Guaranteed Cost Control for Systems with Saturating Actuators and Input Delays

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***Abstract*—This paper considers guaranteed cost control problem for systems with saturating actuators and input delays. By using Razumikhin stability theorem, a linear state feedback control law is found, ensuring not only the closed-loop system's stability but guaranteed cost index. And the problem of designing the optimal guaranteed cost controller is converted to a convex optimization problem within the framework of linear matrix inequalities. Example is given to illustrate the effectiveness of the proposed result.**

***Index Terms*—LMI; Guaranteed cost control; Time delay; Actuator saturation**

I. INTRODUCTION

Nonlinear systems with time-delay constitute basic mathematical models of real phenomena, for instance, in circuits theory, economics and mechanics. Not only dynamical systems with time-delay are common in chemical processes and long transmission lines in pneumatic, hydraulic, or rolling mill systems, but computer controlled systems requiring numerical computation have time-delays in control loops. The presence of time-delays in control loops usually degrades system performance and complicates the analysis and design of feedback controllers. Stability analysis and synthesis of retarded systems is an important issue addressed by many authors and for which surveys can be found in several monographs, see [5], [7]. However, it is also desirable to design a controller which guarantees not only stability but also an adequate level of performance as well. One approach to this problem called guaranteed cost control was firstly presented by Chang and Peng [2]. This approach provides an upper bound on a given performance index. Based on this work, many important results have been proposed in the past fifteen years. However, such systems have to be taken into account in the design of control laws in order to avoid poor performances and even instability of the control systems, see [11], [12]. Stability analysis and synthesis of such systems is an important issue addressed by many authors and some mature methods have been widely used to deal with these problems, for details and references, see [13], [14]. The design of controllers for time-delay systems leads to complex problems lacking of analytical solutions; hence, linear matrix inequality (LMI) techniques are often used to provide computational solutions for continuous-time, see [3], [4] [10] and discrete-time systems, see [6], [15]. In other hand, to deal with time-delay problem, Rzaumikhin theorem is an effective approach that be used extensively.

Another common, but difficult, control problem is to deal with actuator saturation since all control devices are subject to saturation (limited in force, torque, current, flow rate, etc.). This non-linearity cause control systems have to operate under constraints on the magnitude of the control input. These limitations in terms of input constraints must be considered in the controller design. Up to now, the analysis and synthesis of controllers for dynamic systems subject to actuator saturation have been attracting increasingly more attention (see, for example, [1], [8]). And there exist some effective tools to deal with it. However, actuator saturation, time delay are often encountered in control systems. To deal with both problems effectively, appropriate design methods are required.

As far as we know, however, little research has been focused on the guaranteed cost control of systems subject to actuator saturation and input delays. Motivated by the method of [9], we transform the saturation non-linearity into a convex polytope form. In this paper, for a class of saturate systems with delay in states, a method of designing state feedback stabilizing controller guaranteeing an upper bound on a quadratic cost function is proposed. A procedure is given to select the state feedback matrix gain which minimizing this bound. We formulate the problem into a constrained optimization problem with constraints given by a set of linear matrix inequalities.

This paper, divided into 5 sections, begins by formulating the problem and giving some preliminary result in Section 2. We will present our main results in Section 3 and example is given to illustrate design procedure and its effectiveness in Section 4. The paper is concluded in Section 5.

Notation: The following notations will be used throughout the paper. R denotes the set of real numbers, R^+ denotes the set of non-negative real numbers, R^n denotes the n dimensional Euclidean space and $R^{m\times n}$ denotes all $m\times n$ real matrices. The notation $X\geq Y$ (respectively, $X>Y$), where X and Y are symmetric matrices, means that $X-Y$ is positive semidefinite (respectively, positive definite). $C_{n,\tau}=C([-\tau,0],R^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into R^n with the topology of uniform convergence. $||\cdot||$ refers to either the Euclidean vector norm or the induced matrix 2-norm; $||\phi(t)||_c=sup_{-\tau\leq t\leq 0}||\phi(t)||$ stands for the norm of a function $\phi\in C_{n,\tau}$.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Problem Statement

Consider the time-delay system with actuator saturation described in state-space form

$$\dot{x}(t)=Ax(t)+Bu(t)+A_dx(t-\tau) \quad (1)$$

for $t\in[0,\infty)$ and with $x(t)=\phi(t)$ for $t<0$.

In this description, $A,A_d\in R^{n\times n}$, $B\in R^{n\times m}$ are constant matrices, $x(t)\in R^n$ is the state vector, $u(t)\in R^m$ is the control vector, and $\phi:[-\tau,0]\rightarrow R^n$ is a continuous vector-valued function specifying the initial state of the system. Moreover, the control input u in system (1) is subjected to the following constraints,

$$-\bar{u}_i\leq u_i\leq \bar{u}_i \quad (2)$$

so $u(t)$ can be described by $\bar{u}_i\sigma(u)$ without loss of generality.

The function $\sigma:R^m\rightarrow R^m$, is the standard saturation function defined as follows:

$$\sigma(u)=[\sigma(u_1)\ \sigma(u_2)\ \cdots\ \sigma(u_m)]^T$$

$$\sigma(u_i)=sign(u_i)\min\{1,|u_i|\} \qquad i=1,2,\cdots,m$$

For system (1), consider the following cost function:

$$J=\int_0^\infty[x^T(t)Qx(t)+u^T(t)Rx(t)]dt \quad (3)$$

where Q and R are given positive-definite symmetric matrices, and if there exists a scalar V and a controller $u(t)$ such that $J\leq V$, then V is called a guaranteed cost and the controller $u(t)$ is called a guaranteed cost controller.

B. Razumikhin Theorem

For stability analysis of systems with time-delay, the Rzaumikhin Theorem is used extensively. In what follows, we give a brief summary of the theorems simplified to autonomous systems.

Consider the retarded functional differential equation

$$\begin{aligned}\dot{x}(t)&=f(x_t), && t\geq 0\\ x(t)&=\psi(t), && t\in[-\tau,0]\end{aligned} \quad (4)$$

Assume that $\psi\in C_{n,\tau}$ and the map $f(\psi):C_{n,\tau}\mapsto R^n$ is continuous and Lipschitzian in ψ and $f(0)=0$. Also denote the solution of the functional differential (4) with the initial condition $x_0\in C_{n,\tau}$ as $x(t,x_0)$.

Definition 1: The trivial solution $x(t)\equiv 0$ of (4) is said to be asymptotically stable if

1) for every $\delta>0$ there exists an $\epsilon=\epsilon(\delta)$ such that for any $\psi\in B(0,\epsilon)$ the soluntion $x(t,\psi)$ of (4) satisfies $x_t\in B(0,\delta)$ for all $t\geq 0$.

2) for every$\eta>0$ there exists a $T(\eta)$ and a v_0 independent of η such that $\psi\in B(0,v_0)$ implies that $||x_t||_c<\eta$, $\forall t\geq T(\eta)$.

The Razumikhin Theorem give conditions for $x(t)\equiv 0$ to be asymptotically stable. The additional information is incorporated in the following statement of the theorem.

Theorem 1: (Razumikhin Stability Theorem) Suppose that $u(s)$, $v(s)$, $w(s)$ and $p(s)\in R^+\rightarrow R^+$ are scalar, continuous and nondecreasing functions, $u(s),v(s),w(s)$ positive for $s>0$, $u(0)=v(0)=0$ and $p(s)>s$ for $s>0$. If there is a continuous function $V:R^n\rightarrow R$ and a positive number ρ, such that for all $x_t\in M_V(\rho):=\{\psi\in C_{n,\tau}:V(\psi(\theta))\leq\rho,\forall\theta\in[-\tau,0]\}$, the following conditions hold.

1) $u(||x||)\leq V(x)\leq v(||x||)$
2) $\dot{V}(x(t))\leq -w(||x(t)||),\ if\ V(x(t+\theta))<p(V(x(t)))$, $\forall\theta\in[-\tau,0]$

Then, the solution $x(t)\equiv 0$ of the system (4) is asymptotically stable.

C. Some Mathematical Tools

Let f_i be the i-th row of the matrix F. We define the symmetric polyhedron,

$$L(F)=\{x\in R^n:|f_ix|\leq 1,i=1,2,\cdots,m\}$$

If the control u does not saturate for all $i=1,\ldots,m$, that is $x\in L(F)$, then the nonlinear system (1) admits the following linear representation:

$$\dot{x}(t)=(A+BF)x(t)+A_dx(t-\tau).$$

Lemma 1: [8] Let ν be set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. Then there are 2^m elements in ν. Suppose that each element of ν is labeled as $D_i,\ i=1,2,\ldots 2^m$ and denote $D_i^-=I-D_i$. Clearly, D_i^- is also an element of ν if $D_i\in\nu$.

Let $K,H\in R^{m\times n}$ be given. For $x(t)\in R^n$, if $\|Hx\|_\infty\leq 1$, then

$$\sigma(Kx)\in co\{D_iKx+D_i^-Hx:i\in[1,2,\cdots,2^m]\}$$

where $co\{\cdot\}$ denotes the convex hull of a set.

Lemma 2: [12] For any x, $y\in R^n$ and a matrix $M>0$ with compatible dimensions, the following inequality holds

$$2x^Ty\leq x^TMx+y^TM^{-1}y.$$

III. MAIN RESULT

In this section, we will consider the problem of designing a state feedback guaranteed cost controller and the procedure of selecting the controller minimizing the guaranteed cost for the linear time-delay system with saturation.

Theorem 2: For system (1), if there exists a scalar function $V(x(t))$, $V(0) = 0$ with continuous derivative, and a continuous nondecreasing function $w(s), u(s), v(s)$ and $p(s) > s$ for $s > 0$ such that

1) $V(x) > 0$, for any $x \neq 0$.
2) $V(x) \to \infty$, when $||x|| \to \infty$.
3) $u(||x||) \leq V(x) \leq v(||x||)$
4) $x^T(t)Rx(t) + u^T(t)Qu(t) + \dot{V}(x(t), t) \leq -w(||x(t)||)$, if $V(x(t+\theta)) < p(V(x(t))), \forall\theta \in [-\tau, 0]$.

Then the closed system is asymptotically stable and $V(x(0))$ is a guaranteed cost, where $x(0) \in R^n$ is the initial state.

Proof: We get conditions 1, 2, 3 from the Razumikhin Stability Theorem mentioned above. From condition 4, we have that if

$$V(x(t+\theta)) < p(V(x(t))), \forall\theta \in [-\tau, 0],$$

then

$$\begin{aligned}\dot{V}(x(t), t) &\leq -w(||x(t)||) - x^T(t)Rx(t) - u^T(t)Qu(t) \\ &\leq -\hat{w}(||x(t)||)\end{aligned} \tag{5}$$

Considering condition 1, 2, 3 and Razumikhin theorem it can be concluded that the closed system is asymptotically stable.

Integrating both sides of the inequality (5) from 0 to ∞, we have

$$\begin{aligned}\int_0^\infty \dot{V}(x(t))dt &= V(x(\infty)) - V(x(0)) \\ &\leq -\int_0^\infty [x^T(t)Rx(t) + u^T(t)Qu(t)]dt \\ &\quad -\int_0^\infty w(||x(t)||)dt \\ &\leq -\int_0^\infty [x^T(t)Rx(t) + u^T(t)Qu(t)]dt\end{aligned} \tag{6}$$

considering that asymptotically stability leads to $x(\infty) \to 0$, we obtain

$$\int_0^\infty [x^T(t)Rx(t) + u^T(t)Qu(t)]dt \leq V(x(0))$$

This completes the proof.

Now we will give a way to find the form of controller which makes system (1) not only asymptotically stable but satisfies cost function (3).

Theorem 3: For the linear time-delay system with saturation (1). Consider the set $\Omega(P, \rho) := \{x \in R^n : \ x^T Px \leq \rho\}$. $P \in R^{n\times n}$ is a symmetric positive definite matrix and $\rho \in R^+$. If there exist positive definite symmetric matrices $G, X \in R^{n\times n}$ and V,$W \in R^{m\times n}$ such that

$$G - X \leq 0 \tag{7}$$

$$\begin{bmatrix} 1 & v_i \\ v_i^T & G \end{bmatrix} \geq 0 \tag{8}$$

$$\begin{bmatrix} 1 & x_0^T \\ x_0 & G \end{bmatrix} \geq 0 \tag{9}$$

$$\begin{bmatrix} \Gamma & G & W^T \\ G & -\rho Q^{-1} & 0 \\ W & 0 & -\rho R^{-1} \end{bmatrix} < 0 \tag{10}$$

where

$$\begin{aligned}\Upsilon &= AG + B\bar{u}_i(D_i W + D_i^- V) \\ \Gamma &= \Upsilon + \Upsilon^T + G + A_d X A_d^T\end{aligned}$$

and v_i is the i-th row of the matrix V.

Then

$$u(t) = WG^{-1}x(t)$$

is a guaranteed cost control law of system (1) satisfies performance index (3) and $J \leq \rho$.

Proof: Given $P > 0$, consider a quadratic Lyapunov function candidate $V(x) = x^T Px$. First, we have

$$\alpha_1 ||x||^2 \leq V(x) \leq \alpha_2 ||x||^2$$

where

$$\alpha_1 = \lambda_{min}(P), \quad \alpha_2 = \lambda_{max}(P).$$

Let $u(t) = Kx(t)$, the derivative of V is

$$\begin{aligned}\dot{V}(x(t)) &= 2x(t)^T PAx(t) + 2x^T(t)PA_d x(t-\tau) \\ &\quad +2x^T(t)PB\sigma(Kx(t))\end{aligned}$$

By Lemma 1, saturation non-linearity can be substituted by a convex polytope if

$$\Omega(P, \rho) \subset L(H)$$

which is equal to

$$\rho h_i P^{-1} h_i^T \leq 1,$$

h_i is the ith row of matrix H. Utilizing Schur complement, we get

$$\begin{bmatrix} 1 & h_i(\frac{P}{\rho})^{-1} \\ (\frac{P}{\rho})^{-1} h_i^T & (\frac{P}{\rho})^{-1} \end{bmatrix} \geq 0 \tag{11}$$

which can be rewritten as (8) by taking $V = \rho HP^{-1}$. Thus for every $x(t) \in \Omega(P, \rho)$

$$\sigma(Kx) \in co\{D_i Kx + D_i^- Hx : i \in [1, 2, \cdots, 2^m]\}.$$

It follows that for every $x(t) \in \Omega(P, \rho)$, we have

$$\begin{aligned}\dot{V}(x(t)) &= 2x(t)^T P[A + B\bar{u}_i\sigma(D_i F + D_i^- H)]x(t) \\ &\quad +2x^T(t)PA_d x(t-\tau)\end{aligned}$$

From Lemma 2, we have

$$\begin{aligned}2x^T(t)PA_d x(t-\tau)) &\leq x^T(t)PA_d M A_d^T Px(t) \\ &\quad +x^T(t-\tau)M^{-1}x(t-\tau)\end{aligned}$$

Let $P \geq M^{-1}$, we get

$$\begin{aligned}\dot{V}(x(t)) \leq x^T(t)[&(A + B(D_i K + D_i^- H))^T P \\ &+P(A + B(D_i K + D_i^- H)) \\ &+PA_d M A_d^T P]x(t) + V(x(t-\tau))\end{aligned}$$

By Razumikhin Theorem, to prove that $\Omega(P, \rho)$ is an invariant set inside the domain of attraction, it suffices to show that there exist an $\varepsilon > 1$ and a $\delta > 0$ such that

$$\dot{V}(x(t)) \leq -\delta V(x(t))$$

$$if \quad V(x(t+\theta)) < \varepsilon V(x(t)) \qquad \forall\theta \in [-\tau, 0].$$

In the remainder of the proof, we will construct such ε and δ and show that they satisfy the inequality above.

Suppose there exists a $\delta > 0$ such that

$$\begin{aligned} Q + K^T RK + (A + B(D_i K + D_i^- H))^T P \\ +P(A + B(D_i K + D_i^- H)) \\ +PA_d MA_d^T P + (1 + 2\delta)P < 0, \end{aligned}$$

Let $\varepsilon = 1 + \delta$. Now suppose that $V(x(t + \theta)) < \varepsilon V(x(t)), \forall\theta \in [-\tau, 0]$. Then we have

$$\begin{aligned} x^T Qx + u^T Ru + \dot{V}(x) \leq x^T (Q + K^T RK + (A \\ +B(D_i K + D_i^- H))^T P + P(A + B(D_i K + D_i^- H)) \\ +PA_d MA_d^T P + \varepsilon P)x \\ < -\delta V(x(t)). \end{aligned}$$

Let

$$\begin{aligned} \Psi = Q + K^T RK + (A + B(D_i K + D_i^- H))^T P \\ +P(A + B(D_i K + D_i^- H)) + PA_d MA_d^T P + P \end{aligned}$$

system (1) will be asymptotically stable if $\Psi < 0$ and we get $\hat{w} = -\lambda_{max}(\Psi)$. Multiplying $\Psi < 0$ by $\rho^{\frac{1}{2}} P^{-1}$ on the left and on the right, respectively, and let $G = \rho P^{-1}$, $W = \rho KP^{-1}$, $V = \rho HP^{-1}$, $X = \rho M$, we obtain

$$\Upsilon + \Upsilon^T + \rho^{-1} GQG + \rho^{-1} W^T RW + A_d XA_d^T + G < 0 \tag{12}$$

Utilizing Schur complement, we get (10). Inequality (7) can be easily got from the condition $P \geq M^{-1}$, and inequality (9) can be obtained from the condition $x_0^T Px_0 \leq \rho$ by using Schur complement.

Theorem 3 gives some condition for the existence of the guaranteed cost controller with the guaranteed cost index $J \leq \rho$. Now we would like to choose from all the $\varepsilon(P, \rho)$ that satisfy the condition such that the guaranteed cost minimized. This problem can be formulated as

$$min \ \ \rho \quad s.t. \quad (7) - (10) \tag{13}$$

If the above optimization problem has an optimal solution $\tilde{\rho}, \tilde{X}, \tilde{G}, \tilde{W}, \tilde{V}$, then $u(t) = \tilde{W}\tilde{G}^{-1}x(t)$ is a guaranteed cost control law of system (1) satisfies performance index (3) and $J \leq \tilde{\rho}$.

It is clear that (13) is a convex optimization problem with LMI constraints. Therefore, the global minimum of the problem can be reached if it is feasible, and it can be easily solved by using the solver mincx in the LMI Toolbox of MATLAB.

Remark 1: Note that the condition of Theorem 3 does not include any information of time-delay, the theorem provides a delay-independent condition for regional stability of linear uncertain systems with time-delay and input saturation in terms of the feasibility of several linear matrix inequalities.

IV. Illustrative Examples

Example 1: Now consider an continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) + A_d x(t - \tau)$$

with the following parameters:

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.707 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0.3 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.4422 & 0.1711 \\ 3.0447 & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$-5 \leq u_i \leq 5, \ i = 1, 2$$

The associated performance index of the system is

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By applying Theorem 3 and solving the corresponding optimization problem, we get the optimal guaranteed cost controller

$$u(t) = \begin{bmatrix} -6.588 & -1.562 & 1.468 & 7.174 \\ 27.027 & 7.598 & -5.450 & -29.245 \end{bmatrix} x(t)$$

and the guaranteed cost of the uncertain closed-loop system is $J^* = 3.0669$.

V. Conclusion

In this paper, we have presented an LMI-based approach to the optimal guaranteed cost control problem via state feedback control laws for a class of uncertain system subject to control saturating actuators and input delays. By utilizing Razumikhin Theorem and transforming the system with actuator saturation non-linearities into a convex polytope of linear systems, we obtain a effective approach to deal with the problem.

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