Singularity Analysis of Planar Cable-Driven Parallel Robots

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***Abstract*—It is well known that parallel robots may have singular configurations that can result in a loss of full control the mechanisms. This paper analyzes two different categories of singularities of planar cable-driven parallel robots with four or more cables. The unidirectional constraint of cables makes the singularity analysis of cable-driven parallel robots different from that of rigid-link parallel robots even if they have similar kinematical architectures. Based on their natures, singularities of cable-driven parallel robots are classified into two categories: the Jacobian singularity and the force-closure singularity. A Jacobian singularity occurs when the Jacobian matrix of a cable-driven parallel robot loses its full rank. Based on rank analysis of Jacobian matrix, a group of Jacobian singularities is reported with mathematical proof. When the Jacobian matrix of a cable-driven parallel robot has a full rank, the cables' inability to generate tension will lead to force-closure singularities, which can always happen to fully-constrained cable-driven parallel robots. An algorithm of identifying force-closure singularities of planar cable-driven parallel robots is proposed. Understanding of the natures of singularities is important for the design and control of cable-driven parallel robots.**

***Keywords*—singularity analysis, cable robots, parallel robots**

I. INTRODUCTION

Cable-driven parallel robots, referred to as cable robots for short in this paper, have recently attracted much interest in the robotics community for special applications. In addition to the well-known advantages of parallel robots relative to serial robots, cable robots possess some other desirable attributes [1]-[3]. First, for the same physical size, they can have larger workspaces because their joints can reel out a large amount of cables. Second, all of their actuators and transmission systems can always be mounted on the fixed base and thus, they have a higher payload-to-weight ratio, which makes them attractive for high-load or high-acceleration applications. Third, their special designs make them less expensive, modular, and easy to reconfigure. Finally, and also the most important characteristic for model-based controls, they have much simpler dynamics model than their rigid-link counterparts if the inertia of the cables can be ignored because the mass of the cables is usually much smaller than those of the end-effector and the payload.

Cable robots can be classified into two basic types, the under-constrained type and the fully-constrained type, based on the extent to which the end-effector is constrained by the cables [2]-[4]. Fig. 1 shows an example of the two types of cable robots. This paper concerns about cable robots of the fully-constrained type.

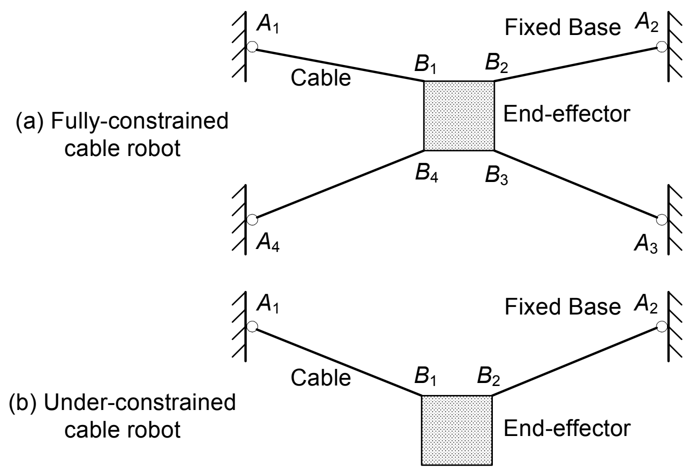

Figure 1. Example of the two types of planar cable robots

Comparing to rigid links, cables present some advantages, but they also have a distinct characteristic having to be considered seriously. That is, cables are characterized by the unidirectional constraint (can pull but cannot push) and thus, they can support tension only. The unidirectional constraint of cables makes the singularity analysis of cable robots different from that of rigid-link parallel robots [5], [6]. For example, an under-constrained cable robot is an under-actuated robot and thus, the rank of its Jacobian matrix will be always smaller than its degrees of freedom. For a rigid-link parallel robot, this would be considered to be singular. However, this situation may not be problematic for an under-constrained cable robot. In fact, it may be preferable because fewer cables decrease the possibility of interference. A fully-constrained cable robot has more cables than its degrees of freedom and thus, the rank of its Jacobian matrix could be equal to its degrees of freedom. For a rigid-link parallel robot, this would be considered to be non-singular. Yet this situation may be problematic for a fully-constrained cable robot due to cables' inability of supporting tension. In this case, the fully-constrained cable robot will not be able to work because the cables cannot generate tension to balance the load exerted on the end-effector. Therefore, the singularity analysis methods for rigid-link parallel robots cannot be directly applied to cable robots as is.

Many studies have been conducted concerning the kinematics, dynamics and control of cable robots in recent years [7-10]. However, the singularity problem of cable robots has received considerably less attention. As long as singular configurations possibly exist in the workspace of a cable robot, special care is needed in the design and control of the robot. Yang et al. [11] proposed the concept of instantaneous center for singularity analysis of planar rigid-link parallel robots. An instantaneous center is defined as a common point of two rigid bodies (or the extended parts of them) at which the two bodies have the same velocity at a time instant. They applied the concept of instantaneous center to the singularity analysis of planar cable robots and proposed a geometrical singularity analysis approach for fully-constrained 3-DOF 4-cable robots [6]. Qiu et al. discussed the force singularity problem of an under-constrained 6-cable suspended structure for the next generation large radio telescope [12]. Force singularity was explored through the determinant of Jacobian matrix. One more cable was added to the system to eliminate force singularity. Verhoeven et al. [13] classified the singularities of a 6-DOF tendon-driven Stewart platform into two types, namely, the under-mobility singularity and the over-mobility singularity, based on the properties of the Jacobian matrices. Tension limits, stiffness, and singularity conditions for the Stewart platform were given. Williams II et al. proposed a novel 3-D cable-based metrology system wherein the sculpting tool was suspended using six cables [14]. The singularity conditions for the forward kinematics of the sculpting tool were analyzed. Trevisani et al. [15] derived both the kinematics and dynamics models for a hybrid serial/parallel architecture. The parallel part of the system is a planar translational cable robot. The singularity conditions of the 2-DOF cable robot were examined by checking the determinant of the 2×2 sub-matrices of the Jacobian matrix. Gosselin and Wang presented the kinematic analysis and design of a cable-driven spherical parallel mechanism [16]. They studied the singularity loci associated with the rank deficiency of the Jacobian matrix of the spherical parallel mechanism.

The singularity analyses of cable robots reported in [13]-[16] were all based on the examination of the Jacobian matrices of cable robots. These Jacobian-based singularity analyses, which consider all cables as rigid links, have not taken into account the unidirectional characteristic of cables and thus, are incomplete for singularity analysis of cable robots, because the Jacobian-based singularity analysis is meaningful only when all the cables are in tension. In other words, if any of the cable is in slack condition, the Jacobian matrix becomes meaningless and thus, the Jacobian-based analyses will become invalid too. References [6], [12] discussed the analyses of singularities caused by the cables' inability to generate tension for fully-constrained 3-DOF 4-cable robots and under-constrained 6-cable structure, respectively. This paper will study the singularity analysis of general fully-constrained planar cable manipulators with four or more cables. The singularities of cable robots are classified into two categories: the Jacobian singularity and the force-closure singularity, depending on whether the Jacobian matrix is singular or the force-closure condition is violated. Then the two categories of singularities are investigated separately. A group of Jacobian singularities is reported with mathematical proof. An algorithm of identifying force-closure singularities is proposed.

II. MODELING OF PLANAR CALBE ROBOTS

The kinematics model of a general 3-DOF n-cable ($n \geq 4$) robot is derived based on the architecture shown in Fig. 2. The end-effector is assumed to be controlled by four or more cables with their driving actuators mounted to the fixed base.

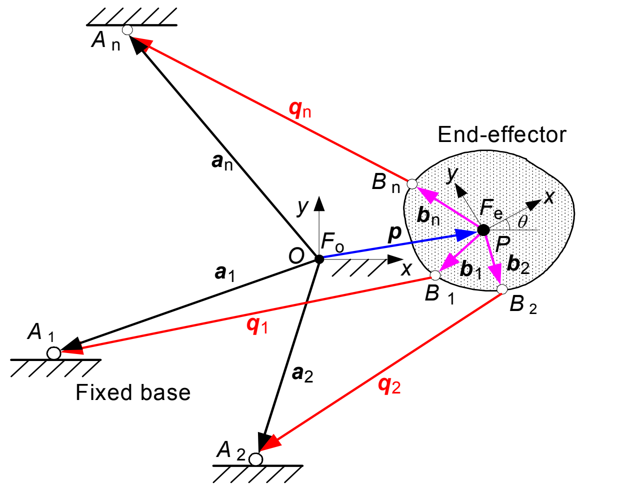

Figure 2. Kinematics notation of a planar 3-DOF n-cable robot

In Fig. 2, $\mathbf{q}_i \in R^3$ ($i = 1,2,\cdots,n$) is the vector along the ith cable and has the same length as the cable. The length of the ith cable is represented by scalar q_i which is also considered as the robot's joint variable. A_i and B_i are the two attaching points of the ith cable on the base and the end-effector, respectively. The positions of the two attaching points are represented by vectors $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively. Obviously, $\mathbf{a}_i$ is a constant vector in the base frame F_o and $\mathbf{b}_i$ is a constant vector in the end-effector frame F_e. θ is the rotation angle between the end-effector frame F_e and the base frame F_o. The origin of the end-effector frame F_e is fixed at a reference point P of the end-effector, which is used to define the position of the end-effector. Based on the kinematics notation defined in Fig. 2, the position of the end-effector can be described as

$$\mathbf{p} = \mathbf{a}_i - \mathbf{q}_i - \mathbf{Q}\mathbf{b}_i \quad \text{for } i = 1,2,\cdots,n \tag{1}$$

where matrix $\mathbf{Q}$ is a rotation matrix defined as

$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{2}$$

From (1), one has

$$\mathbf{q}_i = \mathbf{a}_i - \mathbf{p} - \mathbf{Q}\mathbf{b}_i \quad \text{for } i = 1,2,\cdots,n \tag{3}$$

and

$$q_i^2 = [\mathbf{a}_i - \mathbf{p} - \mathbf{Q}\mathbf{b}_i]^T[\mathbf{a}_i - \mathbf{p} - \mathbf{Q}\mathbf{b}_i] \quad \text{for } i = 1,2,\cdots,n \tag{4}$$

Differentiating (4) with respect to time, one obtains

$$q_i\dot{q}_i = [\mathbf{a}_i - \mathbf{p} - \mathbf{Q}\mathbf{b}_i]^T[-\dot{\mathbf{p}} - \dot{\mathbf{Q}}\mathbf{b}_i] \quad \text{for } i = 1,2,\cdots,n \tag{5}$$

with

$$\dot{\mathbf{Q}} = \mathbf{E}\mathbf{Q}\dot{\theta},\quad \mathbf{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

Substituting (6) into (5) yields

$$q_i \dot{q}_i = [\mathbf{p} + \mathbf{Q}\mathbf{b}_i - \mathbf{a}_i]^T \dot{\mathbf{p}} + [\mathbf{p} - \mathbf{a}_i]^T \mathbf{E}\mathbf{Q}\mathbf{b}_i \dot{\theta} \quad \text{for } i = 1, 2, \cdots, n \tag{7}$$

which can be re-written into the following matrix form

$$\mathbf{B}\dot{\mathbf{q}} = \mathbf{A}\mathbf{t} \tag{8}$$

where

$$\mathbf{B} = \mathrm{diag}(q_1, q_2, \cdots q_n),\quad \dot{\mathbf{q}} = [\dot{q}_1\ \dot{q}_2 \cdots \dot{q}_n]^T,\quad \mathbf{t} = \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\theta} \end{bmatrix} \tag{9}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{p}+\mathbf{Q}\mathbf{b}_1-\mathbf{a}_1 & \mathbf{p}+\mathbf{Q}\mathbf{b}_2-\mathbf{a}_2 & \cdots & \mathbf{p}+\mathbf{Q}\mathbf{b}_n-\mathbf{a}_n \\ \mathbf{b}_1^T\mathbf{Q}^T\mathbf{E}^T(\mathbf{p}-\mathbf{a}_1) & \mathbf{b}_2^T\mathbf{Q}^T\mathbf{E}^T(\mathbf{p}-\mathbf{a}_2) & \cdots & \mathbf{b}_n^T\mathbf{Q}^T\mathbf{E}^T(\mathbf{p}-\mathbf{a}_n) \end{bmatrix}^T \tag{10}$$

In the above equations, matrices $\mathbf{A}$ and $\mathbf{B}$ are the forward and inverse kinematics Jacobian matrices [17] of the cable robot, respectively. $\dot{\mathbf{p}}$ represents the linear velocity of point P on the end-effector; $\dot{\theta}$ is the angular velocity of the end-effector; and vector $\mathbf{t}$ represents the twist vector in R^3 which consists of both the linear and angular velocities of the end-effector.

Based on (8), the solution of the inverse velocity problem can be expressed as

$$\dot{\mathbf{q}} = \mathbf{B}^{-1}\mathbf{A}\mathbf{t} \tag{11}$$

And the solution of the forward velocity problem can be written as

$$\mathbf{t} = \mathbf{A}^{-1}\mathbf{B}\dot{\mathbf{q}} \tag{12}$$

Obviously, the inversion of diagonal matrix $\mathbf{B}$ is very simple and always possible, unless one of the cable lengths vanishes, which is almost impossible in practice. In other words, the inverse kinematics of a cable robot is trivial. However, the inversion of matrix $\mathbf{A}$ is not straightforward, and hence, deserves more attention. Note that the Jacobian matrix of a serial robot is defined based on the linear transformation from the joint velocity vector to the end-effector twist vector. If the same definition is adopt here, the resulting Jacobian matrix will be $\mathbf{A}^{-1}\mathbf{B}$. Apparently, with such a definition, the Jacobian matrix is undefined when matrix $\mathbf{A}$ is rank-deficient. Instead, one can define the Jacobian matrix of a cable robot as the mapping from the end-effector twist vector to the joint velocity vector, namely,

$$\dot{\mathbf{q}} = \mathbf{J}\mathbf{t} \tag{13}$$

where $\mathbf{J}$ denotes the aforementioned $n \times 3$ Jacobian matrix. Such a Jacobian matrix will always be defined, even when matrix $\mathbf{A}$ is rank-deficient. From (11), the Jacobian matrix can be expressed as

$$\mathbf{J} = \mathbf{B}^{-1}\mathbf{A} \tag{14}$$

Based on the definition of the Jacobian matrix in (14), one can derive the relation between the wrench exerted on the end-effector and the cable forces as follows

$$\mathbf{J}^T\mathbf{f} = \mathbf{w} \tag{15}$$

where vector $\mathbf{w}$ is a 3-dimensional resultant wrench vector composed of all the inertia and external wrenches exerted on the end-effector, and vector $\mathbf{f}$ is a n-dimensional vector consisting of all the cable forces. Equation (15) can be readily derived from the principle of conservation of energy.

III. Classification of Singularities

A singular configuration of a parallel robot refers to a particular configuration in which the robot gains or loses one or more degrees of freedom instantaneously. In a singular configuration, the Jacobain matrix $\mathbf{J}$ defined in (14) becomes rank deficient and thus, both solving the twist vector $\mathbf{t}$ from (13) and calculating the cable forces from (15) are impossible. As a result, the robot is out of control. From this point of view, a singular configuration has to be avoided.

Before analyzing the singularities of cables robots, one has to make a classification of them first because different singularities have different natures and thus, may need different treatments in practice. For instance, some singularities of a cable robot are caused by the cables' inability to generate tension in particular configurations. Such a singularity may disappear if the associated configuration is changed but it cannot be avoided by design. Some singularities are caused by inappropriate robot design. Changing configuration may not help remove such a singularity. It can be eliminated by design only.

Based on the combination of singularities of the Jacobian matrices $\mathbf{A}$ and $\mathbf{B}$, Gosselin and Angeles [17] classified singularities of closed-loop kinematic chains into three main groups. Ma and Angeles [18] classified the singularities of a parallel robot into three categories, namely, architecture, configuration, and formulation singularities, based on their natures. Note that these two classifications are regarding rigid-link parallel robots only. Since a cable robot is also a parallel robot, both classifications are applicable to cable robots, too. However, they are not complete for singularities of cable robots because the unidirectional constraint of cables is not considered in both classifications. In other words, whether or not a configuration of a cable robot is singular depends not only on the Jacobain matrix corresponding to this configuration but also on the cables' ability to generate tension in this configuration. For example, even if the corresponding Jacobian matrix has a full rank, a configuration cannot be in the workspace of the cable robot if one or more cables become slack in this configuration. Obviously, such a configuration has a force-transmission problem. Therefore, it follows that both classifications in [17], [18] cannot include all the singularities of cable robots. To be able to identify all the singularities of a cable robot, a new classification of singularities of cable robots is needed. Such a classification consists of two categories, as defined below:

1. Jacobian singularity: A Jacobian singularity occurs in a configuration in which the Jaobian matrix of the cable robot becomes rank-deficient.

2. Force-closure singularity: A force-closure singularity occurs in a configuration in which the Jacobian matrix of the cable robot still has a full rank but the configuration does not satisfy the force-closure condition. A configuration is said to have a force-closure if and only if any external wrench applied to the end-effector can be balanced by a set of cables with tension. In other words, the force-closure condition is satisfied if and only if the inverse dynamics problem in (15) has a feasible solution regardless the external wrench applied to the end-effector [19].

It can be noted that the Jacobian singularity is defined from the kinematical point of view while the force-closure singularity is defined from a dynamical point of view. In general, both categories of singularities involve the force-transmission problem, namely, the cable forces fail to balance the wrench exerted on the end-effector due to either the singularity of the Jacobian matrix or the cables' inability to generate tension at all. Consequently, the cable robot cannot work properly. Obviously, this is undesirable and should be avoided. The next two sections will discuss the analyses of both categories of the singularities.

IV. Analysis of Jacobian Singularities

Since matrix $\mathbf{B}$, as discussed in Section II, is always non-singular in practice, one can deduce from (14) that

$$\text{rank}(\mathbf{J}) = \text{rank}(\mathbf{B}^{-1}\mathbf{A}) = \text{rank}(\mathbf{A}) \tag{16}$$

Equation (16) indicates that the Jacobian matrix $\mathbf{J}$ is singular if and only if matrix $\mathbf{A}$ is singular. In other words, the Jacobian singularities can be identified through the rank analysis of matrix $\mathbf{A}$.

From (10) one can see that matrix $\mathbf{A}$ depends not only on the configuration (represented by $\mathbf{p}$ and $\mathbf{Q}$) but also on the design variables (represented by $\mathbf{a}_i$ and $\mathbf{b}_i$, $i = 1,2,\cdots,n$). Thus, matrix $\mathbf{A}$ may be singular due to either an improper configuration or design. The avoidance of an improper configuration is possible at the trajectory-planning stage while an improper design can be avoided at the design stage only. Moreover, it is found that, with an improper design, a cable robot may fail to work in all or most of its workspace. Hence, a Jacobian singularity caused by the design of the cable robot is more serious and thus, more attention is needed. This section will discuss Jacobian singularities caused by improper designs of planar cable robots.

By a close inspection of various numerical results of the forward kinematics from simulation studies with different designs of cable robots, we find an interesting observation, namely, Jacobian singularities occur throughout the whole workspace if the polygons $A_1A_2\cdots A_n$ and $B_1B_2\cdots B_n$ (called base polygon and end-effector polygon, respectively, see Fig. 2) are similar and have the same orientation. This observation can be stated as the following theorem.

Theorem 1: The Jacobian matrix $\mathbf{J}$ is singular if both the base polygon and the end-effector polygon are similar and have the same orientation.

Based on (16), the singularity of the Jacobain matrix $\mathbf{J}$ can be investigated through the rank analysis of matrix $\mathbf{A}$. Both the base polygon and the end-effector polygon have the same orientation, i.e.,

$$\theta = 0 \quad \text{or} \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{17}$$

Also note that the base polygon is similar to the end-effector polygon, one has

$$\mathbf{a}_i = \alpha\,\mathbf{Q}\mathbf{b}_i = \alpha\,\mathbf{b}_i \quad \text{for } i = 1,2,\cdots,n \tag{18}$$

where α is a nonzero scalar representing the scaling factor between the two similar polygons. Substituting (17) and (18) into (10) yields

$$\mathbf{A} = \begin{bmatrix} \mathbf{p}+(1-\alpha)\mathbf{b}_1 & \mathbf{p}+(1-\alpha)\mathbf{b}_2 & \cdots & \mathbf{p}+(1-\alpha)\mathbf{b}_n \\ \mathbf{b}_1^T\mathbf{E}^T\mathbf{p} & \mathbf{b}_2^T\mathbf{E}^T\mathbf{p} & \cdots & \mathbf{b}_n^T\mathbf{E}^T\mathbf{p} \end{bmatrix}^T \tag{19}$$

where the terms $-\mathbf{b}_i^T\mathbf{E}^T\mathbf{a}_i\,(i = 1,2,\cdots,n)$ in the last column of matrix $\mathbf{A}$ are deleted because they are all zero as shown follows

$$-\mathbf{b}_i^T\mathbf{E}^T\mathbf{a}_i = -\alpha\mathbf{b}_i^T\mathbf{E}^T\mathbf{b}_i = 0 \quad \text{for } i = 1,2,\cdots,n \tag{20}$$

Now, let's define

$$\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{b}_i = \begin{bmatrix} b_{ix} \\ b_{iy} \end{bmatrix} \quad \text{for } i = 1,2,\cdots,n \tag{21}$$

where x and y are the Cartesian coordinates of point P in frame F_o while b_{ix} and b_{iy} are the Cartesian coordinates of attaching point B_i in frame F_e. Substituting (6) and (21) into (19), one obtains

$$\mathbf{A} = \begin{bmatrix} x+(1-\alpha)b_{1x} & x+(1-\alpha)b_{2x} & \cdots & x+(1-\alpha)b_{nx} \\ y+(1-\alpha)b_{1y} & y+(1-\alpha)b_{2y} & \cdots & y+(1-\alpha)b_{ny} \\ -xb_{1y}+yb_{1x} & -xb_{2y}+yb_{2x} & \cdots & -xb_{ny}+yb_{nx} \end{bmatrix}^T \tag{22}$$

The singularity of matrix $\mathbf{A}$ can be proven by applying elementary column operations to it. Because the singularities of matrix $\mathbf{A}$ involves singularities in a sub-region of the workspace, which consists of several configuration sets, the corresponding proof will be presented case by case. Each case is associated with one configuration set.

Case 1: $\alpha = 1$

When $\alpha = 1$, (22) can be reduced to

$$\mathbf{A} = \begin{bmatrix} x & x & \cdots & x \\ y & y & \cdots & y \\ -xb_{1y}+yb_{1x} & -xb_{2y}+yb_{2x} & \cdots & -xb_{ny}+yb_{nx} \end{bmatrix}^T \tag{23}$$

It is clear that matrix $\mathbf{A}$ is singular if $x = 0$ or $y = 0$. In fact, matrix $\mathbf{A}$ is singular even if both x and y are nonzero because the first two columns of matrix $\mathbf{A}$ are proportional in this case. Thus, matrix $\mathbf{A}$ is always singular regardless where point P is. Therefore, the Jacobian matrix $\mathbf{J}$, based on (16), is also singular. It is shown that, no mater where the end-effector is, Jacobain singularities occur as long as the base polygon have the same size as the end-effector polygon.

Case 2: $\alpha \neq 1, x = 0, y = 0$

When $x = y = 0$, (22) can be reduced to

$$\mathbf{A}=\begin{bmatrix}(1-\alpha)b_{1x} & (1-\alpha)b_{2x} & \cdots & (1-\alpha)b_{nx}\\ (1-\alpha)b_{1y} & (1-\alpha)b_{2y} & \cdots & (1-\alpha)b_{ny}\\ 0 & 0 & \cdots & 0\end{bmatrix}^T \tag{24}$$

Obviously, matrix $\mathbf{A}$ is singular because all the components of the third column are zero in this case. Therefore, the Jacobian matrix $\mathbf{J}$, based on (16), is also singular. It is shown that, no mater the sizes of the polygons, Jacobian singularities occur as long as the centers of both polygons are coincident.

Case 3: $\alpha \neq 1, x = 0, y \neq 0$

When $x = 0$, (22) can be reduced to

$$\mathbf{A}=\begin{bmatrix}(1-\alpha)b_{1x} & (1-\alpha)b_{2x} & \cdots & (1-\alpha)b_{nx}\\ y+(1-\alpha)b_{1y} & y+(1-\alpha)b_{2y} & \cdots & y+(1-\alpha)b_{ny}\\ yb_{1x} & yb_{2x} & \cdots & yb_{nx}\end{bmatrix}^T \tag{25}$$

It is clear that matrix $\mathbf{A}$ is singular because the first column and the third column of matrix $\mathbf{A}$ are proportional in this case. Therefore, the Jacobian matrix $\mathbf{J}$, based on (16), is also singular.

Case 4: $\alpha \neq 1, x \neq 0, y = 0$

When $y = 0$, (22) can be reduced to

$$\mathbf{A}=\begin{bmatrix}x+(1-\alpha)b_{1x} & x+(1-\alpha)b_{2x} & \cdots & x+(1-\alpha)b_{nx}\\ (1-\alpha)b_{1y} & (1-\alpha)b_{2y} & \cdots & (1-\alpha)b_{ny}\\ -xb_{1y} & -xb_{2y} & \cdots & -xb_{ny}\end{bmatrix}^T \tag{26}$$

It is clear that matrix $\mathbf{A}$ is singular because the second column and the third column of matrix $\mathbf{A}$ are proportional in this case. Therefore, the Jacobian matrix $\mathbf{J}$, based on (16), is also singular.

Case 5: $\alpha \neq 1, x \neq 0, y \neq 0$

In this case, by applying the elementary column operations to matrix $\mathbf{A}$, one can go through and show that matrix $\mathbf{A}$ is singular. For simplicity, let's take the *i*th row of matrix $\mathbf{A}$ for example

$$\begin{bmatrix}x+(1-\alpha)b_{ix}\\ y+(1-\alpha)b_{iy}\\ -xb_{iy}+yb_{ix}\end{bmatrix}^T\begin{bmatrix}-y\\ x\\ 1-\alpha\end{bmatrix}=0 \quad \text{for}\, i=1,2,\cdots,n \tag{27}$$

Equation (27) indicates that multiplying the three columns of matrix $\mathbf{A}$ by $-y, x$ and $1-\alpha$, respectively, and then adding the first two columns to the third column, all the components of the third column will be zero. Hence, one can infer that matrix $\mathbf{A}$ is singular. Therefore, the Jacobian matrix $\mathbf{J}$, based on (16), is also singular.

From the above discussion one knows that, if the base polygon and the end-effector polygon are similar and have the same orientation, the Jacobian matrix $\mathbf{J}$ is always singular throughout the whole workspace regardless of the sizes of the two polygons and where the end-effector is. Therefore, one has to avoid using similar polygons for both the base and the end-effector in the design of a cable robot.

V. Analysis of Force-Closure Singularities

Because of the unidirectional constraint of cables, maintaining tension in cables is essential for a cable robot to balance arbitrary external wrenches exerted on the end-effector. Even the Jacobian matrix of a cable robot is known to be non-singular in a configuration, i.e., the configuration is not a Jacobain singularity, one still cannot guarantee that such a configuration can hold up in the workspace of the cable robot. Because some of the cables may not be able to generate tension to balance external wrenches in this configuration and thus, the cable robot cannot work properly. In other words, such a configuration does not satisfy the force-closure condition and thus, a force-closure singularity occurs. Hence, it is important to know where the force-closure singularities are such that one can avoid them in the trajectory-planning or control of a cable robot.

A force-closure singularity occurs in a configuration which does not satisfy the force-closure condition, namely, the cables cannot generate tension to balance arbitrary external wrenches exerted on the end-effector [19]. In other words, the force-closure condition is satisfied if and only if the inverse dynamics problem in (15) has a feasible solution regardless the external wrench applied to the end-effector. Such a force-closure condition can be mathematically described as

$$\operatorname{rank}(\mathbf{J}) = 3, \forall\, \mathbf{w} \in R^3,\ \exists\, \mathbf{f} \geq \mathbf{0},\ \ni \mathbf{J}^T\mathbf{f} = \mathbf{w} \tag{28}$$

where $\mathbf{f} \geq \mathbf{0}$ means that each component of vector $\mathbf{f}$ is greater than or equal to zero. Equation (28) indicates that the force-closure condition is satisfied if and only if the row vectors of Jacobian matrix $\mathbf{J}$, denoted by $\mathbf{j}_1, \mathbf{j}_2, \cdots, \mathbf{j}_n$, can positively span R^3. According to [20], this force-closure condition is equivalent to the following theorem.

Theorem 2: Equation (28) has a solution if and only if the nonzero projections of all the *n* row vectors of Jacobian matrix $\mathbf{J}$ on every direction in R^3 do not have the same sign. In other words, the configuration corresponding to Jacobian matrix $\mathbf{J}$ is free of force-closure singularity if and only if the nonzero dot products of vector $\mathbf{v}$ and the row vectors of Jacobian matrix $\mathbf{J}$ have different signs for any nonzero vector $\mathbf{v} \in R^3$.

Although Theorem 2 is originally developed for checking the existence of force-closure, one can employ it to identify force-closure singularities because a configuration is force-closure singular if it cannot satisfy the force-closure condition. However, this theorem, used as is, is inconvenient for identifying force-closure singularity because it requires one to check the sign condition for each and every nonzero vector in R^3. A computationally more attractive and also systematic method is to check only a number of vectors in R^3 which are formed from the row vectors of Jacobian matrix $\mathbf{J}$. Algorithmically, such a method can be implemented as described in the following procedure:

1) Select a set of two linearly independent row vectors of Jacobian matrix $\mathbf{J}$ to form a normal vector $\mathbf{n}$. This is always possible because $\mathbf{J}$ has been assumed to have a full rank. For example, if $\mathbf{j}_1$ and $\mathbf{j}_2$ are the two selected row

vectors. Then $\mathbf{n} = \mathbf{j}_1 \times \mathbf{j}_2$. In fact, the normal vector $\mathbf{n}$ is a candidate for vector $\mathbf{v}$ in Theorem 2.

2) Check the signs of the nonzero dot products of normal vector $\mathbf{n}$ and the remaining column vectors of Jacobian matrix $\mathbf{J}$ (i.e., $\mathbf{j}_3, \mathbf{j}_4 \cdots, \mathbf{j}_n$ in the example). If they have the same sign, one can conclude that the configuration is force-closure singular. Otherwise, go to step 3).
3) Select another set of two linearly independent row vectors of matrix $\mathbf{J}$ and repeat steps 1) and 2). There will be up to C_n^2 sets of two linearly independent row vectors of matrix $\mathbf{J}$ to form up to C_n^2 normal vectors. If all of these C_n^2 normal vectors have passed Step 2), this configuration is not a force-closure singularity.

The difference of the above method from Theorem 2 is that the former requires one to form and check at most C_n^2 normal vectors while the latter requires one to check the sign condition for all nonzero vectors in R^3. Hence, this method is algorithmically more convenient to identify force-closure singularities of general planar cable robots with four or more cables. This work connects the well-developed force-closure theorem with the singularity analysis of cable robots.

VI. Conclusion

Singularity analysis plays an important role in robot design and control. This paper addressed singularity analysis of fully-constrained planar cable robots with four or more cables. Due to the fact that cables can support tension only, it was realized that the singularity analysis of cable robots is different from that of rigid-link parallel robots. Based on their natures, the singularities of cable robots were classified into two categories, namely, the Jacobian singularity and the force-closure singularity. A group of Jacobian singularities was reported with mathematical proof based on rank analysis of Jacobian matrix. An algorithm of identifying force-closure singularities was also proposed. The presented research results can serve as a guideline for the design and control of planar cable robots.

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