Controllability of Multi-agent Systems with Switching Topology

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***Abstract*—The paper studies a class of formation control problem, i.e. the controllability for multi-agent systems. The contribution includes several necessary and/or sufficient conditions for the controllability under multiple leaders and switching topology. The results are not only necessary and/or sufficient, but also indicate to a certain degree how the controllability be impacted by the evolvement of the corresponding dynamic networks and the switched interconnection topologies.**

***Index Terms*—Multi-agent systems, controllability, switched systems, graph theory.**

I. INTRODUCTION

Distributed coordination of networks of dynamic agents has attracted a great deal of attention in recent years[1]-[10]. This is partly due to broad applications of multi-agent systems in, e.g. the cooperative control of unmanned aerial vehicles, and technology improvements allowing smaller, more versatile robots and other types of agents.

The controllability problem was put forward for the first time for multi-agent systems by Tanner in [5], and then developed in [6], [7], [8], [9], [10]. The problem is on how the interconnected systems can be steered to specific positions by regulating the motion of a single system that plays the role of the group leader. This is what the so-called the group can be controlled. This requires the characterization of conditions under which the leaders can move the followers into any desired position or configuration [6]. That is, to derive conditions for a group of systems interconnected via nearest neighbor rules, to be controllable by one of them acting as a leader [5].

It is essentially a kind of formation control problem. The problem is transformed to a classical notion of controllability in [5] with respect to a fixed interconnection topology and a switched controllability problem in [9], [10] with respect to a switching topology. One of the features for the controllability problem studied in [5], [9], [10] is that the leader is assumed unidirectional, i.e. the leader's neighbors still obey the interconnection nearest neighbor rules, but the leader is indifferent, and is free to pick any agent. Accordingly the leader does not participate in the typical configuration updates, and merely acts as an external control signals. The leader is not affected by the members whereas each member is influenced by the leader and the other members.

Central to the investigation of formation control is the nature of interconnection topologies. Some preliminary results on formation control were derived with respect to the fixed topology, which is a necessary step toward the more realistic dynamic setting. For example, in addition to [3], [7], the feasibility problem of achieving a specified geometric formation of a group of unicycles was investigated in [4], where necessary and sufficient graphical conditions for the existence of local information controller to assure the asymptotic convergence of the closed system were derived. Our goal is to consider the formation control, which is reformulated as the controllability problem in this paper, where the dynamics are influenced by switching topologies and leaders. The first result is an algebraic characterization of controllability. The disadvantage of the result is that it does not provide any insights on the impact of dynamic/switching topologies to the controllability. The second result then tries to make up for this shortfall, which shows that the controllability of multi-agent systems comes down to the constructively design of a dynamic evolvement pattern for the topologies of the corresponding dynamic networks. The results are helpful to a further understanding of the relationship between the formation control and the dynamic evolution of interconnection networks.

II. GRAPH THEORY PRELIMINARIES

Some notions in graph theory are recalled in this section.

An undirected graph $\mathcal{G}$ consists of a vertex set $\mathcal{V} = \{1, 2, \cdots, N+1\}$ and an edge set $\mathscr{E} = \{(i,j) : \ i,j \in \mathcal{V}\}$, where an edge is an unordered pair of distinct vertices of $\mathcal{V}$. Two vertices i and j are neighbors if $(i,j) \in \mathscr{E}$, and the neighboring relation is indicated with $j \sim i$. In this case we say that j is a neighbor of i. The number of neighbors of each vertex is its valency or degree. A path $i_0 i_1 \cdots i_s$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \cdots, s$, and a graph $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. The adjacency matrix $\mathcal{A}(\mathcal{G})$ of $\mathcal{G}$ is an $|\mathcal{V}| \times |\mathcal{V}|$ matrix of whose ijth entry is 1 if (i,j) is one of $\mathcal{G}$'s edges and 0 if it is not. Any undirected graph can be represented by its adjacency matrix, $\mathcal{A}(\mathcal{G})$, which is a symmetric matrix with 0-1 elements. The valency matrix $\triangle(\mathcal{G})$ of a graph $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i,j)-entry is the valency of vertex i.

The incidence matrix In$(\mathcal{G})$ of $\mathcal{G}$ is an $|\mathcal{V}| \times |\mathscr{E}|$ matrix, with one row for each node and one column for each edge. Suppose edge $e = (i,j)$. Then column e of In$(\mathcal{G})$ is zero except for the i-th and j-th entries, which are $+1$ and -1, respectively. The Laplacian matrix $L(\mathcal{G})$ of a graph $\mathcal{G}$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an undirected, unweighted graph without graph loops (i,i) or multiple edges from one node to another, is an $|\mathcal{V}| \times |\mathcal{V}|$ symmetric matrix with one row and column for each node defined by

$$L(\mathcal{G})_{i,j} = \begin{cases} d_i, & \text{if } i = j \text{ (number of incident edges)} \\ -1, & \text{if } i \neq j \text{ and } \exists \text{ edge } (i,j) \\ 0, & \text{otherwise.} \end{cases}$$

Given a graph $\mathcal{G}$, its associated matrices $In(\mathcal{G})$ and $L(\mathcal{G})$ have the following properties: (a) $L(\mathcal{G})$ is always symmetric and positive semidefinite; (b) zero is always a eigenvalue of $L(\mathcal{G})$ with $\mathbf{1}_n$, the vector of ones, being the associated eigenvector, and the algebraic multiplicity of the zero eigenvalue is equal to the number of connected components in the graph; (c) $In(\mathcal{G})(In(\mathcal{G}))^T = L(\mathcal{G})$, and $L(\mathcal{G}) = \triangle(\mathcal{G}) - \mathcal{A}(\mathcal{G})$.

III. Problem formulation and main results

Consider a multi-agent system consisting of $N + n_l$ agents with simple, first order dynamics:

$$\mathfrak{M} : \begin{cases} \dot{x}_i = u_i, & i = 1, \dots, N \\ \dot{x}_{N+j} = u_{N+j}, & j = 1, \dots, n_l \end{cases} \tag{1}$$

where x_{N+j} are leaders. The dimension of x_i could be arbitrary, as long as it is the same for all agents. For the simplicity of presentation, we will analyze only for the one-dimensional case. The analysis is valid for any dimension n, with the difference being that expressions should be rewritten in terms of Kronecker products. Once the linkages between agents are known, an interconnection graph can be defined to describe the interconnection network.

Definition 1. *[5] The interconnection graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is being defined as an undirected graph consisting of :*

- *a set of nodes, $\mathcal{V} = \{v_1, \dots, v_N, v_{N+1}, \dots, v_{N+n_l}\}$, indexed by the agents in the group, and*
- *a set of edges, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V} | n_i \sim n_j\}$, containing unordered pairs of nodes that correspond to interconnected agents.*

Interconnections come true through the input u_i

$$u_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j), \quad i = 1, \cdots, N; j = 1, \cdots, N + n_l, \tag{2}$$

where $\mathcal{N}_i = \{j \mid v_i \sim v_j; j \neq i\}$ is the set of indices of the agents that are interconnected to v_i, i.e., the neighboring set of v_i. Interconnections with the leader are now assumed unidirectional: the leader's neighbors still obey (2), but the leader is indifferent, and is free to pick $u_{N+j}, j = 1, \dots, n_l$ arbitrarily. With $x = (x_1, \cdots, x_{N+n_l})^T$ being the stack vector of all the agent states, we will have

$$\dot{x} = -Lx, \tag{3}$$

where L is the Laplacian matrix of the graph of interconnections. Rename the agents and then the multi-agent system reads

$$\mathfrak{M} : \begin{cases} y_i \triangleq x_i, & i = 1, \dots, N \\ z_j \triangleq x_{N+j}, & j = 1, \dots, n_l \end{cases}$$

with y being the stack vector of all y_i, z the stack vector of all z_j, and u the stack vector of all u_{N+j}, $j = 1, \dots, n_l$, one can write the system in the form:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = -\begin{bmatrix} F & R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

where F is the matrix obtained from L after deleting the last n_l rows and n_l columns, and R is the $N \times n_l$ submatrix consisting of the first N elements of the deleted columns. Then the dynamics of the followers that correspond to the y component of the equation can be extracted as

$$\dot{y} = -Fy - Rz. \tag{4}$$

Remark 1. *The selection of leaders $x_{N+j}, j = 1, \dots, n_l$, is indifferent, and it is free to pick any agents. The subsequent analysis is effective for any selected leaders.*

Definition 2. *A follower subgraph $\mathcal{G}_f$ of the interconnection graph is the subgraph induced by the follower set $\mathcal{V}_f$. Similarly, A leader subgraph $\mathcal{G}_f$ is the subgraph induced by the leader set $\mathcal{V}_l$.*

Definition 3. *The multi-agent system (1) is said to be controllable under leaders $x_{N+j}, j = 1, \dots, n_l$, and fixed topology if system (4) is controllable.*

Since the interconnection graph $\mathcal{G}$ is time variant, the dynamic (4) can be viewed more reasonably as a system in switching networks, which can be written in the form

$$\dot{y} = -F_{\sigma(t)} y - R_{\sigma(t)} z, \tag{5}$$

where $\sigma(t) : \mathbb{R}^+ \rightarrow \mathscr{M} \triangleq \{1, 2, \cdots, M\}$ is the switching signal/sequence to be designed. Given a switching signal $\sigma(t) : [t_0, t_f] \rightarrow \mathscr{M}$, we refer to $t_0, t_1, \cdots, t_{s-1}$ with $t_0 < t_1 < \cdots < t_{s-1}$ as the switching time sequence, and $\sigma(t_0) = i_0$, $\sigma(t_1) = i_1$, $\cdots, \sigma(t_{s-1}) = i_{s-1}$ as the switching index sequence. Let $h_i \triangleq t_{i+1} - t_i$, $i = 0, 1, \cdots, s-1$, and $t_s \triangleq t_f$. We denote by $\pi \triangleq \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$ a switching signal. The length of π is s. Throughout the paper, we denote by $L(\pi)$ the length of π. To investigate the controllability under switched dynamic networks and selected leaders, we give the following definitions.

Definition 4. *The multi-agent system (1) is said to be controllable under leaders $x_{N+j}, j = 1, \dots, n_l$ and switched topology if system (5) is controllable.*

The system (5) is controllable if for any nonzero state $y \in \mathbb{R}^N$, there exist a switching sequence π and input z such that $y(0) = y$, and $y(t_f) = 0$. We denote by $\mathcal{C}$ the controllable state set of system (5).

Definition 5. *$\{F_1, \cdots, F_M\}$ is said to be the switching topology set of system (1).*

Since a given $\sigma(t)$ represents an evolvement of the interconnection topology, we give the following definition.

Definition 6. *A given switching signal $\sigma(t)$ is said to be a **dynamic evolvement pattern** of the corresponding dynamic networks of the multi-agent system (1). A dynamic evolvement pattern is said to be periodic, if there is a subset $\{j_1, \cdots, j_s\}$ of $\mathscr{M}$ such that the switching index sequence is $\{j_1, \cdots, j_s, j_1, \cdots, j_s, \cdots\}$; otherwise it is said to be aperiodic.*

The interconnection topology is embodied in F_i. The switching topology set $\{F_1, \cdots, F_M\}$ contains all the possible topology structures. The switching signal $\sigma(t)$ describes the dynamic behavior of networks. Naturally, the problem that how controllability is impacted by the evolution of switching dynamical networks deserves careful study. In particular, we will denote ourselves to the study of how to determine $\sigma(t)$ so that the interconnection system is controllable.

Definition 7. *Assume $\mathcal{G}_1, \mathcal{G}_2$ are two subgraphs induced from the original graph $\mathcal{G}$. It is said that $\mathcal{G}_1$ and $\mathcal{G}_2$ are **linked** if there is a path between one of the nodes of $\mathcal{G}_1$ and one of the nodes of $\mathcal{G}_2$.*

We denote by $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$, the γ connected components in the follower subgraph $\mathcal{G}_f$. In subsequent arguments, the following assumption is made.

Assumption 1. *The leader subgraph $\mathcal{G}_l$ is linked to each of the connected components $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$ of the follower subgraph $\mathcal{G}_f$.*

It is worth noting that the assumption does not require the interconnection graph $\mathcal{G}$ be connected.Accordingly it is a less conservative condition than connectedness.

Let $L = (a_{ij})$ be the $(N+n_l) \times (N+n_l)$ Laplacian matrix of $\mathcal{G}$ associated with multi-agent systems (1). Assume that $L_{i_1,\ldots,i_\eta}$ is such a submatrix obtained by *deleting* the i_1th,..., i_ηth rows and i_1th,..., i_ηth columns of L, $i_1, \ldots, i_\eta \in \{1, \ldots, N+n_l\}$. The following is required for investigation of the controllability.

Lemma 1. *Under Assumption 1, $L_{N+1,\ldots,N+n_l}$ is a positive definite $N \times N$ matrix.*

The proof of this lemma is omitted due to the space limitation. The readers are referred to [16] for the detailed proof of this result.

Given a matrix $A \in \mathbb{R}^{N\times N}$, and a linear subspace $\mathcal{W} \subseteq \mathbb{R}^N$, We denote $\langle A|\mathcal{W}\rangle \triangleq \sum_{i=1}^N A^{i-1}\mathcal{W}$. It follows that $\langle A|\mathcal{W}\rangle$ is a minimum A-invariant subspace that contains $\mathcal{W}$. Given $B \in \mathbb{R}^{N\times p}$, let Im$B$ denote the image space of B. For notational simplicity, we denote by $\langle A|B\rangle$ the $\langle A|\text{Im}B\rangle$. For system (5), consider the nested subspace sequence defined by

$$\mathcal{W}_1 = \sum_{k=1}^M \langle -F_k| - r_k\rangle, \mathcal{W}_{s+1} = \sum_{k=1}^M \langle -F_k|\mathcal{W}_s\rangle, s = 1, 2, \cdots \tag{6}$$

The following result is on the controllability of system (5).

Lemma 2. *System (5) is controllable if and only if $\mathcal{W}_N = \mathbb{R}^N$.*

Proof: The result is a direct consequence of Theorem 1 in [14], or the main result in [15] and [11]. ■

Theorem 1. *Consider an interconnected system with n_l leaders and switching networks described by (5). Denote $R_i = [r_{1i}, \ldots, r_{n_l i}]$. Then z can control the dynamics of all the other states if the following conditions are satisfied:*

1) *With respect to each F_i, the eigenvalues of F_i are distinct from each other, $i = 1, \cdots, M$.*
2) *With respect to each F_i, the eigenvectors of F_i are not orthogonal to r_{ki}; $k = 1, \ldots, n_l$; $i = 1, \cdots, M$.*

Proof: In order to facilitate the statement, we prove the result only for the situation $N = 3, n_l = 2$, and $M = 2$. The general case can be proved in the same manner. In what follows, $\mathcal{W}_N$ will be calculated at first.

Since F_i is symmetric, it can be expressed as

$$-F_i = -U_iD_iU_i^T = U_i\widehat{D}_iU_i^T \triangleq H_i, \quad i = 1, \cdots, M,$$

where $\widehat{D}_i \triangleq -D_i$, U_i is an orthogonal matrix. Denote $(-F_i, -R_i) \triangleq (H_i, B_i)$, one has

$$\langle H_i|B_i\rangle = \sum_{j=1}^N H_i^{j-1}\text{Im}B_i = U_i\sum_{j=1}^N \widehat{D}_i^{j-1}\text{Im}\widehat{B}_i,$$

where $\widehat{B}_i \triangleq U_i^TB_i$. Set $\widehat{D}_i \triangleq diag\left\{\hat{d}_{1i}, \ldots, \hat{d}_{Ni}\right\}$, $\widehat{B}_i \triangleq \left[\widehat{b}_{i1}, \widehat{b}_{i2}\right]$, and $\widehat{b}_{ik} \triangleq \left[\hat{b}_{ik}^{(1)}, \ldots, \hat{b}_{ik}^{(N)}\right]^T$, it can be seen that

$$\langle H_i|B_i\rangle = \text{Im}\Gamma_i \tag{7}$$

with

$$\Gamma_i \triangleq U_i\left[\begin{bmatrix} \hat{b}_{i1}^{(1)} & \hat{d}_{1i}\hat{b}_{i1}^{(1)} & \cdots & \hat{d}_{1i}^{N-1}\hat{b}_{i1}^{(1)} \\ \hat{b}_{i1}^{(2)} & \hat{d}_{2i}\hat{b}_{i1}^{(2)} & \cdots & \hat{d}_{2i}^{N-1}\hat{b}_{i1}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{b}_{i1}^{(N)} & \hat{d}_{Ni}\hat{b}_{i1}^{(N)} & \cdots & \hat{d}_{Ni}^{N-1}\hat{b}_{i1}^{(N)} \end{bmatrix}\right.,$$

$$\left.\begin{bmatrix} \hat{b}_{i2}^{(1)} & \hat{d}_{1i}\hat{b}_{i2}^{(1)} & \cdots & \hat{d}_{1i}^{N-1}\hat{b}_{i2}^{(1)} \\ \hat{b}_{i2}^{(2)} & \hat{d}_{2i}\hat{b}_{i2}^{(2)} & \cdots & \hat{d}_{2i}^{N-1}\hat{b}_{i2}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{b}_{i2}^{(N)} & \hat{d}_{Ni}\hat{b}_{i2}^{(N)} & \cdots & \hat{d}_{Ni}^{N-1}\hat{b}_{i2}^{(N)} \end{bmatrix}\right]$$

$$= [U_i, U_i]\begin{bmatrix} \Lambda_{i1} & \\ & \Lambda_{i2} \end{bmatrix}\begin{bmatrix} \Xi_i & \\ & \Xi_i \end{bmatrix},$$

where

$$\Lambda_{ik} \triangleq \begin{bmatrix} \hat{b}_{ik}^{(1)} & & \\ & \ddots & \\ & & \hat{b}_{ik}^{(N)} \end{bmatrix}, \Xi_i \triangleq \begin{bmatrix} 1 & \hat{d}_{1i} & \cdots & \hat{d}_{1i}^{N-1} \\ 1 & \hat{d}_{2i} & \cdots & \hat{d}_{2i}^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \hat{d}_{Ni} & \cdots & \hat{d}_{Ni}^{N-1} \end{bmatrix}$$

As a consequence,

$$\mathcal{W}_1 = \mathrm{Im}\Theta_1, \tag{8}$$

where

$$\begin{aligned}\Theta_1 \triangleq & [U_1, U_1, U_2, U_2]\, diag\{\Lambda_{11}, \Lambda_{12}, \Lambda_{21}, \Lambda_{22}\} \\ & \times diag\{\Xi_1, \Xi_1, \Xi_2, \Xi_2\}.\end{aligned}$$

Next, we consider $\mathcal{W}_2$. By definition, it is given by

$$\begin{aligned}\mathcal{W}_2 &= \langle H_1|\mathcal{W}_1\rangle + \langle H_2|\mathcal{W}_1\rangle \\ &= \langle H_1|B_1\rangle + \langle H_2|B_2\rangle + H_1\langle H_2|B_2\rangle + H_1^2\langle H_2|B_2\rangle \\ &\quad + H_2\langle H_1|B_1\rangle + H_2^2\langle H_1|B_1\rangle.\end{aligned} \tag{9}$$

Let $U_{ij}^T \triangleq U_i^T U_j$. Then U_{ij} is an orthogonal matrix since U_i, U_j are orthogonal matrices. To express $\mathcal{W}_2$ further, the following matrix Θ_2 is introduced.

$$\begin{aligned}\Theta_2 \triangleq & \Big[U_1, U_1, U_2\widehat{D}_2U_{21}^T, U_2\widehat{D}_2U_{21}^T, U_2\widehat{D}_2^2U_{21}^T, U_2\widehat{D}_2^2U_{21}^T, \\ & U_2, U_2, U_1\widehat{D}_1U_{12}^T, U_1\widehat{D}_1U_{12}^T, U_1\widehat{D}_1^2U_{12}^T, U_1\widehat{D}_1^2U_{12}^T\Big] \\ & \times diag\,[\Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \\ & \qquad \Lambda_{22}, \Lambda_{21}, \Lambda_{22}] \\ & \times diag\,[\Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2]\end{aligned}$$

It follows from (7),(8) and (9) that

$$\mathcal{W}_2 = \mathrm{Im}\,\Theta_2. \tag{10}$$

Now we are in a position to compute $\mathcal{W}_N = \mathcal{W}_3$. By definition, one has

$$\begin{aligned}\mathcal{W}_3 &= \langle -F_1|\mathcal{W}_2\rangle + \langle -F_2|\mathcal{W}_2\rangle = \langle H_1|\mathcal{W}_2\rangle + \langle H_2|\mathcal{W}_2\rangle \\ &= \mathcal{W}_2 + H_1\mathcal{W}_2 + H_1^2\mathcal{W}_2 + H_2\mathcal{W}_2 + H_2^2\mathcal{W}_2 \\ &= \mathcal{W}_2 + H_1H_2\langle H_1|B_1\rangle + H_1H_2^2\langle H_1|B_1\rangle \\ &\quad + H_1^2H_2\langle H_1|B_1\rangle + H_1^2H_2^2\langle H_1|B_1\rangle \\ &\quad + H_2H_1\langle H_2|B_2\rangle + H_2H_1^2\langle H_2|B_2\rangle + H_2^2H_1\langle H_2|B_2\rangle \\ &\quad + H_2^2H_1^2\langle H_2|B_2\rangle.\end{aligned} \tag{11}$$

To proceed, the following two matrices are defined.

$$\begin{aligned}&\Theta_{3,1} \\ &\triangleq \Big[U_1\widehat{D}_1U_{12}^T\widehat{D}_2U_{12}, U_1\widehat{D}_1U_{12}^T\widehat{D}_2U_{12}, U_1\widehat{D}_1U_{12}^T\widehat{D}_2^2U_{12}, \\ & U_1\widehat{D}_1U_{12}^T\widehat{D}_2^2U_{12}, U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2U_{12}, U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2U_{12}, \\ & U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2^2U_{12}, U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2^2U_{12}\Big] \\ & \times diag\,\{\Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}\} \\ & \times diag\,\{\Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1\}\end{aligned}$$

and

$$\begin{aligned}&\Theta_{3,2} \\ &\triangleq \Big[U_2\widehat{D}_2U_{21}^T\widehat{D}_1U_{21}, U_2\widehat{D}_2U_{21}^T\widehat{D}_1U_{21}, U_2\widehat{D}_2U_{21}^T\widehat{D}_1^2U_{21}, \\ & U_2\widehat{D}_2U_{21}^T\widehat{D}_1^2U_{21}, U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1U_{21}, U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1U_{21}, \\ & U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1^2U_{21}, U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1^2U_{21}\Big] \\ & \times diag\,\{\Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}\} \\ & \times diag\,\{\Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2\}\end{aligned}$$

It can be seen from (7) that

$$\begin{aligned}\mathrm{Im}\Theta_{3,1} =& H_1H_2\langle H_1|B_1\rangle + H_1H_2^2\langle H_1|B_1\rangle \\ & + H_1^2H_2\langle H_1|B_1\rangle + H_1^2H_2^2\langle H_1|B_1\rangle\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathrm{Im}\Theta_{3,2} =& H_2H_1\langle H_2|B_2\rangle + H_2H_1^2\langle H_2|B_2\rangle \\ & + H_2^2H_1\langle H_2|B_2\rangle + H_2^2H_1^2\langle H_2|B_2\rangle\end{aligned} \tag{13}$$

Combining (10),(11),(12) with (13) yields

$$\mathcal{W}_3 = \mathrm{Im}\,[\Theta_2, \Theta_{3,1}, \Theta_{3,2}].$$

Furthermore, by computation, one has

$$\begin{aligned}\Theta \triangleq & [\Theta_2, \Theta_{3,1}, \Theta_{3,2}] \\ = & \Big[U_1, U_1, U_2\widehat{D}_2U_{21}^T, U_2\widehat{D}_2U_{21}^T, U_2\widehat{D}_2^2U_{21}^T, U_2\widehat{D}_2^2U_{21}^T, \\ & U_2, U_2, U_1\widehat{D}_1U_{12}^T, U_1\widehat{D}_1U_{12}^T, U_1\widehat{D}_1^2U_{12}^T, U_1\widehat{D}_1^2U_{12}^T, \\ & U_1\widehat{D}_1U_{12}^T\widehat{D}_2U_{12}, U_1\widehat{D}_1U_{12}^T\widehat{D}_2U_{12}, U_1\widehat{D}_1U_{12}^T\widehat{D}_2^2U_{12}, \\ & U_1\widehat{D}_1U_{12}^T\widehat{D}_2^2U_{12}, U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2U_{12}, U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2U_{12}, \\ & U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2^2U_{12}, U_1\widehat{D}_1^2U_{12}^T\widehat{D}_2^2U_{12}, U_2\widehat{D}_2U_{21}^T\widehat{D}_1U_{21}, \\ & U_2\widehat{D}_2U_{21}^T\widehat{D}_1U_{21}, U_2\widehat{D}_2U_{21}^T\widehat{D}_1^2U_{21}, U_2\widehat{D}_2U_{21}^T\widehat{D}_1^2U_{21}, \\ & U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1U_{21}, U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1U_{21}, U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1^2U_{21}, \\ & U_2\widehat{D}_2^2U_{21}^T\widehat{D}_1^2U_{21}\Big] \\ & \times diag\,\{\Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \\ & \Lambda_{22}, \Lambda_{21}, \Lambda_{22}\Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \\ & \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}\} \\ & \times diag\,\{\Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \\ & \qquad \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \\ & \qquad \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2\}.\end{aligned}$$

It can be found that

$$\begin{aligned}\Theta = & [U_1, U_1, U_2, U_2, U_2, U_2, U_2, U_2, U_1, U_1, U_1, U_1, U_1, U_1, \\ & U_1, U_1, U_1, U_1, U_1, U_1, U_2, U_2, U_2, U_2, U_2, U_2, U_2, U_2] \\ & \times diag\Big\{I, I, \widehat{D}_2, \widehat{D}_2, \widehat{D}_2^2, \widehat{D}_2^2, I, I, \widehat{D}_1, \widehat{D}_1, \widehat{D}_1^2, \widehat{D}_1^2, \\ & \quad \widehat{D}_1, \widehat{D}_1, \widehat{D}_1, \widehat{D}_1, \widehat{D}_1^2, \widehat{D}_1^2, \widehat{D}_1^2, \widehat{D}_1^2, \widehat{D}_2, \widehat{D}_2, \widehat{D}_2, \widehat{D}_2, \\ & \qquad \widehat{D}_2^2, \widehat{D}_2^2, \widehat{D}_2^2, \widehat{D}_2^2\Big\} \\ & \times diag\,\{I, I, U_{21}^T, U_{21}^T, U_{21}^T, U_{21}^T, I, I, U_{12}^T, U_{12}^T, U_{12}^T, \\ & \quad U_{12}^T, U_{12}^T, U_{12}^T, U_{12}^T, U_{12}^T, U_{12}^T, U_{12}^T, \\ & \quad U_{12}^T, U_{12}^T, U_{21}^T, U_{21}^T, U_{21}^T, U_{21}^T, U_{21}^T, U_{21}^T, U_{21}^T, U_{21}^T\} \\ & \times diag\Big\{I, I, I, I, I, I, I, I, I, I, I, I, \widehat{D}_2, \widehat{D}_2, \widehat{D}_2^2, \widehat{D}_2^2,\end{aligned}$$

$$\widehat{D}_2, \widehat{D}_2, \widehat{D}_2^2, \widehat{D}_2^2, \widehat{D}_1, \widehat{D}_1, \widehat{D}_1^2, \widehat{D}_1^2, \widehat{D}_1, \widehat{D}_1, \widehat{D}_1^2, \widehat{D}_1^2\Big\}$$
$$\times\, diag\,\{I, I, I, I, I, I, I, I, I, I, I, I, U_{12}, U_{12}, U_{12}, U_{12}, U_{12}, U_{12}, U_{12}, U_{12}, U_{12}, U_{21}, U_{21}, U_{21}, U_{21}, U_{21}, U_{21}, U_{21}, U_{21}\}$$
$$\times\, diag\,\{\Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}\Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}\}$$
$$\times\, diag\,\{\Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2\}\,.$$

Since F_i, and then $\widehat{D}_i = -D_i$ is nonsingular due to Lemma 1, and both U_i and U_{ij} are orthogonal matrices, one has

$$rank\,\Theta = N \ \text{ if } \ |\Lambda_{ik}| \neq 0, \text{ and } |\Xi_i| \neq 0,$$

$i = 1, \ldots, M; k = 1, \ldots, n_l$. Denote $U_i \triangleq [u_{1i}, u_{2i}, \cdots, u_{Ni}]$, $R_i = [r_{1i}, \ldots, r_{n_l i}]$. Then $\hat{b}_{ik}^{(j)} = -u_{ji}^T r_{ki}$; $i = 1, \ldots, M$; $k = 1, \ldots, n_l; j = 1, \ldots, N$. It follows that

$$\begin{aligned}
&|\Lambda_{ik}| \neq 0,\ i = 1, \ldots, M; k = 1, \ldots, n_l.\\
\Longleftrightarrow\ &\hat{b}_{ik}^{(j)} = -u_{ji}^T r_{ki} \neq 0, \quad i = 1, \ldots, M; k = 1, \ldots, n_l;\\
&j = 1, \ldots, N,\\
\Longleftrightarrow\ &\text{For each } F_i, \text{ the eigenvectors of } F_i \text{ are not orthogonal}\\
&\text{to } r_{ki};\ k = 1, \ldots, n_l;\ \ i = 1, \cdots, M,
\end{aligned}$$

and

$$\begin{aligned}
&|\Xi_i| \neq 0,\ i = 1, \cdots, M,\\
\Longleftrightarrow\ &\begin{vmatrix} 1 & \hat{d}_{1i} & \cdots & \hat{d}_{1i}^{N-1} \\ 1 & \hat{d}_{2i} & \cdots & \hat{d}_{2i}^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \hat{d}_{Ni} & \cdots & \hat{d}_{Ni}^{N-1} \end{vmatrix}\\
&= \prod_{1 \le q < p \le N} \left(\hat{d}_{pi} - \hat{d}_{qi}\right) \neq 0,\ i = 1, \cdots, M\\
\Longleftrightarrow\ &\text{For each } F_i, \text{ the eigenvalues of } F_i \text{ are distinct}\\
&\text{from each other, }\ i = 1, \cdots, M.
\end{aligned}$$

The above analysis shows that $rank\Theta = N$ if Conditions 1, 2 are fulfilled. Since $\dim \mathcal{W}_N = N \Longleftrightarrow rank\Theta = N$, the result then follows from Lemma 2. The proof of the general situation can be conducted in the same way. The difference consists in the expression of $\mathcal{W}_N$, which becomes more and more complex as the state dimension N and the number of subsystems M increase. ■

Although Theorem 1 presents a condition on controllability, it does not exhibit any information on the evolution of dynamic interconnection networks. In what follows we will consider the design of switching signals. To this end, the following definition is necessary.

Definition 8. *Given a dynamic evolvement pattern* $\pi = \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$, *denote* $t_f = \sum_{j=0}^{s-1} h_j$, *the controllable state set* $\mathcal{C}(\pi)$ *of* π *with respect to system (5) is defined by*

$$\mathcal{C}(\pi) = \{y|\ \textit{there exists an input } z(t), t \in [0, t_f], \textit{such that } y(0) = y \textit{ and } \ y(t_f) = 0\}.$$

Clearly, $\mathcal{C} = \bigcup_{\forall \pi} \mathcal{C}(\pi)$. To state the result, we need to introduce some notations. Let $\mu = \min\{k|\ \mathcal{W}_k = \mathcal{W}_{k+1}, k = 1, 2, \cdots\}$, where $\mathcal{W}_k$ is the subspace iteratively defined in (6). It can be seen that μ can be equivalently defined by $\mu = \min\{k|\ \dim \mathcal{W}_k = \dim \mathcal{W}_{k+1},\ k = 1, 2, \cdots\}$. Denote $d_k = \dim \mathcal{W}_k$, $\ell_1 = d_1$, $\ell_k = d_k - d_{k-1}$, $k = 2, 3, \cdots, \mu$, and $d = \dim \mathcal{W}_N$. Obviously, $d_\mu = d$, and $\mathcal{W}_\mu = \mathcal{W}_N$. Let $\rho_k = \dim \langle -F_k | -r_k \rangle, k \in \mathscr{M}$, and $\beta = \max_{k \in \mathscr{M}}\{\rho_k\}$. We have the following observations.

Lemma 3. *The multi-agent system (1) is controllable under switching topology and the leaders* $x_{N+j}, j = 1, \ldots, n_l$; *if and only if there exists an aperiodic dynamic evolvement pattern* π_b *such that the controllable state set of* π_b, *i.e.* $\mathcal{C}(\pi_b)$ *satisfies*

$$\mathcal{C}(\pi_b) = \mathbb{R}^N.$$

Moreover the evolvement pattern π_b *can be constructively designed with its length, i.e. the number of switchings involved in* π_b, *satisfying*

$$\frac{\mu(\mu+1)}{2} \le L(\pi_b) \le \sum_{k=1}^{\mu} k\ell_k - \beta + 1. \tag{14}$$

Proof: It follows from the Theorem 1 in [12](or the Theorem 1 in [13]) that for systems (5), there exists a switching signal π_b with its length satisfying (14) such that $\mathcal{C}(\pi_b) = \mathcal{C}$, where $\mathcal{C}$ is the controllable subspace of systems (5), namely the controllable subspace of systems (1) under the leader x_{N+1} and switching topology. Accordingly, the multi-agent system (1) is controllable if and only if $\mathcal{C}(\pi_b) = \mathbb{R}^N$. Moreover, due to the proof of the Theorem 1 in [12](or the Theorem 1 in [13]), the evolvement pattern π_b can be designed according to the following steps:

1) Compute $\mathcal{W}_1$, and choose a group of basis vectors $\xi_1, \cdots, \xi_{d_1}$ for $\mathcal{W}_1$. A concrete procedure is as follows: Firstly, choose a group of basis vectors $\xi_{1,\cdots,}\xi_{\tau_{1,1}}$ for $\langle -F_1 | -R_1 \rangle$. Then expand them to $\xi_{1,\cdots,}\xi_{\tau_{1,1}}$, $\xi_{\tau_{1,1}+1,\cdots,}\xi_{\tau_{2,1}}$, which form a basis for $\langle -F_1 | -R_1 \rangle + \langle -F_2 | -R_2 \rangle$. Continuing this process, one can find a basis $\xi_1, \cdots, \xi_{\tau_{1,1}}, \xi_{\tau_{1,1}+1}, \cdots, \xi_{\tau_{2,1}}, \cdots, \xi_{\tau_{l_1-1,1}+1}, \cdots, \xi_{\tau_{l_1,1}}$ for $\mathcal{W}_1$, where $\tau_{l_1,1} = d_1$.
2) The choosing process divides the basis vectors $\xi_1, \cdots, \xi_{d_1}$ into l_1 groups, namely, $\{\xi_1, \cdots,\ \xi_{\tau_{1,1}}\}$, $\{\xi_{\tau_{1,1}+1}, \cdots, \xi_{\tau_{2,1}}\}$, $\cdots, \{\xi_{\tau_{l_1-1,1}+1}, \cdots, \xi_{\tau_{l_1,1}}\}$. With respect to each group, one can design a switching signal. Consequently there are totally l_1 switching signals $\pi_{1,1}, \cdots, \pi_{l_1,1}$ designed for $\mathcal{W}_1$.
3) Since $\mathcal{W}_1 \subset \mathcal{W}_2$, the basis vectors of $\mathcal{W}_1$ can be expanded to $\xi_1, \cdots, \xi_{d_1}, \xi_{d_1+1}, \cdots, \xi_{d_2}$ which form a basis of $\mathcal{W}_2$. Because $\xi_{d_1+1}, \cdots, \xi_{d_2}$ belong to $\mathcal{W}_2 \backslash \mathcal{W}_1$, $\xi_{d_1+1}, \cdots, \xi_{d_2}$ can be divided into l_2 groups of vectors, namely, $\{\xi_{d_1+1}, \cdots, \xi_{\tau_{1,2}}\}$, $\{\xi_{\tau_{1,2}+1}, \cdots, \xi_{\tau_{2,2}}\}$, $\cdots$, $\{\xi_{\tau_{l_2-1,2}+1}, \cdots, \xi_{\tau_{l_2,2}}\}$. With respect to each group, one

can design a switching signal. Consequently there are totally l_2 switching signals $\pi_{1,2}, \cdots, \pi_{l_2,2}$ designed for $\mathcal{W}_2$.

4) Repeating the same process as step 3) for $\mathcal{W}_s$, $s = 3, \cdots, \mu$, one can design l_s switching signals $\pi_{1,s}, \cdots, \pi_{l_s,s}$ for $\mathcal{W}_s$. Then, one can set $\pi_s = \pi_{1,s} \wedge \cdots \wedge \pi_{l_s,s}$.
5) Finally, the desired aperiodic dynamic evolvement pattern π_b can be designed as follows:

$$\begin{aligned}\pi_b &= \pi_1 \wedge \cdots \wedge \pi_\mu \\ &= (\pi_{1,1} \wedge \cdots \wedge \pi_{l_1,1}) \wedge \cdots \wedge (\pi_{1,\mu} \wedge \cdots \wedge \pi_{l_\mu,\mu}).\end{aligned}$$

We refer to [12], [13] for the concrete design process and expressions of $\pi_{1,s}, \cdots, \pi_{l_s,s}$, $s = 1, \cdots, \mu$. ■

Let $\pi_{h_1,\cdots,h_M} \triangleq \{(1, h_1) \cdots (M, h_M)\}$, $h_i > 0, i = 1, \cdots, M$. One has the following result.

Lemma 4. *The multi-agent system (1) is controllable under switching topology and the leaders $x_{N+j}, j = 1, \ldots, n_l$; if and only if there is a periodic dynamic evolvement pattern $\pi^{\wedge d}_{h_1,\cdots,h_M}$ such that*

$$\mathcal{C}\left(\pi^{\wedge d}_{h_1,\cdots,h_M}\right) = \mathbb{R}^N.$$

Proof: The Theorem 2 in [11] shows that $\mathcal{C}\left(\pi^{\wedge d}_{h_1,\cdots,h_M}\right) = \mathcal{C}$. The result then follows from this fact. ■

Clearly, the number of switchings involved in $\pi^{\wedge d}_{h_1,\cdots,h_M}$ is dM. To sum up, we state the following result.

Theorem 2. *The multi-agent system (1) is controllable under switching topology and the leaders $x_{N+j}, j = 1, \ldots, n_l$; if and only if there is a dynamic evolvement pattern π such that $\mathcal{C}(\pi) = \mathbb{R}^N$. If π is aperiodic, it can be constructively designed according to 1)-5) with the number of switchings satisfying (14). If π is a periodic one, it can be in the form of $\pi^{\wedge d}_{h_1,\cdots,h_M}$ with the number of switchings not more than dM.*

Remark 2. *Theorem 2 implies that the dynamic evolvement of switching networks plays an important role in the formation control of multi-agent systems. It is shown that not only the controllability can be characterized by the algebraic condition in Theorem 1, but also the associated dynamic evolvement pattern can be constructively designed according to Theorem 2. Note that all the results hold under the Assumption 1, which is much less conservative the condition of connectedness.*

IV. Conclusion

In this paper we study the controllability of multi-agent systems in the framework of leader-follower, in which the followers are interconnected via nearest neighbor rules, and the leader takes the role of control input. Necessary and/or sufficient conditions are derived as well as the dynamic evolvement patterns are constructively designed for the system to be controllable. The results show in some sense that the formation of multi-agent systems could be greatly affected by the evolution of dynamic interconnection topologies.

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