Continuous Attractors of a Class of Recurrent Neural Networks without Lateral Inhibition

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***Abstract*—Researches on neural population coding have revealed that continuous stimuli, such as orientation, moving direction, and the spatial location of objects could be encoded as continuous attractors in neural networks. The dynamical behaviors of continuous attractors are interesting properties of recurrent neural networks. This paper proposes a class of recurrent neural networks without lateral inhibition. Since there is no general rule to determine the stability of the network without specifying the excitatory connections, individual conditions can be calculated analytically for some particular cases. It shows that the networks can possess continuous attractors if the excitatory connections are in gaussian shape. Simulation examples are employed for illustration.**

I. INTRODUCTION

Continuous attractors of recurrent neural networks (RNNs) have attracted extensive interests in recent years. Continuous attractors refer to continuous collections of equilibrium points of a network. Researches on neural population coding have revealed that continuous stimuli, such as orientation, moving direction, and the spatial location of objects could be encoded as continuous attractors in neural networks. Under some conditions, RNNs may possess continuous attractors, see for examples, [2][4][5][6][12][13].

The continuous attractors of RNNs have been investigated by many authors. Continuous attractor neural networks are of central importance in computational neuroscience as there are strong indications that such mechanisms are used frequently for information processing in the brain. In some neurobiological models, continuous attractors have been used to represent continuous quantities like working memory in prefrontal cortex [9], orientation of a visual stimulus [1], eye position [2], head direction [3], and so on. The study of continuous attractor neural network models is important in order to see if such models can explain measured effects or if the experimental data indicate that other mechanisms must be at work in the brain.

In continuous attractor network models, if the initial input pattern is given, through the cooperation between close nodes and the competition between distant nodes, and inhibition among all of the neural nodes, locations with the strongest support will win the competition. Thereby, a combination of the initial activity of single nodes is in combination with the activity of neighboring nodes, and the network models implement a specific version of a winner-take-all algorithm. This winner-take-all algorithm could be of practical use in many applications. Thus, such networks that might not only be utilized in the brain, but might allow in practical applications, such as winner-take-all or winner-take-most algorithms. The application of continuous neural network models in technology was explored in a variety of papers [7][10][11].

Many additions to the basic continuous neural network model have been proposed in recent years that are destined for further explorations into various application areas. Generally, among these models, continuous attractors are difficult to be studied analytically, especially for those that generate unimodal profiles of activity [1][3]. However, using appropriate theoretical framework, an attractor solution can be calculated explicitly for certain models, see for examples, [12][13].

This paper proposes a class of RNNs without lateral inhibition. Since there is no general rule to determine the stability of the network without specifying the excitatory connections, individual conditions can be calculated analytically for some particular cases. Especially, we study the dynamical behavior in the case of synaptic connections have a gaussian-like shape. It shows that if the excitatory connections are in gaussian-like shape, the network can possess continuous attractors. Convergence of the network only depends on the the cooperation between close nodes and the competition between distant nodes.

The remain part of the paper is organized as follows. In Section II, a model of a class of RNNs is proposed and some preliminaries are given. Dynamics behaviors of the proposed network in some particular cases, especially for continuous attractors are studied in Section III. Simulations are given in Section IV. Finally, Section V presents the conclusions.

II. PRELIMINARIES

The proposed RNN model without lateral inhibition is described by the following nonlinear differential equation:

$$\frac{dx_i}{dt} = -x_i + \left(\sum_j w_{ij} x_j + c_i \right)^m \tag{1}$$

for $t \geq 0$ and $i = 1, 2, ...n$, where c_i represents an external input whose value is independent of the network's activity, m

is a positive constant. Here, a neighboring neuron j can drive neuron i excite through synaptic connection w_{ij}, but cannot decrease its gain.

For some of the following theoretical derivations, it is useful to rewrite the networks (1) in an essentially equivalent form that involves continuous functions rather than vectors and matrices. The key is to consider a relatively large number of neurons, so that sum over cells can be replaced by integral. In this case, the vector of neuron's activity turns into a function $x(a)$, where a acts the neuron's index or label; similarly, the connection matrix turns into a function $w(a,b)$ of two variables. Thus, the continuous version of the network (1) is

$$\frac{dx(a)}{dt} = -x(a) + \left(\rho \int_{-\infty}^{+\infty} w(a,b)x(b)db + c(a)\right)^m \tag{2}$$

for $t \geq 0$, where ρ is the density of neurons. This factor ensures that for any quantity x, the sum over neurons $\sum_j x_j$ is equal to the integral $\rho \int_{-\infty}^{+\infty} x(b)db$. Equations (1) and (2) will be used interchangeably, as they are basically identical.

As mentioned above, in these expressions, $w(a,b)$ is the excitatory connections, and $c(a)$ is the external input. In the rest of the paper, We explore the dynamics of this model for various connection patterns w. We show that these equations typically have several stable solutions, and especially, when $w(a,b)$ is gaussian shape, the networks possess continuous attractors.

Next, some definitions and a useful lemma are given.

Definition 1: x^* is an equilibrium point of the networks (1) if $x(0) = x^*$ implies $x(t) = x^*$, for $t \geq 0$.

Definition 2: x^* is a stable equilibrium point of the networks (1) if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, such that $\mid x(0) - x^* \mid < \delta(\varepsilon)$ implies that $\mid x(t) - x^* \mid < \varepsilon$ for all $t \geq 0$.

Lemma 1: It holds that

$$\int_{-\infty}^{+\infty} \exp\left(-s^2\right) dx = \sqrt{\pi}.$$

Proof: Define

$$I = \int_{-\infty}^{+\infty} \exp\left(-s^2\right) dx,$$

then,

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty} \exp\left(-s^2\right) ds \int_{-\infty}^{+\infty} \exp\left(-\theta^2\right) dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left(-(s^2+\theta^2)\right) dxdy \\ &= \int_{0}^{2\pi} \int_{0}^{+\infty} \exp\left(-r^2\right) \cdot rdr \\ &= \pi. \end{aligned}$$

Thus, $I = \sqrt{\pi}$. The proof is completed. ■

III. Equilibrium point and convergence analysis

A key property of equation (2) is that it allows a solution in which all neurons' activity are equal; refer to this uniform activity as X. If the activity is same for all neurons, the network equations are reduced to a single, nonlinear equation

$$\frac{dX}{dt} = -X + (w_{tot}X + c)^m \tag{3}$$

for $t \geq 0$. Two conditions are necessary for this to be possible: first, the external input should be the same for all units. This is why the term $c(a)$ now appears as a constant c. Second, the quantity

$$w_{tot} \equiv \rho \int_{-\infty}^{+\infty} w(a,b)db \tag{4}$$

must also be independent of a. Because $w(a,b)$ corresponds to synaptic weight from neuron b to neuron a, this means that the total synaptic input to all neurons must be the same. Notice that this is a normalization condition, it does not restrict the distribution of synaptic weights, just the total amount works.

Theorem 1: If X^* is an equilibrium of the network (3), it must be stable under the condition:

$$m \cdot w_{tot} \left(w_{tot}X^* + c\right)^{m-1} < 1. \tag{5}$$

Proof: From definition 1 and the network model (3), the equilibrium point X^* must satisfy

$$X^* = \left(w_{tot}X^* + c\right)^m. \tag{6}$$

To determine whether X^* is stable or not, set the steady state plus a small perturbation, $X \longrightarrow X^* + \delta X$. Then derive the first-order development at point X^* as follows

$$\frac{d\delta X}{dt} = \delta X \left(-1 + m \cdot w_{tot} \left(w_{tot}X^* + c\right)^{m-1}\right)$$

for $t \geq 0$. The requirement for stability is that the coefficient multiplying δX must be negative, which leads to the condition (5). The proof is completed. ■

The results in Theorem 1 are under the condition that all x_i vary identically; for the system (1) to be stable, however, the procedure is more complicated if each neuron is perturbed by an independent amount. This is discussed in the following theorem.

Theorem 2: If x^* is an equilibrium point of the networks (1), it must be stable under the following condition:

$$\alpha \cdot w_{tot} < 1, \tag{7}$$

where

$$\alpha = m \cdot \left(\sum_j w_{ij}x_j + c_i\right)^{m-1}.$$

Proof: Substitute x_j for $x^* + \delta x_j$ in Eq.(1). Again use Eq.(6) and linearize to obtain

$$\frac{d\delta x_i}{dt} = -\delta x_i + \alpha \sum_j w_{ij}\delta x_j. \tag{8}$$

Rewrite the equation in vector form,

$$\frac{d\delta x}{dt} = L\delta x = (-1 + \alpha W)\delta x. \tag{9}$$

The largest eigenvalue of L is equal to $-1+\alpha w_{tot}$, imposing that it be smaller than 0 leads to the condition (7). The proof is completed. ∎

Clearly, suppose $W = w_0 U$ or $w_0 I$, where U is matrix with all entries equal to 1, I is an identity matrix. From (7), it is easy to give the corresponding stable condition of (1).

Recurrent neural networks may possess continuous attractors. The dynamical behaviors of continuous attractors are interesting properties of RNNs. Next, we will show that if the excitatory connections have a gaussian shape, the network (2) can also possess continuous attractors.

Theorem 3: Suppose excitatory connections have a gaussian shape with standard deviation σ, such that

$$w(a,b) = w_{\max} \exp\left(-\frac{(a-b)^2}{2\sigma^2}\right), \tag{10}$$

where $w_{\max}$ is a constant. Then, given any $z \in R$,

$$x(a,t) = x_{\max}(t) \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right), t \geq 0 \tag{11}$$

is a trajectory of (2) with $c(a) = C \cdot \exp\left(-\frac{(a-z)^2}{2m\sigma_1^2}\right)$, where C is a constant, and

$$\sigma_1^2 = \frac{\sigma^2}{m-1}, \tag{12}$$

$x_{\max}(t)$ is a solution of the following equation

$$\frac{dx_{\max}}{dt} = -x_{\max} + \left(\frac{1}{\sqrt{m}} w_{tot} \cdot x_{\max} + C\right)^m \tag{13}$$

for $t \geq 0$.

Proof: Substitute (11) into Eq.(2), we have

$$\begin{aligned} &\frac{dx_{\max}}{dt} \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right) \\ = \; & -x_{\max} \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right) \\ & + \left(\rho w_{\max} x_{\max} \int_{-\infty}^{+\infty} \exp\left(-\frac{(a-b)^2}{2\sigma^2} - \frac{(b-z)^2}{2\sigma_1^2}\right) db \right. \\ & \left. + C \cdot \exp\left(-\frac{(a-z)^2}{2m\sigma_1^2}\right)\right)^m \end{aligned}$$

for $t \geq 0$.

Since,

$$\begin{aligned} & \int_{-\infty}^{+\infty} w(a,b) x(b) db \\ = \; & w_{\max} \cdot x_{\max} \int_{-\infty}^{+\infty} \exp\left(-\frac{(a-b)^2}{2\sigma^2} - \frac{(b-z)^2}{2\sigma_1^2}\right) db \\ = \; & w_{\max} \cdot x_{\max} \exp\left(-\frac{(a-z)^2}{2(\sigma^2+\sigma_1^2)}\right) \\ & \times \int_{-\infty}^{+\infty} \exp\left(\frac{(a-z)^2}{2(\sigma^2+\sigma_1^2)} - \frac{(a-b)^2}{2\sigma^2} - \frac{(b-z)^2}{2\sigma_1^2}\right) db \\ = \; & w_{\max} \cdot x_{\max} \exp\left(-\frac{(a-z)^2}{2(\sigma^2+\sigma_1^2)}\right) \\ & \times \int_{-\infty}^{+\infty} \exp - \frac{\left(b - \frac{a\sigma_1^2 + z\sigma^2}{\sigma^2+\sigma_1^2}\right)^2}{2\left(\frac{\sigma\sigma_1}{\sqrt{\sigma^2+\sigma_1^2}}\right)^2} db \\ = \; & w_{\max} \cdot x_{\max} \exp\left(-\frac{(a-z)^2}{2(\sigma^2+\sigma_1^2)}\right) \cdot \sqrt{2\pi} \cdot \frac{\sigma\sigma_1}{\sqrt{\sigma^2+\sigma_1^2}} \end{aligned}$$

Because of

$$\begin{aligned} & \rho \int_{-\infty}^{+\infty} w(a,b) db \\ = \; & \rho \int_{-\infty}^{+\infty} w_{\max} \exp - \frac{(a-b)^2}{2\sigma^2} db \\ = \; & \sqrt{2\pi}\rho \cdot w_{\max}\sigma = w_{tot} \end{aligned}$$

From Lemma 1 and (13), it follows that,

$$\begin{aligned} & \frac{dx_{\max}}{dt} \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right) \\ = \; & -x_{\max} \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right) \\ & + \left(w_{tot} \cdot x_{\max} \frac{1}{\sqrt{m}} + C\right)^m \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right) \end{aligned}$$

for $t \geq 0$. Reducing the above equation to a scalar equation for the amplitude of the guassian shape, Eq.(12) can be given. The proof is completed. ∎

Theorem 4: Suppose that

$$w(a,b) = w_{\max} \cdot \exp\left(-\frac{(a-b)^2}{2\sigma^2}\right),$$

where $w_{\max}$ is some constant. If $x^*_{\max}$ $(\neq 0)$ is a stable equilibrium point of (13), then,

$$S = \left\{ x(a) \,\middle|\, x(a) = x^*_{\max} \cdot \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right), a \in R, z \in R \right\}$$

is a continuous attractor of (2).

Proof: By Theorem 3, clearly, to each fixed value of z, if $x^*_{\max}(\neq 0)$ is a stable equilibrium point of (13), $x^*_{\max} \cdot \exp\left(-\frac{(a-z)^2}{2\sigma_1^2}\right)$ is a stable equilibrium of (2). Then, through continuous variation of z, we can get S as a continuous attractor of the network (1). The proof is completed. ∎

IV. Simulations

In this section, some simulations will be provided to illustrate and verify the theory developed.

Example 1: Consider the one-dimensional neural network:

$$\frac{dx}{dt} = -x + (0.5x + 0.25)^2 \tag{14}$$

for $t \geq 0$. Clearly, $x_1^* = 1.5 + \sqrt{2}$, $x_2^* = 1.5 - \sqrt{2}$ are the equilibrium points of the network (14).

By the condition (5) in Theorem 1, it can checked that x_1^* is not a stable equilibrium point of the network (14), but, x_2^* is stable.

Figure 1 shows the simulation result for convergence of the network (14) for 100 trajectories starting from randomly

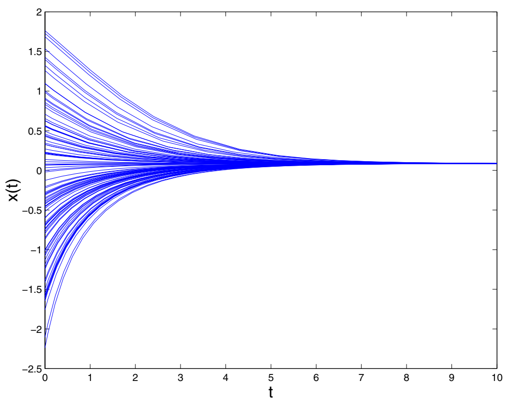

Fig. 1. Convergence of network in (14)

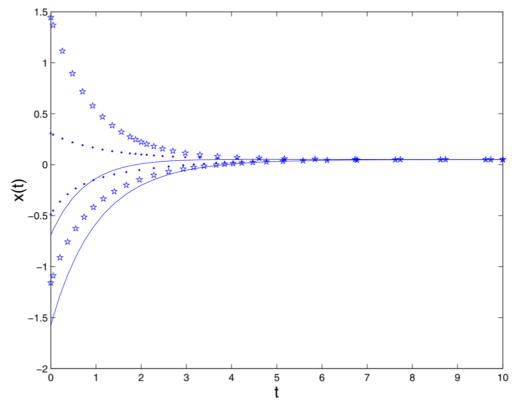

Fig. 2. Convergence of networks in (15)

selected initial points. It can be observed that x_2^* is a stable equilibrium point.

Example 2: Consider the following three-dimensional neural network:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \left(\begin{pmatrix} 0 & 0.2 & 0.3 \\ 0.4 & 0 & 0.1 \\ 0.3 & 0.2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}\right)^2 \quad (15)$$

for $t \geq 0$. Clearly,

$$W = \begin{pmatrix} 0 & 0.2 & 0.3 \\ 0.4 & 0 & 0.1 \\ 0.3 & 0.2 & 0 \end{pmatrix}.$$

thus, $w_{tot} = 1.5$, because of the condition (7) in Theorem 2, it is evident that

$$x^* = \begin{pmatrix} 1.5 - \sqrt{2} \\ 1.5 - \sqrt{2} \\ 1.5 - \sqrt{2} \end{pmatrix}.$$

is a stable equilibrium of the network (15).

Figure 2 shows the simulation result for the convergence of the network (15). Three kinds of curves represent the convergence of three neurons respectively. It can be observed that every trajectory converges to an equilibrium point. This well verifies Theorem 2.

From Eq.(13), if $C = 0$, the equilibrium point x^* must satisfy

$$-x^* + \left(w_{tot}\frac{1}{\sqrt{m}}x^*\right)^m = 0,$$

then use the expression above and linearize to obtain the following stable condition:

$$m < 1.$$

Thus, it is easy to see that, the network (13) has stable equilibrium points in the case of $m = \frac{1}{3}$.

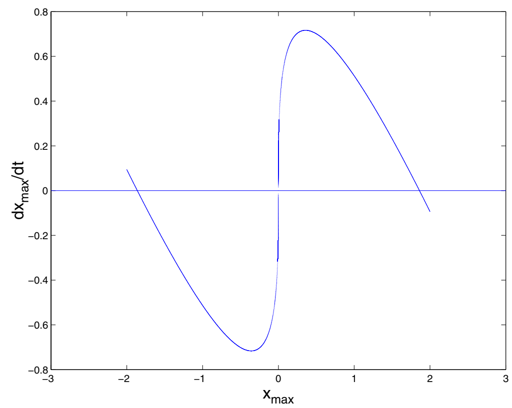

Fig. 3. Equilibrium of network in (16)

Example 3: Consider the following one-dimensional neural network:

$$\frac{dx_{\max}}{dt} = -x_{\max} + \left(\frac{6}{\sqrt{3}} \cdot x_{\max}\right)^{\frac{1}{3}} \quad (16)$$

for $t \geq 0$. It is easy to calculate out the equilibrium points of Eq.(16), zero is one equilibrium point, another two equilibrium point are $\sqrt{2\sqrt{3}}$ and $-\sqrt{2\sqrt{3}}$. Obviously, two nonzero equilibrium points are stable.

The dynamics of (16) can be visualized by plotting the derivative $dx_{\max}/dt$ against $x_{\max}$ as in Figure 3. Derivative becomes zero when the curve crosses the horizontal line, the crossing points $\sqrt{2\sqrt{3}}$, 0 and $-\sqrt{2\sqrt{3}}$ are equilibrium points to the network.

Figure 4 shows the simulation results for convergence of the network (16) for 100 trajectories originating from randomly selected initial points. It can be observed that the two nonzero equilibrium points are stable.

However, from Eq.(12), there must hold $m > 1$. In this case, $C \neq 0$ is a necessary condition for the network (1) having continuous attractors.

Example 4: Consider the network (2) with $m = 2$:

$$\frac{dx(a)}{dt} = -x(a) + \left(\rho\int_{-\infty}^{+\infty} w(a,b)x(b) + c(a)\right)^2 \quad (17)$$

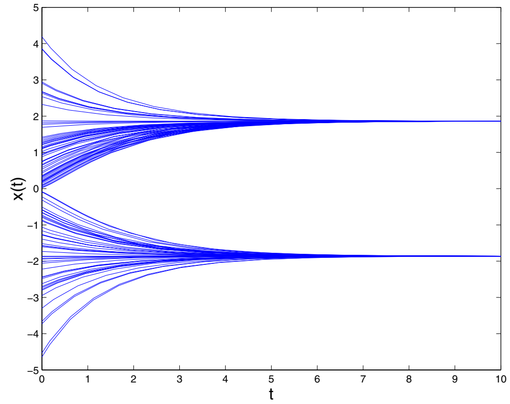

Fig. 4. Convergence of network in (16)

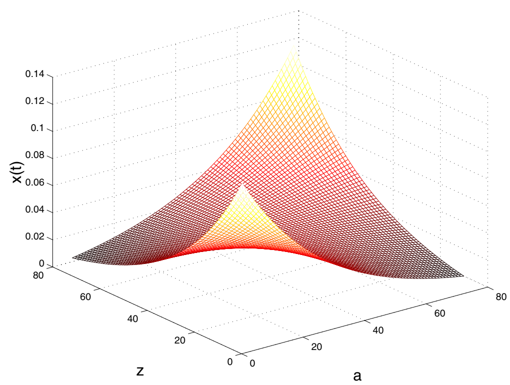

Fig. 5. Stable equilibrium points for (17) with continuous variation of z

for $t \geq 0$, where $c(a) = C \exp\left(-\dfrac{(a-z)^2}{2\sigma^2}\right)$, then for $t \geq 0$

$$x(a) = x_{\max}(t) \exp\left(-\frac{(a-z)^2}{2\sigma^2}\right) \tag{18}$$

is a solution of the network (17), and $x_{\max}(t)$ satisfies

$$\frac{dx_{\max}}{dt} = -x_{\max} + \left(\frac{\sqrt{2}}{2} w_{tot} x_{\max} + C\right)^2, t \geq 0 \tag{19}$$

Suppose $\rho = 1$, $C = 0.1$ and $w_{tot} = \sqrt{2}$, it is easy to check that $x^*_{\max} = 0.0254$ is a stable equilibrium point of (19). Fig.5 shows the result of (18) with continuous variation of a and z. Fig.6 shows the simulation results of a continuous attractor of the network (17) with three neurons which indice equal to 0.1, 0.2 and 0.3 respectively. Each point in the figure indicates a stable equilibrium point.

V. Conclusion

This paper proposes a class of RNNs without lateral inhibition. Since there is no general rule to determine the stability of the network without specifying the excitatory connections, individual conditions are calculated analytically for some particular cases. Especially, we study the dynamical behavior in the case of synaptic connections having a gaussian-like shape. It shows that the networks can possess continuous attractors

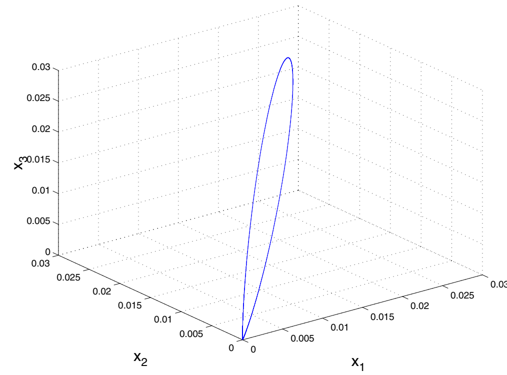

Fig. 6. Continuous attractors of network in (17) with three neurons

if the excitatory connections are in gaussian shape. For all specific examples discussed here, there was good qualitative agreement between theories and simulations. This model is under going research both in theory and applications. It is believed that more interesting results on this model will come out.

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