Global Asymptotic Stability of Stochastic Neural Networks with Time-Varying Delays

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Abstract—**This paper is concerned with asymptotic stability of stochastic neural networks with time-varying delay. Distinct difference from other analytical approach lies in "linearization" of neural network model, by which the considered neural network model is transformed into a linear time-variant system. A sufficient condition is derived such that for all admissible disturbance, the considered neural network is asymptotic stability in the mean square. The stability criterion is formulated by means of the feasibility of a LMI, which can be easily checked in practice. Finally, a numerical example is given to illustrate the effectiveness of the developed method.**

***Keywords*—neural network, stochastic system, time-varying delays, linear matrix inequality**

I. INTRODUCTION

Different classes of neural networks with or without delays have been extensively studied in the past few years, due to its practical importance and successful applications in many areas such as combinatorial optimization, signal processing and pattern recognition [1]. Moreover, it should be pointed out that the applications of neural networks rely heavily on the dynamical behaviors of the networks. Therefore, stability analysis of neural networks has been investigated and a great number of approaches have been proposed. Time delay is inevitably encountered in neural network, since the interactions between different neurons are asynchronous. Time delays are a source of instability and bad performance of neural networks. Recently, many research interests have been attracted to the stability analysis for delayed neural networks. A great deal of result related to this issue have been reported, see [2-4] and reference therein.

In recent years, the stability analysis issues for neural networks in these presence of parameter uncertainties or stochastic perturbations have caught some people's notice [5,6]. Connection weights of the neurons depend on certain resistance and capacitance values that include uncertainties, and in real nervous systems the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters, and other probabilistic causes. A neural network could be stabilized or destabilized by some stochastic inputs [7], which implies that it is of practical significance to study the stability for delayed stochastic neural networks. However, the stability analysis of stochastic neural networks is more difficult than that of traditional neural networks.

In [8], the asymptotical stability was studied for uncertain stochastic neural networks with discrete and distributed delays. in [9]exponential stability was studied for uncertain stochastic neural networks with multiple delays, in [10] delay-dependent stability was studied for uncertain stochastic neural networks with time-varying delay.

Motivated by the aforementioned discussion, this paper focuses on the asymptotical stability problem for stochastic neural networks with time-varying delay. An "linearization" approach is employed to shift nonlinear system into an interval linear time-varying system under the appropriate assumption on the activation functions. A stability criterion is developed by using the Lyapunov stability theory and the LMI technique. The LMI condition can be efficiently solved by LMI Control Toolbox, and no turning of parameters is required [11].

Notations*:* for convenience, some notations are introduced. For a real square matrix X, the notation $X>0(X<0)$ means that X is symmetric and positive definite (negative definite). I is the identity matrix with appropriate dimension. The superscript ''T'' represents the transpose. For $\tau>0$, $\wp([-\tau,0];R^n)$ denotes the family of continuous functions φ from $[-\tau,0]$ to R^n with the norm $\|\varphi\|=\sup_{-\tau\le\vartheta\le0}|\varphi(\vartheta)|$. Let $(\Omega,F,\{F_t\}_{t>0},P)$ be a complete probability space with a filtration $\{F_t\}_{t\ge0}$ satisfying the usual conditions (i.e. it is right continuous and F_0 contains all P-pull sets); $L^p_{F_0}([-h,0];R^n)$ the family of all F_0 -measurable $\wp([-\tau,0];R^n)$ -valued random variables $\xi=\{\xi(\theta):-\tau\le\theta\le0\}$ such that $\sup_{-\tau\le\theta\le0}E|\xi(\theta)|^p<\infty$ where E{.} stands for the mathematical expectation operator with respect to the given probability measure P; $\wp^{2,1}(R^n\times R^+;R^+)$ the family of all nonnegative functions $V(x,t)$ on $R^n\times R^+$ which are continuously twice differentiable in x and differentiable in t.

II. PROBLEM FORMULATION

the neural network with time-varying delay and stochastic perturbations can be described as follows:

$$dx(t) = [-Dx(t) + AG(x(t)) + BG(x(t-\tau(t))]dt + \sigma(t, x(t), x(t-\tau(t)))d\omega(t) \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \ldots x_n(t))^T$ is the state vector, $D = diag(d_1, d_2, \ldots, d_n)$ has positive entries $d_i > 0$, A and B are known constant matrices with appropriate dimensions , $G(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T$ is the neuron activation function vector with g(0)=0, $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_m(t))^T$ is an m-dimensional Brownian motion defined on $(\Omega, F, \{F_t\}_{t>0}, P)$. $\tau(t)$ is time-varying delay. $\sigma(x, y_1, y_2, \ldots, y_n, t) \in R^{n\times m}$ is locally Lipschitz continuous and satisfies the linear growth condition as well. Moreover , σ satisfies

$$\begin{aligned} &trace[\sigma^T(x(t), x(t-\tau(t), t)P\sigma(x(t), x(t-\tau(t), t)] \\ &\le x^T(t)\Sigma_1 x(t) + x^T(t-\tau(t))\Sigma_2 x(t-\tau(t)) \end{aligned} \quad (2)$$

where $\Sigma_1 > 0$ and $\Sigma_2 > 0$ are known constant matrices with appropriate dimensions.

In order to obtain our main results, the assumptions are always made throughout this paper.

(H1) The active function g(.) is bounded, and there exists constants L_i>0 such that ,for any $x, y \in R, i = 1, 2, \cdots, n, | g_i(x) - g_i(y) | \le L_i | x - y |$,

(H2) The time-varying delay $\tau_i(t) : [0, \infty) \to [0, h]$ are continuous and differentiable functions with $| \dot{\tau}_i(t) | \le \varepsilon < 1, i = 1, 2, \cdots n.$

Recall the assumption (H1) on the activation functions and g(0)=0, we can define, for $i = 1, 2, \cdots, n$

$$s_i(t) = \begin{cases} \frac{g_i(x_i(t))}{x_i(t)}, x_i(t) \ne 0, \\ 0, x_i(t) = 0, \end{cases} \quad (3)$$

Obviously, $s_i(t)$ is piecewise continuous on R. From (3) and the assumption (H1),we have $-1 \le s_i \le 1$. Furthermore, system (1) can be rewritten as, respectively,

$$\begin{aligned} dy(t) &= [-Dy(t) + AS(t)y(t) + BS(t-\tau(t))y(t-\tau(t)]dt \\ &+ \sigma(t, x(t), x(t-\tau(t)))d\omega(t) \end{aligned} \quad (4)$$

Where $S(t) = diag(s_i(t))_{n\times n}$

In Ref.[12], the fact

$$x(t-\tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t+\xi)d\xi = x(t) - \int_{-\tau}^{0} [Ax(t+\theta) + A_d x(t+\xi-\tau)]d\xi \quad (5)$$

We used to transform the system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \quad (6)$$

into a distributed delay system:

$$\dot{x}(t) = (A+C)x(t) + (A_d - C)x(t-\tau) - C\int_{-\tau}^{0} [Ax(t+\theta) + A_d x(t+\theta-\tau)]d\theta \quad (7)$$

Where, C is a parameter matrix which makes the stability result less restrictive to some degree. Such process is generically called a parameterized first-order model transformation since only one-integration over one delay interval is used herein. Obviously, the new system (7) may exhibit some additional dynamics. However, the stability of (7) implies the stability of (6) for all $\tau \in [0, h]$ (but the inverse is not always true). We refer the readers to Ref. [12] for the further discuss to the original system (5).

Applying the model transformation above to the considered systems (4), we derive

$$\begin{aligned} \dot{y}(t) &= (-D + AS(t) + C)y(t) + (BS(t-\tau(t)) - C)y(t-\tau(t)) \\ &-C\int_{t-\tau(t)}^{t} \begin{pmatrix} (-D + AS(\xi))y(\xi) \\ + BS(\xi - \tau(t))y(\xi - \tau(t)) \end{pmatrix} d\xi + \sigma(t, x(t), x(t-\tau(t)))d\omega(t) \end{aligned} \quad (8)$$

For the analysis made above, the stability of (8) implies the stability of (4),and hence, in what follows we mainly focus on the stability analysis for system (8).

Remark 1. The assumption (2) on the stochastic disturbance term, $\sigma^T(x(t), x(t-\tau(t), t)$ .has been used in recent papers dealing with stochastic neural networks, see[9] and references therein.

We are now in a position to introduce the notion of global asymptotic for the stochastic neural network (4) with time-varying delay.

Before starting the main results, we first need the following lemmas.

Lemma 1. Given any real matrices $\Sigma_1, \Sigma_2, \Sigma_3$ of appropriate dimensions and a scalar $\varepsilon > 0$ such that $0 < \Sigma_3 = \Sigma_3^T$. Then , the following inequality holds:

$$\Sigma_1^T\Sigma_2 + \Sigma_2^T\Sigma_1 \le \varepsilon\Sigma_1^T\Sigma_3\Sigma_1 + \varepsilon^{-1}\Sigma_2^T\Sigma_3^{-1}\Sigma_2.$$

Lemma 2. (Schur Complement). The following linear matrix inequality(LMI)

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0$$

Where $Q(x) = Q^T(x), R(x) = R^T(x),$ and S(x) depend affinely on x, is equivalent to

$$R(x) > 0, \ \ Q(x) - S(x)R^{-1}(x)S^T(x) > 0.$$

III. STOCHASTIC STABILITY ANALYSIS

Definition1. For the neural network (4) and every $\xi \in L^2_{F_0}([-h, 0]; R^n)$, the trivial solution (equilibrium point) is globally, asymptotically stable in the mean square if the following holds:

$$\lim_{t\to\infty} E\,|\,x(t;\xi)\,|^2 = 0 \tag{9}$$

$It\hat{o}'s$ formula [13] plays a key role in the stability analysis of stochastic systems. To facilitate the reader, some related results are cited here (see [16] for details). For a general stochastic system

$dx(t) = h_1(x(t),t)dt + h_2(x(t),t)d\omega(t)$ on $t \ge t_0$ with initial value $x(t_0) = x_0 \in R^n$, where $\omega(t)$ is m-dimensional Brownian motion defined on $(\Omega, F, \{F_t\}_{t>0}, P)$, $h_1 : R^n \times R^+ \to R^n$ and $h_2 : R^n \times R^+ \to R^{n\times m}$. Let $V \in \wp^{2,1}(R^n \times R^n; R^+)$, an operator LV is defined from $R^n \times R^n$ to R by

$$LV(x,t) = V_t(x,t) + V_x(x,t)h_1(x,t) + \tfrac{1}{2}trace[h_2^T(x,t)V_{xx}(x,t)h_2(x,t)],$$

Where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t},$$
$$V_x(x,t) = (\frac{\partial V(x,t)}{\partial x_1},\dots,\frac{\partial V(x,t)}{\partial x_n}),$$
$$V_{xx}(x,t) = (\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j})_{n\times n}.$$

Then, one can have

$$EV(x(t),t) = EV(x_0,t_0) + E\int_{t_0}^{t} LV(x(s))ds$$

The main purpose of this paper is to present some LMI-based stability criteria under which the global asymptotical stability in the mean square of system (4) can be checked effectively by using the Matlab LMI Control Toolbox[11].

Theorem 1. If there exist positive definite matrices P , R_2 , R_3 , diagonal positive matrices R_1 , R_4,, R_5, R_6 and constant matrix $W \in R^{n\times n}$ such that the following LMI holds:.

$$\begin{bmatrix} \Omega & -PA & -sPB & -sW & -\sqrt{h}WD & -\sqrt{h}WA & -\sqrt{h}WB \\ -A^TP & -R_1 & 0 & 0 & 0 & 0 & 0 \\ -sB^TP & 0 & -R_2 & 0 & 0 & 0 & 0 \\ -sW^T & 0 & 0 & -R_3 & 0 & 0 & 0 \\ -\sqrt{h}D^TW^T & 0 & 0 & 0 & -R_4 & 0 & 0 \\ -\sqrt{h}A^TW^T & 0 & 0 & 0 & 0 & -R_5 & 0 \\ -\sqrt{h}B^TW^T & 0 & 0 & 0 & 0 & 0 & -R_6 \end{bmatrix} < 0 \tag{10}$$

where $\Omega = -PD - D^TP + W + W^T + R1 + R2 + R_3 + hR_34 + hR_5 + hR_6 + \Sigma_1 + s^2 * \Sigma_2$ and $s = \sqrt{(1-\varepsilon)^{-1}}$, W=PC , then the dynamics of the neural network (4) is globally, asymptotically stable in the mean square.

Proof. Define a Lyapunov-Krasovskii functional candidate $V(x(t),t) \in \wp^{2,1}(R^n \times R^n; R^+)$ by

$$V(x(t),t) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) \tag{11}$$

$$V_1(x(t),t) = x^T(t)Px(t)$$

$$V_2(x(t),t) = \int_{t-\tau(t)}^{t} x^T(\xi)(R_2 + R_3 + s^2\Sigma_2)x(\xi)d\xi + \int_{-h}^{0}(\int_{t+\theta}^{t} x^T(\xi)(R_4 + R_5)x(\xi)d\xi)d\theta$$

$$V_3(x(t),t) = \int_{-2h}^{-h}(\int_{t+\theta}^{t} x^T(\xi)R_6x(\xi)d\xi)d\theta$$

Employing $It\hat{o}$ differential rule, one can deduce that

$$\begin{aligned} LV_2(x(t),t) &= x^T(t)(R_2 + R_3 + s^2\Sigma_2)x(t) \\ &+(1-\varepsilon)x^T(t-\tau(t))(R_2 + R_3 + s^2\Sigma_2)x(t-\tau(t)) \\ &+\int_{-h}^{0} x^T(t)(R_4 + R_5)x(t)d\xi - \int_{-h}^{0} x^T(t+\xi)(R_4 + R_5)x(t+\xi)d\xi \\ &= hx^T(t)(R_4 + R_5)x(t) + \int_{t-h}^{t} x^T(\xi)(R_4 + R_5)x(\xi)d\xi \\ &\le hx^T(t)(R_4 + R_5)x(t) + \int_{t-\tau(t)}^{t} x^T(\xi)(R_4 + R_5)x(\xi)d\xi \end{aligned} \tag{12}$$

$$\begin{aligned} LV_3(y(t)) &= \int_{-2h}^{h} x^T(t)R_6x(t)d\xi - \int_{-2h}^{h} x^T(t+\xi)R_6x(t+\xi)d\xi \\ &= hx^T(t)R_6x(t) + \int_{t-h}^{t} x^T(\xi-h)R_6x(\xi-h)d\xi \\ &\le hx^T(t)R_6x(t) + \int_{t-\tau(t)}^{t} x^T(\xi-\tau(t))R_6x(\xi-\tau(t))d\xi \end{aligned} \tag{13}$$

$$\begin{aligned} LV_1(y(t)) &= 2y^T(t)P\{(-D + AS(t) + C)y(t) + (BS(t-\tau(t)) - C)y(t-\tau(t)) \\ &-C\int_{t-\tau(t)}^{t}((-D + AS(t))y(\xi) + BS(\xi-\tau(t))y(\xi-\tau(t)))d\xi\} \\ &+trace(\sigma^T(x(t),x(t-\tau(t)),t)P\sigma(x(t),x(t-\tau(t)),t)) \end{aligned} \tag{14}$$

$$\begin{aligned} LV(x(t),t) &= x^T(t)\{-PD - D^TP + W + W^T + R_1 + R_2 + R_3 + hR_4 + hR_4 + hR_6 + \Sigma_1 + s^2\Sigma_2 \\ &+PAR_1^{-1}A^TP + (1-\varepsilon)^{-1}PBR_2^{-1}B^TP + (1-\varepsilon)^{-1}WR_3^{-1}W^T + hWDR_4^{-1}D^TW^T \\ &+hWAR_5^{-1}A^TW^T + hWBR_6^{-1}B^TW^T\}x(t) = x^T(t)\Omega x(t) \end{aligned} \tag{15}$$

Where

$$\begin{aligned} \Omega &= -PD - D^TP + W + W^T + R_1 + R_2 + R_3 + hR_4 + hR_4 + hR_6 + \Sigma_1 + s^2\Sigma_2 \\ &+PAR_1^{-1}A^TP + (1-\varepsilon)^{-1}PBR_2^{-1}B^TP + (1-\varepsilon)^{-1}WR_3^{-1}W^T + hWDR_4^{-1}D^TW^T \\ &+hWAR_5^{-1}A^TW^T + hWBR_6^{-1}B^TW^T \end{aligned} \tag{16}$$

From (10) , we known that $\Omega < 0$. Taking the mathematical expectation of both sides of (), we have

$$E[LV(x(t),t)] \le E[x^T(t)\Omega x(t)] \le -\varepsilon E\,|\,x(t)\,|^2 \tag{17}$$

It indicates that system (4) is globally asymptotically stable in the mean square. This completes the proof.

In Theorem 1, if we select model transformation matrix C as matrix B in system (11), consequently, W=PB, and A=0,that is the model was a pure delay model, then from Theorem 1 we have the following corollary:

Corollary1. Suppose that the assumptions (H) are satisfied. Then system (1) is globally stable if there exist positive definite matrices P , R_2 , R_3 and diagonal positive matrices R_1 , R_4 such that the following LMI holds:

$$\begin{bmatrix} \Omega_1 & -sPB & -sPB & -\sqrt{h}PBD & -\sqrt{h}PBB \\ -sB^TP & -R_2 & 0 & 0 & 0 \\ -sB^TP & 0 & -R_3 & 0 & 0 \\ -\sqrt{h}D^TB^TP & 0 & 0 & -R_4 & 0 \\ -\sqrt{h}B^TB^TP & 0 & 0 & 0 & -R_6 \end{bmatrix} < 0 \qquad (18)$$

$$\Omega_1 = -PD - D^TP + PB + B^TP + R2 + R_3 + hR_34 + hR_6$$

IV. NUMERICAL EXAMPLE

Example1. Consider a neural network with time-varying delay (3) with parameters

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0.125 & 0.25 \\ 0.25 & 0.125 \end{bmatrix} \quad B = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$$
$$\Sigma_1 = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix}$$

There exists at least a feasible solution to the conditions in Theorem 1, where h=10, $\varepsilon = 0.2$.

$$P = \begin{bmatrix} 33.6165 & 4.0292 \\ 4.0292 & 33.6165 \end{bmatrix} \quad R_1 = \begin{bmatrix} 10.2448 & 0 \\ 0 & 10.2448 \end{bmatrix}$$
$$R_2 = \begin{bmatrix} 3.6547 & 0 \\ 0 & 3.6547 \end{bmatrix} \quad R_3 = \begin{bmatrix} 0.7307 & 0 \\ 0 & 0.7307 \end{bmatrix}$$
$$R_4 = \begin{bmatrix} 0.6417 & 0 \\ 0 & 0.6417 \end{bmatrix} \quad R_5 = \begin{bmatrix} 0.6413 & 0 \\ 0 & 0.6413 \end{bmatrix}$$
$$R_6 = \begin{bmatrix} 12.0638 & 0 \\ 0 & 12.0638 \end{bmatrix} \quad W = \begin{bmatrix} -0.2714 & 0.0688 \\ 0.0688 & -0.2714 \end{bmatrix}$$

It follows from Theorem 1 that the stochastic neural network is globally asymptotically stable in the mean square.

V. CONCLUSION

The global asymptotical stability analysis problem for stochastic neural network with time-varying delay has been studied in this paper. Distinct difference from other analytical approach lies in "linearization" of neural network model, by which the considered neural network model is transformed into a linear time-variant system. Then, a process, which is called parameterized first-order model transformation, is used to transform the linear process. Novel criteria for global asymptotic stability of the unique equilibrium point of stochastic neural network with time-varying delays are obtained. The stability criterion is expressed by means of LMI, which can be readily tested by some standard numerical package. Therefore, the developed result is practical.

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REFERENCES

[1] G. Joya, M. A. Atencia, F. Sandoval, Hopfield neural networks for optimization: study of the different dynamics, Neurocomputing 43 (2002) pp.219–237.

[2] X. Liao, G. Chen, E.SancheZ. LMI-based approach for asymptotical stability analysis of delayed neural network. IEEE Tans. Circuits and Systems-I 2002,49(7), pp. 1033-1039.

[3] X.Liao, C.Li. An LMI approach to asymptotical stability of multi-delayed interval neural networks. Physica D,200(2005), pp.139-155.

[4] Q.zhang, X.Wei, J.Xu, Delay-dependent global stabiltiy condition for delayed Hopfield neural network. Nonlinear Analysis: Real world Applications 8. (2007),pp.997-1002

[5] S. Blythe, X. Mao, X.X. Liao, Stability of stochastic delay neural networks, J. Franklin Inst. 338 (2001) ,pp.481–495.

[6] S. Blythe, X.R. Mao, A. Shah, Razumikhin-type theorems on stability of stochastic neural networks with delays, Stochast. Anal. Appl. 19(2001) 85–101.

[7] X.X. Liao, X. Mao, Exponential stability and instability of stochastic neural networks, Stochast. Anal. Appl. 14 (1996), pp.165–185.

[8] Z.Wang, Y.Liu, K.Fraser,X.Liu, Stochastic stability of uncertain neural networks with discrete and distributed delays. Phsyics A 354 (2006),pp.288-297.

[9] H.Huang, J.Cao, Exponential stability analysis of uncertain stochastic neural networks with multiple delays. Nonlinear Analysis: Real world Application 8 (2007),pp. 646-653.

[10] H.Huang, G.Feng, Delay-dependent stability for uncertain stochastic neural networks with time-varying delay. Physica A 381 (2007),pp.93-103.

[11] S. Boyd, L. EI Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, PA, 1994.

[12] Niculescu S.-I. Delay effects on stability: A robust approach. Spirnger-Verlag,Germany, 2001.

[13] Bernet Oksendal, Stochastic Differential Equations,. Fourth Edition, Springer-Verlag.

[14] C.Li., X.Liao, K.Wong, Delay-dependent and delay-independent stability criteria for cellular neural networks with delays, International Journal of Bifurcation and Chaos 16 (2006), pp.3323-3340.

[15] L.Wan, J. Sun, Mean square exponential stability of stochastic delayed Hopfield neural networks, Phys. Lett. A 343 (2005), pp.306–318.

[16] Z. Wang, S. Lauria, J. Fang, X. Liu, Exponential stability of uncertain stochastic neural networks with mixed time-delays, Chaos, Solitons Fractals 32 (2007), pp. 62–72