Convergence Analysis of Background Neural Networks with Two Subnetworks

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***Abstract*— The convergence of background neural networks with two identical, unstructured subnetworks that inhibit each other is studied. The convergence properties include global attractivity and complete stability analysis. Furthermore, it shows that shifting the background level affects the existence and stability of the equilibrium point. Adding noise to the networks, it will fluctuate around an equilibrium point.**

I. Introduction

When we need to remember a phone number announced on the radio, some neural process in the brain enables our short-term memory; without the context information indicating that it is important to store it, the same number vanishes promptly from the mind. Clearly, context determines whether a motor responses or not. The background model of neural networks was proposed in [6]. The neural activity evoked by artificial stimulation which would normally have a strong impact on the perceptual decision, has no effect when it is produced slightly earlier or later relative to the normal sequence of events. It shows that small changes in this background input level may shift a network from a relatively quiet state to some other state. Thus we can see that the background may act as a switch that allows networks to be turned on or off. In this paper, we use the model which have two identical, unstructured subnetworks that inhibit each other and discuss the background neural networks in which a uniform background input may determine whether a random network has one or two stable firing levels. All neurons in a subnetwork could fire at a approximately firing rate, with the interaction between two groups.

Among the dynamical properties of the neural networks, the convergence analysis of neural networks is very important in some practical applications [1], [2], [3], [7]. Convergence analysis for neural networks has been extensively studied recently in [4], [5], [8], [9]. In [10], it was analyzed that the dynamical of the background neural networks with one group. In this paper, we consider two subnetworks that inhibit each other. This means that they decrease each other's gain. If one of the rates is constant, the dynamics of the other is the same as those discussed in [10].

Using theoretical analysis and computer simulations, we show that a uniform background plays an important role in the background neural networks. Specifically, we discuss the convergence of the coupled systems, including the existence of equilibrium point, global attractivity and complete stability analysis. Global attractivity is very useful for determining the final behavior of network's trajectories. Complete stability describes a kind of convergence characteristics of networks.

The rest of paper is organised as follows. In Section II, preliminaries and some basic definitions are given. Global attractivity using inequalities analysis is discussed in Section III. Complete stability is studied in Section IV. In Section V, simulations are presented. Finally, Section VI gives the conclusions.

II. Preliminaries

Consider two identical, unstructured subnetworks that inhibit each other. Here unstructured means constant or random synaptic weights, and each subnetwork is described by a uniform rate.

$$\begin{cases} \tau \dfrac{dR_1}{dt} = -R_1 + \dfrac{(wR_1 + h_1)^2}{s + \upsilon N(R_1^2 + R_2^2)} \\ \tau \dfrac{dR_2}{dt} = -R_2 + \dfrac{(wR_2 + h_2)^2}{s + \upsilon N(R_1^2 + R_2^2)} \end{cases} \tag{1}$$

for $t \geq 0$, where R_1, R_2 represents the firing rate of each subnetwork respectively. $\tau > 0$ is a time constant and $\upsilon > 0$ is the synaptic connection strength. $s > 0$ is a saturation constant and N is the total number of neurons. $h_1 \geq 0$ and $h_2 \geq 0$ are the background inputs. And the total synaptic input $w \geq 0$ to all neurons is the same.

Lemma 1: It holds that

$$\frac{(wR_i(t) + h_i)^2}{s + \upsilon N R_i^2(t)} \leq \frac{w^2}{\upsilon N} + \frac{h_i^2}{s}, (i = 1, 2) \tag{2}$$

for all $t \geq 0$.

Proof: Clearly,

$$\begin{aligned} & \frac{(wR_i(t) + h_i)^2}{s + \upsilon N R_i^2(t)} - \frac{w^2}{\upsilon N} - \frac{h_i^2}{s} \\ = & \frac{s\upsilon N (wR_i(t) + h_i)^2 - s(s + \upsilon N R_i^2(t))w^2}{s\upsilon N(s + \upsilon N R_i^2(t))} \\ & - \frac{\upsilon N (s + \upsilon N R_i^2(t))h_i^2}{s\upsilon N (s + \upsilon N R_i^2(t)} \\ = & -\frac{(sw - h_i \upsilon N R_i(t))^2}{s\upsilon N(s + \upsilon N R_i^2(t))} \\ \leq & \; 0 \end{aligned}$$

for $i = 1, 2$ and $t \geq 0$. The proof is completed. ■

Definition 1: The network (1) is bounded if each trajectory is bounded.

Definition 2: Let S be a compact subset of R^n. Denote the $\epsilon-$neighborhood of S by S_ϵ. The compact set S is called a global attractive set of a network if for any $\epsilon > 0$, all trajectories of that network ultimately enter and remain in S_ϵ.

Definition 3: The neural network (1) is called completely stable, if each trajectory of (1) converges to an equilibrium point.

Denote a vector norm in R^2 by

$$\|x\| = \sqrt{\sum_{i=1}^{2} |x_i|^2}$$

for any $x \in R^2$.

Denote a matrix norm in $R^{2\times 2}$ by

$$\|W\|_{m_2} = \left(\sum_{j=1}^{2}\sum_{i=1}^{2} |w_{ij}|^2\right)^{1/2}$$

for any $w_{ij} \in R^{2\times 2}$.

III. GLOBAL ATTRACTIVITY

In this section, boundedness and global attractivity of the network (1) will be studied.

Theorem 1: The network (1) is always bounded. Moreover,

$$S = \left\{ R \,\middle|\, 0 \leq R_i \leq \frac{w^2}{\upsilon N} + \frac{h_i^2}{s},\ (i = 1, 2) \right\}$$

is a global attractive set of the network (1).

Proof: By Lemma 1, it holds that

$$\begin{aligned} R_i(t) &\leq R_i(0)e^{-\frac{t}{\tau}} \\ &\quad + \frac{1}{\tau}\int_0^t e^{-\frac{t-\theta}{\tau}} \cdot \frac{(wR_i(\theta) + h_i)^2}{s + \upsilon N(R_1^2(\theta) + R_2^2(\theta))} d\theta \\ &\leq R_i(0) + \frac{1}{\tau}\int_0^t e^{-\frac{t-\theta}{\tau}} \cdot \frac{(wR_i + h_i)^2}{s + \upsilon N R_i^2(\theta)} d\theta \\ &\leq R_i(0) + \frac{1}{\tau}\int_0^t e^{-\frac{t-\theta}{\tau}} \cdot \left(\frac{w^2}{\upsilon N} + \frac{h_i^2}{s}\right) d\theta \\ &\leq R_i(0) + \frac{w^2}{\upsilon N} + \frac{h_i^2}{s} \end{aligned}$$

for $i = 1, 2$ and $t \geq 0$. This shows that the network (1) is bounded. Next, it will prove that S is the global attractivity of the network (1).

Suppose

$$\lim_{t\to+\infty} sup R_i(t) = \xi_i, (i = 1, 2).$$

Since each $R_i(t) \geq 0 (i = 1, 2)$ is upper bounded, then $\xi_i < +\infty, (i = 1, 2)$. It will prove that

$$\xi_i < \frac{w^2}{\upsilon N} + \frac{h_i^2}{s}, (i = 1, 2). \tag{3}$$

For the sake of convenience in the subsequent discussion, and without loss of generality, assume that $\xi_l = \max_{1\leq j\leq 2}\{\xi_j\}$. Suppose (3) is not true, i.e.,

$$\xi_l > \max_{1\leq i\leq 2}\left\{\frac{w^2}{\upsilon N} + \frac{h_i^2}{s}\right\}. \tag{4}$$

It is clear that we can choose a small $\epsilon > 0$ such that

$$\xi_l > \max_{1\leq i\leq 2}\left\{\frac{w^2}{\upsilon N} + \frac{h_i^2}{s} + \epsilon\right\}.$$

By the basic property of the upper limit, there exists a $t_1 \geq 0$ such that

$$R_i(t) \leq \xi_i + \epsilon \leq \xi_l + \epsilon, (i = 1, 2)$$

for all $t \geq t_1$.

We will prove that there exists a $t_2 \geq t_1$ such that

$$\dot{R}_i(t) < 0 \tag{5}$$

for $i = 1, 2$ and $t \geq t_2$. If (5) is not true, there exists a $t_3 > t_1$ such that

$$\dot{R}_i(t_3) \geq 0, R_i(t_3) \geq \xi_l - \epsilon, (i = 1, 2).$$

However, it follows that

$$\begin{aligned} \dot{R}_i(t_3) &\leq -(\xi_l - \epsilon) + \frac{w^2}{\upsilon N} + \frac{h_i^2}{s} \\ &= -\xi_l + \epsilon + \frac{w^2}{\upsilon N} + \frac{h_i^2}{s} \\ &\leq 0 \end{aligned}$$

for $i = 1, 2$, which is a contradiction, thus (5) is true. By (5), it shows that $R_i(t)$ is monotonically decreasing. Thus the limit of $R_i(t) (i = 1, 2)$ exists, that is,

$$\lim_{t\to+\infty} R_i(t) = \lim_{t\to+\infty} \sup R_i(t) = \xi_l, i = (1, 2).$$

Then, we can obtain that

$$\begin{aligned} &\lim_{t\to+\infty} \sup R_i(t) \\ = &\lim_{t\to+\infty} |R_i(t)| \\ = &\lim_{t\to+\infty} |R_i(0)|e^{-\frac{t}{\tau}} \\ &+ \lim_{t\to+\infty} \frac{1}{\tau} \cdot \int_0^t \frac{e^{-\frac{t-\theta}{\tau}} \cdot (wR_i(\theta) + h_i)^2}{s + \upsilon N(R_1^2(\theta) + R_2^2(\theta))} d\theta \\ \leq &\lim_{t\to+\infty} |R_i(0)|e^{-\frac{t}{\tau}} \\ &+ \lim_{t\to+\infty} \frac{1}{\tau} \int_0^t e^{-\frac{t-\theta}{\tau}} \frac{(wR_i(\theta) + h_i)^2}{s + \upsilon N R_i(\theta)^2} d\theta \\ \leq &\frac{w^2}{\upsilon N} + \frac{h_i^2}{s} \end{aligned}$$

for $i = 1, 2$ and $t \geq 0$. This shows that S is a global attractive set of the network (1). The proof is completed. ■

Theorem 2: The network (1) is always bounded. Moreover,

$$S = \left\{ R \,\middle|\, \|R(t)\| \leq 2\left(\frac{w^2}{\upsilon N} + \frac{\|h\|^2}{s}\right) \right\}$$

is a global attractive set of the network (1).

Proof: Let

$$W = \begin{pmatrix} \frac{wR_1+h_1}{s+\upsilon NR^TR} & 0 \\ 0 & \frac{wR_2+h_2}{s+\upsilon NR^TR} \end{pmatrix}$$

and

$$h = (wR_1+h_1, wR_2+h_2)^T.$$

The equivalent vector form of the network (1) is

$$\tau\dot{R}(t) = -R(t) + Wh \tag{6}$$

for $t \geq 0$, and $R(t) = (R_1(t), R_2(t))^T$. The solution of the network (1) is represented as

$$R(t) = R(0)e^{-\frac{t}{\tau}} + \frac{1}{\tau}\int_0^t e^{-\frac{t-\theta}{\tau}} \cdot W \cdot h \cdot d\theta$$

for $t \geq 0$. Thus,

$$\begin{aligned} \|R(t)\| &\leq \|R(0)\|e^{-\frac{t}{\tau}} + \frac{1}{\tau}\int_0^t e^{-\frac{t-\theta}{\tau}} \cdot W \cdot h \cdot d\theta \\ &\leq \|R(0)\|e^{-\frac{t}{\tau}} \\ &\quad + \frac{1}{\tau}\int_0^t \frac{e^{-\frac{t-\theta}{\tau}}}{s+\upsilon NR^T(\theta)R(\theta)} \cdot \|W\|_{m_2} \cdot \|h\| \cdot d\theta \\ &= \|R(0)\|e^{-\frac{t}{\tau}} \\ &\quad + \frac{1}{\tau}\int_0^t e^{-\frac{t-\theta}{\tau}} \cdot \frac{\sum_{i=1}^{2}(wR_i(\theta)+h_i)^2}{s+\upsilon NR^TR} d\theta \\ &\leq \|R(0)\|e^{-\frac{t}{\tau}} \\ &\quad + \frac{2}{\tau}\int_0^t e^{-\frac{t-\theta}{\tau}} \cdot \max_{i=1,2}\left(\frac{w^2}{\upsilon N} + \frac{h_i^2}{s}\right) d\theta \\ &\leq \|R(0)\|e^{-\frac{t}{\tau}} + 2\left(\frac{w^2}{\upsilon N} + \frac{\|h\|^2}{s}\right) \end{aligned}$$

for $i = 1, 2$ and $t \geq 0$. Then,

$$\lim_{t\to+\infty} \sup \|R(t)\| \leq 2\left(\frac{w^2}{\upsilon N} + \|h\|^2\right).$$

This shows that S globally attracts the network (1). The proof is completed. ■

IV. Complete stability

In this section, the complete stability of (1) will be studied.

Theorem 3: The network (1) is completely stable.

Proof: Constructing an energy function

$$\begin{aligned} E(t) &= \frac{1}{2}\sum_{i=1}^{2} R_i^2(t) - \frac{w^2}{\upsilon N}\sum_{i=1}^{2} R_i^2(t) \\ &\quad - \sum_{i=1}^{2} \frac{wh_i}{\upsilon N}\ln(s+\upsilon NR_i^2(t)) \\ &\quad - \sum_{i=1}^{2} \frac{\upsilon Nh_i^2 - sw^2}{\upsilon N\sqrt{s\upsilon N}} \arctan\left(\sqrt{\frac{\upsilon N}{s}}R_i(t)\right) \end{aligned} \tag{7}$$

for all $t \geq 0$. Clearly, since $R(t) = (R_1(t), R_2(t))^T$ is bounded, $E(t)$ is also bounded. It follows that

$$\begin{aligned} \dot{E}(t) &= R_1(t)\dot{R}_1(t) + R_2(t)\dot{R}_2(t) - \frac{w^2}{\upsilon N}(\dot{R}_1(t) + \dot{R}_2(t)) \\ &\quad - \left[\frac{2wh_1R_1(t)}{s+\upsilon NR_1^2(t)}\dot{R}_1(t) + \frac{2wh_2R_2(t)}{s+\upsilon NR_2^2(t)}\dot{R}_2(t)\right] \\ &\quad - \frac{\upsilon Nh_1^2 - sw^2}{\upsilon N(s+\upsilon NR_1^2(t))}\dot{R}_1(t) \\ &\quad - \frac{\upsilon Nh_2^2 - sw^2}{\upsilon N(s+\upsilon NR_2^2(t))}\dot{R}_2(t) \\ &= \left[R_1(t) - \frac{(wR_1(t)+h_1)^2}{s+\upsilon NR_1^2(t)}\right]\dot{R}_1(t) \\ &\quad + \left[R_2(t) - \frac{(wR_2(t)+h_2)^2}{s+\upsilon NR_2^2(t)}\right]\dot{R}_2(t) \\ &\leq \left[R_1(t) - \frac{(wR_1(t)+h_1)^2}{s+\upsilon N(R_1^2(t)+R_2^2(t))}\right]\dot{R}_1(t) \\ &\quad + \left[R_2(t) - \frac{(wR_2(t)+h_2)^2}{s+\upsilon N(R_1^2(t)+R_2^2(t))}\right]\dot{R}_2(t) \\ &= -\tau\dot{R}_1(t)\dot{R}_1(t) - \tau\dot{R}_2(t)\dot{R}_2(t) \\ &= -\tau\left[\dot{R}_1(t)^2 + \dot{R}_2(t)^2\right] \\ &= -\tau\|\dot{R}(t)\|^2 \\ &\leq 0 \end{aligned}$$

for all $t \geq 0$. Thus $E(t)$ is monotonically decreasing. Since $E(t)$ is bounded, there must exist a constant E_0 such that

$$\lim_{t\to+\infty} E(t) = E_0 < +\infty.$$

Then,

$$\begin{aligned} \int_0^{+\infty} \|\dot{R}(t)\|^2 dt &= \lim_{t\to+\infty}\int_0^t \|\dot{R}(\theta)\|^2 d\theta \\ &\leq \lim_{t\to+\infty}\int_0^t \frac{-\dot{E}(\theta)}{\tau} d\theta \\ &= -\frac{1}{\tau} \cdot \lim_{t\to+\infty} E(t) \\ &= \frac{-E_0}{\tau} \\ &< +\infty. \end{aligned}$$

Since $R(t)$ is bounded, from the network (6), it follows that $\dot{R}(t)$ is bounded. Then, $R(t)$ is uniformly continuous on $[0, +\infty)$. Again, from (6), it follows that $\|\dot{R}(t)\|^2$ is also uniformly continuous on $[0, +\infty)$. Thus, it must hold that

$$\lim_{t\to+\infty} \|\dot{R}(t)\|^2 = 0.$$

Since $R(t)$ is bounded, every subsequent of $R(t)$ must contain convergent subsequence. Let $R(t_m)$ be any of such a convergent subsequence. There exists a $R^* \in R^2$ such that

$$\lim_{t\to+\infty} R(t_m) = R^*.$$

Then, from the network (6), we have

$$-\frac{R^*}{\tau} + \frac{(wR^*+h)^2}{\tau(s+\upsilon NR^{*2})} = \lim_{t_m\to+\infty} \dot{R}(t_m) = 0.$$

Clearly, R^* must be an equilibrium point. The proof is completed. ■

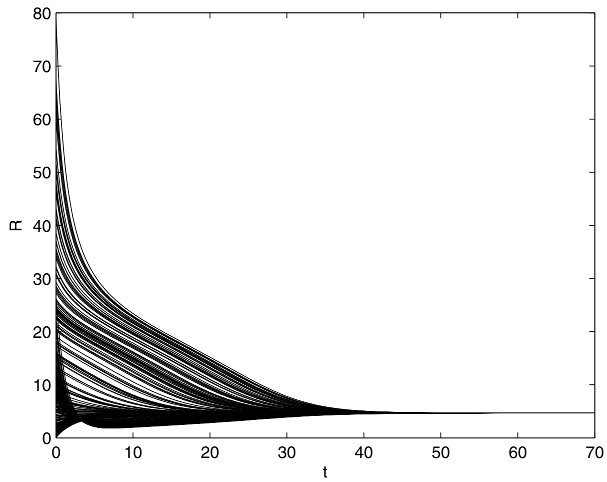

Fig. 1. Convergence of (1) with $h_1 = h_2 = 8.5$.

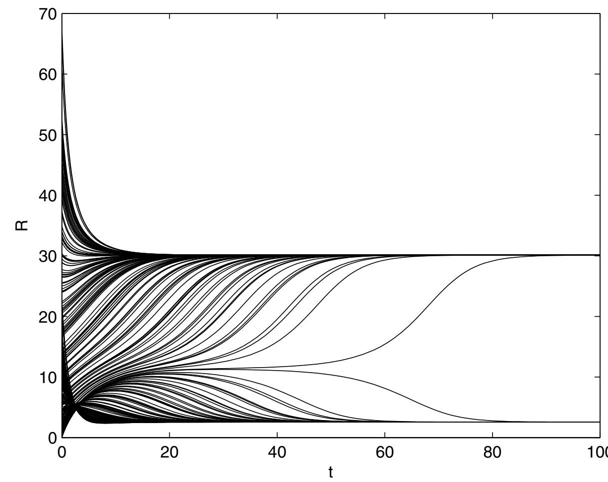

Fig. 2. Convergence of (1) with $h_1 = h_2 = 10.15$.

V. Simulations

In this section, an example is provided to illustrate the above theory.

Let us consider the network (1) with $w = 1.154, \upsilon N = 0.0295, \tau = 1, s = 40$.

Taking $h_1 = h_2 = 8.5$, by Theorem 1, the network (1) is bounded. By simple calculations, the global attractive set is given by

$$S = \{R|0 \le \|R\| \le 46.9492\},$$

which globally attracts all the trajectories of the network. By Theorem 3, this network is completely stable. Thus each trajectory of (1) must converge to a stable equilibrium point. By simulation we can see that firing rate is about 4 spikes per second.

Taking $h_1 = h_2 = 10.15$, the boundedness and global attractivity of the network (1) can also be obtained by Theorems 1 and 2. There are two stable steady states, each with a low and a high rate around 3 and 30 spikes per second, respectively, and an unstable point as disappearance. The global attractive set is given by

$$S = \{R|0 \le \|R\| \le 47.7185\}.$$

Figures 1 and 2 show the simulation results for completely stability of the network (1). As time is increasing, all the trajectories converge to a stable equilibrium point or two stable equilibrium points. Figures 3 and 4 show that projection trajectories of the network (1) on phase plane. They show that global attractivity and completely of the network (1).

VI. Conclusions

In this paper, convergence of the background neural networks has been studied. The background neural networks are composed of two subnetworks which have uniform firing rate and background input. Three basic dynamical problems are addressed for the kind of network: boundedness, global

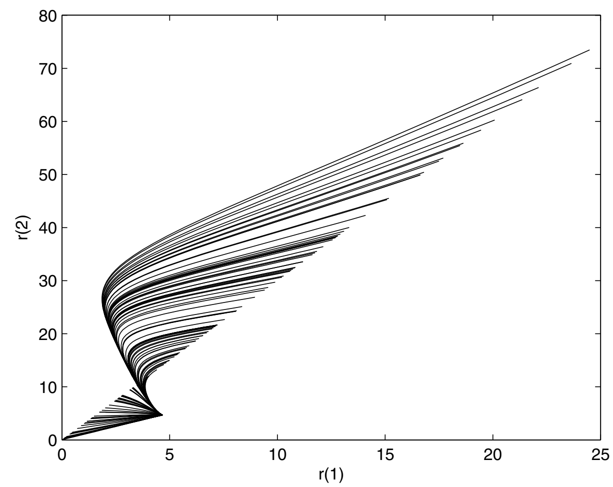

Fig. 3. Projection trajectories of (1) on phase plane with $h_1 = h_2 = 8.5$.

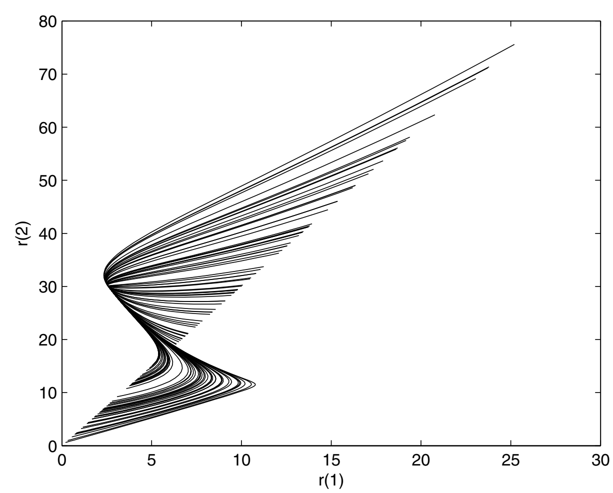

Fig. 4. Projection trajectories of (1) on phase plane with $h_1 = h_2 = 10.15$.

attractivity, and complete stability. Simulations are used for illustrating.

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