPT criterion, the CCN criterion, and the matrix reorderings:

Observation 5.1. For any $m \otimes n$ bipartite density matrix ρ_{AB}, if both dimensions *m* and *n* are squares of nature numbers, then we have

$$\rho_{AB}^{T_1}=\Upsilon\Gamma(\rho_{AB})=\Xi\Omega(\rho_{AB})=\Delta\Theta(\rho_{AB})=\Lambda\Psi(\rho_{AB})\ .$$

Observation 5.2. For any $m \otimes n$ bipartite density matrix ρ_{AB}, if both dimensions *m* and *n* are squares of nature numbers, then we have

$$\rho_{AB}^{T_2}=\Xi\Gamma(\rho_{AB})=\Upsilon\Omega(\rho_{AB})=\Lambda\Theta(\rho_{AB})=\Delta\Psi(\rho_{AB})\ .$$

Observation 5.3. For any $m \otimes n$ bipartite density matrix ρ_{AB}, if both dimensions *m* and *n* are squares of nature numbers, then we have

$$\widetilde{\rho}=\Omega(\rho_{AB})\ .$$

Observation 5.4. For any $m \otimes n$ bipartite density matrix ρ_{AB}, if both dimensions *m* and *n* are squares of nature numbers, then we have

$$\rho_{AB}^{T}=\Theta\Gamma(\rho_{AB})=\Gamma\Omega(\rho_{AB})=\Psi\Theta(\rho_{AB})=\Omega\Psi(\rho_{AB})\ .$$

Observation 5.5. For any $m \otimes n$ bipartite density matrix ρ_{AB}, if both dimensions m and n are squares of nature numbers, then we have

$$\rho_{AB} = \Omega\Theta(\rho_{AB}) = \Theta\Omega(\rho_{AB}) = \Psi\Psi(\rho_{AB}) = \Gamma\Gamma(\rho_{AB}) .$$

VI. The Trace Norms in $C^3 \otimes C^3$ Composite Quantum Systems

Now we shall give four examples of bound entangled states and compute the trace norms of all matrix reorderings to illuminate the equivalence in $C^3 \otimes C^3$ composite quantum systems as follows:

Example 6.1. The Horodecki $3 \otimes 3$ bound entangled state [35]

Let $a = 0.8$, we have

$$\rho_{0.8} = \frac{5}{37}\left(\begin{array}{ccc|ccc|ccc} 0.8 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0.8 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0.8 & 0 & 0 & 0 & 0.8 & 0 & 0.3 & 0 & 0.9 \end{array}\right).$$

By a simple calculation we obtain

$$\left\|\rho_{0.8}^{T_1}\right\|_1 = 1 ,\ \left\|\tilde{\rho}_{0.8}\right\|_1 = 1.0009 ,\ \left\|\rho_{0.8}\right\|_1 = 1 .$$

The reminder matrix reorderings are equal to either 1 or 1.0009. Therefore, the entanglement of $\rho_{0.8}$ could be detected by the CCN criterion, not by the PPT criterion and all twelve elements of the group generated by all matrix reorderings are equivalent to either the elements corresponding to the PPT or CCN criteria or identity.

Example 6.2. The BDMSST $3 \otimes 3$ bound entangled state [36]

$$\rho = \frac{1}{72}\left(\begin{array}{ccc|ccc|ccc} 7 & 7 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ 7 & 7 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ -2 & -2 & 7 & -2 & -2 & 7 & -2 & -2 & -2 \\ \hline -2 & -2 & -2 & 7 & -2 & -2 & 7 & -2 & -2 \\ -2 & -2 & -2 & -2 & 16 & -2 & -2 & -2 & -2 \\ -2 & -2 & 7 & -2 & -2 & 7 & -2 & -2 & -2 \\ \hline -2 & -2 & -2 & 7 & -2 & -2 & 7 & -2 & -2 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 & 7 & 7 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 & 7 & 7 \end{array}\right).$$

By a routine calculation we obtain

$$\left\|\rho^{T_1}\right\|_1 = 1 ,\ \left\|\tilde{\rho}\right\|_1 = 1.0874 ,\ \left\|\rho\right\|_1 = 1 .$$

The reminder matrix reorderings are equal to either 1 or 1.0874. Hence the entanglement of ρ could be detected by the CCN criterion, not by the PPT criterion and all twelve elements of the group generated by all matrix reorderings are equivalent to either the elements corresponding to the PPT or CCN criteria or identity.

Example 6.3. The BP $3 \otimes 3$ bound entangled state [37, 30]

$$\rho_c = \frac{1}{12}\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

A simple calculation yields

$$\left\|\rho_c^{T_1}\right\|_1 = 1 ,\ \left\|\tilde{\rho}_c\right\|_1 = 1.1667 ,\ \left\|\rho_{AB}\right\|_1 = 1 .$$

The reminder matrix reorderings are equal to either 1 or 1.1667. Thus the entanglement of ρ_c could be detected by the CCN criterion, not by the PPT criterion and all twelve elements of the group generated by all matrix reorderings are equivalent to either the elements corresponding to the PPT or CCN criteria or identity.

Example 6.4. The GMTA $3 \otimes 3$ bound entangled state [38]

Let $p = 0.875$, we have

$$\rho = \frac{1}{576}\left(\begin{array}{ccc|ccc|ccc} 57 & 49 & -14 & -14 & -14 & -14 & -14 & -14 & -14 \\ 49 & 57 & -14 & -14 & -14 & -14 & -14 & -14 & -14 \\ -14 & -14 & 57 & -14 & -14 & 49 & -14 & -14 & -14 \\ \hline -14 & -14 & -14 & 57 & -14 & -14 & 49 & -14 & -14 \\ -14 & -14 & -14 & -14 & 120 & -14 & -14 & -14 & -14 \\ -14 & -14 & 49 & -14 & -14 & 57 & -14 & -14 & -14 \\ \hline -14 & -14 & -14 & 49 & -14 & -14 & 57 & -14 & -14 \\ -14 & -14 & -14 & -14 & -14 & -14 & -14 & 57 & 49 \\ -14 & -14 & -14 & -14 & -14 & -14 & -14 & 49 & 57 \end{array}\right).$$

A routine calculation yields

$$\left\|\rho^{T_1}\right\|_1 = 1 ,\ \left\|\tilde{\rho}\right\|_1 = 0.9884 ,\ \left\|\rho\right\|_1 = 1 .$$

The reminder matrix reorderings are equal to either 1 or 0.9884. Consequently, the entanglement of ρ could be detected neither by the CCN criterion, not by the PPT criterion and all twelve elements of the group generated by all matrix reorderings are equivalent to either the elements corresponding to the PPT or CCN criteria or identity.

VII. Conclusions

Operational characterization of entanglement is an open problem. So far, the strongest operational separability criteria are the positive partial transpose (PPT) criterion and the computable cross norm (CCN) criterion. Both of them are based on an appropriate rearrangement of the matrix elements from original density matrix. What is the relation between the rearrangement of the matrix elements of the PPT criterion and the CCN criterion? From the general matrix algebra point of view, we can connect them with the matrix reorderings. There are also some drawbacks for our discussion due to limitation of the matrix orderings found by Poluikis and Hill in [32] and Oxenrider and Hill in [33]. In this letter we mainly discuss the $m \times n$ bipartite density matrix ρ_{AB}, where both dimensions m and n are squares of nature numbers. One question is leave:

What is the relationship between the rearrangement of the matrix elements of the PPT criterion and the CCN criterion if either dimensions *m* or *n* is not square of nature number?

ACKNOWLEDGMENT

This work was supported by the Youth Science and Technology Foundation of UESTC and the National Fundamental Research Program of China (No. 2004CB318103).

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