ACTIVE CONTOURS BASED ON CHAMBOLLE'S MEAN CURVATURE MOTION

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ABSTRACT

This paper proposes an algorithm to solve most of existing active contour problems based on the approach of mean curvature motion proposed by Chambolle in [1] and the image denoising model of Rudin, Osher and Fatemi (ROF) introduced in [2]. More precisely, the motion of active contours is discritized by the ROF model applied to the signed distance of the evolving contour. The advantage of this new discretization scheme is to use a time step much larger than in standard explicit schemes, which means that less iterations are needed to converge to the steady state solution. We present results on 2-D natural images.

***Index Terms*—** Image segmentation, object extraction, active contour, ROF model, signed distance function.

1. INTRODUCTION

Object extraction is one of the most fundamental issues in the fields of image processing and computer vision. Its objective is to extract semantically important objects from given images such as medical structures in medical images. Promising mathematical frameworks to solve the object extraction problem are the variational approaches and the partial differential equations (PDEs). One well-known variational object extraction model is the active contour or snake model, initially proposed by Kass, Witkin and Terzopoulos in [3]. This model extracts objects in images based on the detection of edges, i.e. locations with sharp intensity changes. Other active contour models have been developed to detect objects with homogeneous intensity regions.

In this paper, we propose an algorithm to solve most of existing active contour models, based on boundary or region detection, with the variational approach presented by Chambolle in [1]. In his paper, Chambolle proposes an algorithm for mean curvature motion of a hypersurface based on the Total Variation (TV) norm as introduced by Rudin, Osher and Fatemi in [2] to solve the image denoising problem. We will extend his algorithm from mean curvature motion to the most well-known active contour models.
The main contributions of this paper are as follows:

1. New formulation of active contour models based on the ROF model,
2. Time step in the discretization segmentation flow is much larger than in standard explicit approaches, which means that less iterations are needed to converge to the steady state solution,
3. Applications to image segmentation with geodesic active contours [4, 5], active contours without edges [6], active contours based on the Kullback-Leigbler divergence [7] and to shape reconstruction from a set of unorganized points [8].

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2. CHAMBOLLE'S MEAN CURVATURE MOTION

In this section, we describe the method introduced by Chambolle in [1] to compute the generalized mean curvature motion (MCM) of a hypersurface evolving in $\mathbb{R}^N$. The mean curvature motion/flow is one of the most fundamental PDEs used in image processing. It is usually used in applications such as image segmentation or image denoising to introduce a regularization term in order to compute a steady, smooth and unique solution. The MCM basically determines a family of gradually smoother versions of the original 1-codimensional hypersurface embedded in the N-D Euclidean space. More precisely, the MCM corresponds to the gradient flow, with respect to the L^2 scalar product, of the hyperarea functional which decreases the hyperarea of a hypersurface as fast as possible by moving its boundary in the normal direction with the mean curvature as velocity speed.

A standard approach to solve the MCM is based on the computation of the Euler-Lagrange equation of the hyperarea functional $\int_{\partial E} d\mathcal{H}^{N-1}$, where ∂E is the boundary of a set $E \subset \mathbb{R}^N$ and $\mathcal{H}^{N-1}$ is the $N-1$ dimension of the Hausdorff measure corresponding to the length for $N = 2$ and area for $N = 3$. The Euler-Lagrange equation of $\int_{\partial E} d\mathcal{H}^{N-1}$ coupled with a descent gradient method leads to the MCM as follows:

$$\partial E_t = \kappa \mathcal{N}_E, \quad (1)$$

where ∂E_t is the derivative of the hypersurface ∂E w.r.t. the artificial flow time t, $\mathcal{N}_E$ is the normal to ∂E, $\kappa = \sum_{i=1}^{N-1} \kappa_i$ corresponds to the mean curvature of ∂E and κ_i are the $N-1$ principal curvatures. An efficient way to numerically solve the flow (1) is to use the level set approach developed by Osher and Sethian [9], which consists of embedding the evolving hypersurface $\partial E(t)$ as the zero level set of a function $u(x,t)$ and solving the second order elliptic PDE: $u_t = \nabla.(\frac{\nabla u}{|\nabla u|})\ |\nabla u|$, where u_t is the derivative of u w.r.t. the time t and $\nabla.(\frac{\nabla u}{|\nabla u|})$ is the mean curvature of constant level sets $\{x : u(x,t) = \lambda\}, \lambda \in \mathbb{R}$.

Another approach to solve the MCM has been proposed by Almgren, Taylor and Wang in [10]. They propose a variational model to discretize in time the MCM of a hypersurface. In other words, they solve a minimization problem to compute the hypersurface at time $(k+1)h$ from the surface at time $kh, k \in \mathbb{N}$ and $h > 0$ being the time step of the discretized flow. They show the consistency of their approach with smooth evolutions for the MCM but their minimizing solution is not unique given an initial hypersurface. Chambolle propose in [1] to overcome the lack of uniqueness by defining a new variational model, based on the ROF model and the signed distance function of the evolving hypersurface, satisfying the monotonicity property which is related with uniqueness. Monotonicity means that if sets $E \subseteq E'$ then sets $T_h(E) \subseteq T_h(E')$, where $T_h(E)$ is a transformation produced by a flow s.a. the MCM on a set E during time h. Chambolle prove that his variational model is an approximation of the generalized MCM in the sense of minimal barriers [1].

The variational model introduced by Chambolle to discretize the MCM is, for $h > 0$, as follows:

$$\min_{u \in L^2(\Omega)} F_C(u) = \int_\Omega |\nabla u| + \frac{1}{2h}\int_\Omega (u(x) - d_E(x))^2 dx, \quad (2)$$

where the first term $\int_\Omega |\nabla u|, \Omega \subset \mathbb{R}^N$ is the well-known TV-norm of function u where ∇u is the distributional derivative of u and $|.|$ is the Euclidean norm in $\mathcal{R}^N$, Ω is a bounded open subset of $\mathcal{R}^N$ and $d_E(x)$ is the signed distance function of a closed set $E \subset \Omega$ defined by $d_E(x) = d(x, E) - d(x, \mathbb{R}^N \setminus E)$, where $d(x, E) = \inf_{y \in E} |x - y|$ is at x the smallest distance from E. It is clear that variational model (2) corresponds to the ROF model with the signed distance as the given image. Even though the ROF model has been initially developed for image denoising [2], Chambolle uses it to approximate the MCM of the boundary of the set E. Let follow the development given by Chambolle. Define the transformation T_h by letting $T_h(E) = \{x : u(x) < 0\}$ whose u is the minimizing solution of (2). Then, the Euler-Lagrange equation of (2) is $-\nabla.(\frac{\nabla u}{|\nabla u|}) + \frac{1}{h}(u - d_E) = 0$, which implies if $x \in \partial T_h(E)$ that $u(x) = 0$ and

$$d_E(x) = -h\left[\nabla.\left(\frac{\nabla u}{|\nabla u|}\right)\right]_{\partial T_h(E)}(x) = -h\,\kappa_{\partial T_h(E)}(x), \quad (3)$$

where $\kappa_{\partial T_h(E)}$ is the mean curvature of the hypersurface $\partial T_h(E)$. If x_0 is the projection of x on ∂E and $\mathcal{N}(x_0)$ is the exterior normal to the set E at x_0, then we have $x = x_0 + d_E(x)\mathcal{N}(x_0)$ and with (3), the following equation:

$$x = x_0 - h\,\kappa_{\partial T_h(E)}\,\mathcal{N}(x_0),$$

which corresponds to the discretization of the MCM given in Equation (1), with time step h. Minimizing solution of (2) thus provides an approximation of the MCM with time step h.

3. NEW ALGORITHM FOR SNAKES

In this section, we extend the previous work of Chambolle to the well-known image segmentation model called snakes or geodesic/ geometric active contours (GAC). As we said in the introduction, the snake model has been introduced in computer vision by Kass, Witkin and Terzopoulos in [3] and then improved by Caselles, Kimmel and Sapiro in [4] and Kichenassamy, Kumar, Olver, Tannenbaum and Yezzi in [5]. It basically detects objects of interest in images by deforming a curve toward object boundaries, which correspond to sharp intensity variations, i.e. large intensity gradients. More precisely, the GAC model is a variational model which consists of finding the curve C which minimizes the following geometrically intrinsic energy:

$$F_{GAC}(C) = \int_0^{L(C)} g_b(|\nabla f(C(s))|)\,ds, \quad (4)$$

where ds is the Euclidean element of length, $L(C)$ is the length of the curve C and function g_b is an edge indicator function that vanishes at object boundaries such as $g_b(|\nabla f|) = \frac{1}{1+\beta|\nabla f|^2}$, where f is the given image and β is an arbitrary positive constant. Hence, energy functional (4) is actually a new length obtained by weighting the Euclidean element of length ds by the function g_b which contains information concerning the boundaries of objects. The calculus of variations provides us the Euler-Lagrange equation of functional F_{GAC} and the gradient descent method gives us the PDE flow that minimizes as fast as possible F_{GAC}:

$$C_t = \big(g_b\kappa - \langle\nabla g_b, \mathcal{N}\rangle\big)\mathcal{N}, \quad (5)$$

where C_t is the derivative of C w.r.t. the artificial time parameter t, κ and $\mathcal{N}$ are respectively the curvature and the normal to the curve C. The evolution equation of GAC, defined by the PDE (5), is well-defined in term of viscosity solution [4].

We now introduce the variational model, based on the ROF model, to approximate the motion of the GAC:

$$\min_{u \in L^2(\Omega)} F_1(u) = \int_\Omega g_b(x)|\nabla u| + \frac{1}{2h}(u - d_E)^2 dx, \quad (6)$$

where the first term $\int_\Omega g_b|\nabla u|$ is the *weighted* TV-norm of function u where g_b is the edge detector function of the GAC model and d_E is the signed distance function of a closed set $E \subset \Omega$. We follow the same explanations given in Section 2 to show that variational model (6) approximate the evolution of GAC. Define the transformation T_h by letting $T_h(E) = \{x : u(x) < 0\}$ whose u is the minimizing solution of (6). Then, the Euler-Lagrange equation of (6) is

$$\begin{aligned}&-\nabla.\big(g_b\tfrac{\nabla u}{|\nabla u|}\big) + \tfrac{1}{h}(u - d_E) = \\ &-g_b\nabla.\big(\tfrac{\nabla u}{|\nabla u|}\big) - \langle\nabla g_b, \tfrac{\nabla u}{|\nabla u|}\rangle + \tfrac{1}{h}(u - d_E) = 0,\end{aligned}$$

where $\langle .,. \rangle$ is the Euclidean scalar product. This implies if $x \in \partial T_h(E)$ that $u(x) = 0$ and

$$\begin{aligned} d_E(x) &= -h\left[g_b\nabla.\left(\tfrac{\nabla u}{|\nabla u|}\right) + \langle\nabla g_b, \tfrac{\nabla u}{|\nabla u|}\rangle\right]_{\partial T_h(E)}(x) \\ &= -h\left(g_b\,\kappa_{\partial T_h(E)} - \langle\nabla g_b, \mathcal{N}_{\partial T_h(E)}\rangle\right)(x),\end{aligned}$$

where $\kappa_{\partial T_h(E)}$ and $\mathcal{N}_{\partial T_h(E)} = -\frac{\nabla u}{|\nabla u|}_{\partial T_h(E)}$ are respectively the mean curvature and the normal of the hypersurface $\partial T_h(E)$. If x_0 is the projection of x on ∂E and $\mathcal{N}(x_0)$ is the exterior normal to the set E at x_0, then we have $x = x_0 + d_E(x)\mathcal{N}(x_0)$ and with (7), the following equation:

$$x = x_0 - h\left(g_b\kappa_{\partial T_h(E)} - \langle\nabla g_b, \mathcal{N}_{\partial T_h(E)}\rangle\right)\mathcal{N}(x_0),$$

which corresponds to the discretization of the GAC motion given in Equation (5), with time step h. Minimizing solution of (6) thus provides an approximation of the evolution flow of GAC with time step h.

4. NEW ALGORITHM FOR REGION-BASED ACTIVE CONTOURS

In this section, we extend the previous segmentation model to region-based active contours. Region-based active contours are active contour models which find objects of interest in images from homogeneous intensity criteria such as intensity statistics. A well-known example is the Chan and Vese model of active contours without edges (ACWE) [6]. In this model, the active contour evolves in such a way that the difference between the inside (respectively outside) gray-level value and the inside (resp. outside) mean intensity value is minimized. The evolution equation of ACWE is given by:

$$C_t = \Big(\kappa + \lambda\underbrace{\big((\mu_{in} - f)^2 - (\mu_{out} - f)^2\big)}_{V_1}\Big)\mathcal{N}, \quad (7)$$

where κ and $\mathcal{N}$ are the mean curvature and the normal to C, f is the given image and μ_{in}, μ_{out} are the mean intensity values inside and outside the evolving active contour.

Other statistical moments such as the variance descriptor can be used with the active contour model to carry out the segmentation task, see e.g. [11], but the probability density function (PDF) descriptor looks to be one of the most powerful descriptors at this time, see e.g. [12, 7]. The active contour model developed in [7] is based on the Kullback-Leibler (KL) divergence measure, which determines the difference between two PDFs, and the computation of PDFs based on the Parzen method [13]. The evolution equation of active contours minimizing the KL difference $\int_{\mathbb{R}} q_{in}(I) \log \frac{q_{in}(I)}{q_{out}(I)} + q_{out} \log \frac{q_{out}}{q_{in}} dI$ computed in [7] is as follows:

$$C_t = \Big(\kappa + \lambda \underbrace{(\nu_{in} - \nu_{out})}_{V_2}\Big) \mathcal{N}, \tag{8}$$

where the two speed terms are given by:

$$\begin{cases} \nu_{in} = \int_{\mathbb{R}} \frac{1}{|\Omega_{in}|}\Big(1 - \frac{q_{out}(I)}{q_{in}(I)} + \log \frac{q_{in}}{q_{out}}\Big) \cdot \\ \qquad \big(K(I - I(s)) - q_{in}(I)\big)\, dI \\ \nu_{out} = \int_{\mathbb{R}} \frac{1}{|\Omega_{out}|}\Big(1 - \frac{q_{in}(I)}{q_{out}(I)} + \log \frac{q_{out}}{q_{in}}\Big) \cdot \\ \qquad \big(K(I - I(s)) - q_{out}(I)\big)\, dI \end{cases}, \tag{9}$$

where $\Omega_{in}, \Omega_{out}$ correspond to the evolving regions inside and outside the active contour, $|\Omega.|$ is the area of region $\Omega.$, K is the 1-D Gaussian kernel and q_{in}, q_{out} are the PDFs inside and outside the evolving active contour computed according the Parzen method [13] as follows:

$$\begin{cases} q_{in}(I) = \frac{1}{|\Omega_{in}|} \int_{\Omega_{in}} K(I - I(x))dx \\ q_{out}(I) = \frac{1}{|\Omega_{out}|} \int_{\Omega_{out}} K(I - I(x))dx \end{cases}. \tag{10}$$

Evolution equations (7) and (8) extract objects in images based on the mean descriptor and the PDF descriptor. They are composed, as all region-based active contour models, of two terms: a regularization term based on the mean curvature and a data-dependent term. Hence, the general evolution for different kind of region-based active contours is as follows:

$$C_t = \big(\kappa + \lambda V_r\big) \mathcal{N}, \tag{11}$$

where $V_r = V_1$ for ACWE and $V_r = V_2$ for active contours based on the KL measure.

We propose the following variational model to approximate the evolution equation of region-based active contours:

$$\min_{u \in L^2(\Omega)} F_2(u) = \int_\Omega g_b(x)|\nabla u| + \frac{1}{2h}(u - d_E)^2 + \lambda V_r u, \tag{12}$$

where the edge detector function g_b of Section 3 is introduced here in order to merge the edge detection model of GAC [4, 5] with the region-based active contours [6, 11, 12, 7] which detect homogeneous regions. Let show that the variational model (12) approximate the evolution of region-based active contours coupled with the GAC model. Define the transformation T_h by letting $T_h(E) = \{x : u(x) < 0\}$ whose u is the minimizing solution of (12). Then, the Euler-Lagrange equation of (12) is

$$-\nabla.(g_b \frac{\nabla u}{|\nabla u|}) + \frac{1}{h}(u - d_E) + \lambda V_r, \tag{13}$$

which implies if $x \in \partial T_h(E)$ that $u(x) = 0$ and $d_E(x) =$

$$\begin{array}{l} -h\left[g_b \nabla.\Big(\frac{\nabla u}{|\nabla u|}\Big) + \langle \nabla g_b, \frac{\nabla u}{|\nabla u|}\rangle + \lambda V_r\right]_{\partial T_h(E)}(x) \\ -h\Big(g_b\, \kappa_{\partial T_h(E)} - \langle \nabla g_b, \mathcal{N}_{\partial T_h(E)}\rangle + \lambda V_{r\partial T_h(E)}\Big)(x), \end{array} \tag{14}$$

where $\kappa_{\partial T_h(E)}$ and $\mathcal{N}_{\partial T_h(E)}$ are respectively the mean curvature and the normal of the hypersurface $\partial T_h(E)$ and $V_{r\partial T_h(E)}$ correspond to the value of the speed function at the location $\partial T_h(E)$. If x_0 is the projection of x on ∂E and $\mathcal{N}(x_0)$ is the exterior normal to the set E at x_0, then we have $x = x_0 + d_E(x)\mathcal{N}(x_0)$ and with (14), the following equation:

$$x = x_0 - h\left(g_b \kappa_{\partial T_h(E)} - \langle \nabla g_b, \mathcal{N}_{\partial T_h(E)}\rangle + \lambda V_{r\partial T_h(E)}\right)\mathcal{N}(x_0),$$

which corresponds to the discretization of the motion of the region-based active contours given in Equation (11) when $g_b = 1$, with time step h. Minimizing solution of (12) thus provides an approximation of the evolution flow of the region-based active contours s.a. [6, 11, 12, 7] merged with the GAC [4, 5].

5. NUMERICAL SCHEMES

We show in this section how to numerically solve the evolution of the GAC and the region-based active contours in this new framework. As explained by Chambolle in [1], given an initial set $E_0 \subset \Omega$, $h > 0$, then for every $t > 0$, the boundary of the following set: $E^h(t) = (T_h)^{[t/h]}(E_0)$ where [.] is the integer part and T_h are the transformation defined in Sections 2, 3 and 4, converge when $h \to 0$ to the evolution equation of the GAC and the region-based active contours.

From a numerical point of view, it means that the active contour at time $(k+1)h, k \in \mathbb{N}$ is given by the zero level set of the function u obtained by iterating the two following steps, given $u = d_{E_0}$ at $k = 0$:

1. computation of the signed distance function d_E from the zero level set of the function u at time kh,
2. solving variational problem (12) to get u at time $(k+1)h$.

The signed distance function of a set E is computed with the fast-marching algorithm developed in [14]. The minimization problem (12) is solved by introducing a convex regularization of the variational problem (12) as follows:

$$\min_{u,v \in (L^2(\Omega))^2} F_2^r(u,v) = \int_\Omega g_b(x)|\nabla u| + \frac{1}{2h}(u - d_E)^2 + \lambda V_r v + \frac{1}{2\theta}(u - v)^2,$$

where v is a new function with parameter $\theta > 0$. Since Functional F_2^r is convex, its minimizer can be computed by minimizing F_2^r w.r.t. u and v separatively and iterating until convergence. Thus, the following minimization problems are considered:

1. v being fixed, we search u as a solution of:
$$\min_u \int_\Omega g_b(x)|\nabla u| + \frac{1}{2h}(u - d_E)^2 + \frac{1}{2\theta}(u - v)^2, \tag{15}$$
2. u being fixed, we search v as a solution of:
$$\min_v \int_\Omega \lambda V_r v + \frac{1}{2\theta}(u - v)^2. \tag{16}$$

The solution of (15) is also given by this minimization problem:

$$\min_u \int_\Omega g_b(x)|\nabla u| + \frac{1}{2\alpha}(u - w)^2, w = \frac{\theta d_E + hv}{h + \theta}, \alpha = \frac{h + \theta}{h\theta},$$

which is a ROF model coupled with the TV-norm weighted by the edge detector function g_b whose solution is given in [15] by

$$u = w - \alpha \operatorname{div} p,$$

where $p = (p^1, p^2)$ is given by the fixed point method: $p^0 = 0$ and

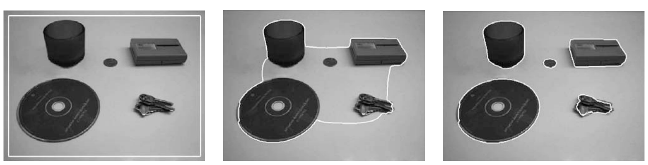
$$p^{n+1} = \frac{p^n + \delta t \nabla\big(\operatorname{div} p^n - w/\alpha\big)}{1 + \frac{\delta t}{g_b(x)}|\nabla\big(\operatorname{div} p^n - w/\alpha\big)|}, n \in \mathbb{N}.$$

The solution of (16) is given by: $v = u + \theta\lambda V_r$.

6. RESULTS

We present some results and outlines the advantage of this new framework to solve the active contour problem. In the proposed work, the time step h of the discretized motion of active contours [4, 5, 6, 11, 12, 7] is not restricted to small values like in standard explicit discretization schemes usually used to carry out the minimization task. In our approach, the time step h can be larger, which means that less iterations are needed to converge to the steady state solution.

As a first example, we apply the GAC model, developed in Section 3, to the image given on Figure 1. We introduce the traditional balloon force to the GAC model, which can be done with Section 4 choosing $V_r = ct.g_b(x), ct > 0$. The time step is taken equal to $h = 200$ and the number of iterations needed to converge to the steady state solution is 24, which at least 10 times less than the explicit scheme. Then, we apply the ACWE model, developed in Section 4, to the medical image given on Figure 2. The time step is $h = 100$ and the number of iterations is 8 (standard scheme takes 10 times more of iterations). We apply the active contour model based on the KL divergence measure, developed in Section 4, to the zebra image given on Figure 3, where the ACWE can not segment the textures. The time step is $h = 100$ and the number of iterations is 8 (standard scheme is also 10 times longer). Finally, the GAC model is applied to the problem of shape reconstruction from unorganized points [8] on Figure 4. We extract some points on the brain ventricle and its shape is reconstructed with the model developed in Section 3 choosing $g_b(x) = d(x, G)$, G being the set of points and d the distance function.

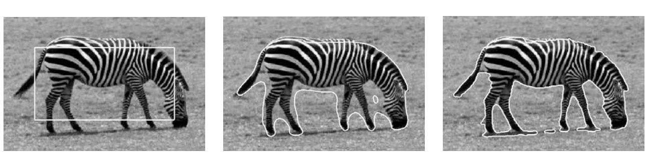


(a) Initial Position. (b) # Iterations = 10. (c) # Iterations = 24.

Fig. 1. GAC Model [4, 5] with a Balloon Force.

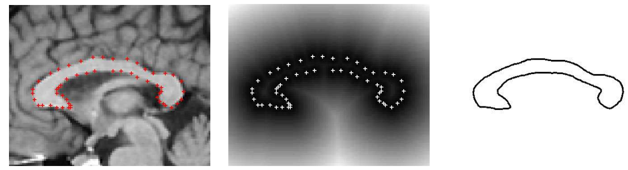
(a) Initial Position. (b) # Iterations = 3. (c) # Iterations = 8.

Fig. 2. ACWE Model [6].



(a) Initial Position. (b) # Iterations = 2. (c) # Iterations = 8.

Fig. 3. Active Contour Model based on the KL Divergence [7].



(a) Set of Points. (b) Distance Function of Set of Points. (c) Shape Reconstruction.

Fig. 4. GAC Model [4, 5] applied to Shape Reconstruction.

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