VARIATIONAL BAYESIAN BLIND IMAGE DECONVOLUTION WITH STUDENT-T PRIORS.

Dimitris Tzikas, Aristidis Likas and Nikolaos Galatsanos

Department of Computer Science, University of Ioannina, Ioannina, GR 45110, Greece.
E-mail:{tzikas, arly, galatsanos}@cs.uoi.gr

ABSTRACT

In this paper we present a new Bayesian model for the blind image deconvolution (BID) problem. The main novelties of this model are three. The first one is the use of a sparse kernel-based model for the point spread function (PSF) that allows estimation of both PSF shape and support. The second one is a robust distribution of the BID model errors and the third novelty is an image prior that preserves edges of the reconstructed image. Sparseness, robustness and preservation of edges is achieved by using priors that are based on the Student-t probability density function (pdf). The Variational methodology is used to solve the corresponding Bayesian model. Numerical experiments are presented that demonstrate the advantages of this model as compared to previous Gaussian based ones.

***Index Terms*—** Bayesian, Variational, Blind Deconvolution, Kernel Model, Sparse Prior, Robust Prior, Student-t Prior

1. INTRODUCTION

In blind image deconvolution (BID) both the initial image and the blurring point spread function (PSF), are unknown. Thus for this problem the observed data are not sufficient to uniquely specify the unknown parameters. In order to resolve this ambiguity, prior knowledge (constraints) has to be used for both the image and the PSF.

A recent and successful approach to apply constraints to the image and the PSF is the use of the Bayesian methodology. Unfortunately, because of the non-linearity of the data generation model in BID, Bayesian inference presents several computational difficulties, since the posterior distribution of the unknown parameters can not be computed analytically. These difficulties can be overcome using the variational Bayesian methodology [1, 2]. However, the previous Bayesian models [1, 2] used for BID were based on Gaussian stationary statistics. Thus, they can not estimate reliably the support of the PSF, are not robust to large imaging model errors and the edges in the reconstructed images are blurred and display ringing artifacts.

In this paper we propose a new Bayesian model for the BID problem that ameliorate on the above shortcomings. This model introduces three novelties.

- First, using the proposed PSF model, we can estimate both the support and shape of the PSF. More specifically, a sparse kernel based model is used for the unknown PSF, in a similar manner as for the Relevance Vector Machine (RVM) [3]. This model has the property to prune out unnecessary kernels, thus providing an effective mechanism to estimate the PSF spatial support.
- Second, the proposed model is robust to errors of the BID model. This is achieved by assuming non Gaussian errors modeled by a pdf with heavy tails.
- Third, an edge preserving image prior is employed, which is based on the assumption that the local image differences also follow a non Gaussian pdf with heavy tails.

The pdf used to enforce sparseness of the PSF prior, robustness to the BID model errors and edge preservation via the local image differences of the image, is the Student-t pdf [4].

The herein proposed Bayesian model is too complex to be solved exactly. Therefore, we resort to the variational approximation methodology [5]. This approximation methodology assumes a family of approximate posterior distributions and then finds the best approximation of the true posterior within this family. This methodology has been succesfully used in many Bayesian inference problems.

2. BID MODEL

We assume that the observed image $g(x)$ has been generated by convolving an unknown image $f(x)$ with an also unknown PSF $h(x)$. To account for errors additive independent noise $n(x)$ is also assumed. This model is written as

$$g(x) = f(x) * h(x) + n(x), \tag{1}$$

where $x = (x_1, x_2) \in \Omega_I$, $\Omega_I \subset \Re^2$ is the support of the image and $*$ denotes two-dimensional circular convolution.

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Equivalently, this can be written in vector form as

$$g = f * h + n, \quad (2)$$

where g, f, h and n are $N \times 1$ lexicographicaly ordered vectors (N is the number of pixels) of the intensities of the degraded image, observed image, blurring PSF and additive noise respectively. Here, we introduce the $N \times N$ block-circulant matrices F and H, which implement two-dimensional convolution with the vectors f and h respectively. Then, the above equation can been written as

$$g = Fh + n = Hf + n. \quad (3)$$

2.1. PSF kernel model

We model the PSF as the linear combination of basis functions:

$$h(x) = \sum_{i=1}^{N} w_i \phi_i(x), \quad (4)$$

where $\phi_i(x) = K(x, x_i)$ is a kernel function centered at pixel $x_i = (x_{i1}, x_{i2}) \in \Omega_I$ and $w = (w_1, \ldots, w_N)^T$ are the parameters of the linear combination. We denote with $h = (h(x_1), \ldots, h(x_N))^T$ the vector of values of the PSF $h(x)$ at each image pixel x_i and with $\phi_i = (\phi_i(x_1), \ldots, \phi_i(x_N))^T$ the corresponding basis vector for $\phi_i(x)$. Then the PSF vector h is modeled as the linear combination of the basis vectors ϕ_i:

$$h = \sum_{i=1}^{N} w_i \phi_i. \quad (5)$$

We further assume that $K(x, x_i) = K(x - x_i)$, thus equation (5) can be written as:

$$h = \Phi w = W\phi = \phi * w, \quad (6)$$

where Φ and W are $N \times N$ block-circulant matrices that implement two-dimensional convolution with $\phi = \phi_1$ and w respectively.

Thus, the data generation model (2) can be written as:

$$g = F\Phi w + n = \Phi W f + n. \quad (7)$$

2.2. PSF sparseness

A hierarchical prior that enforces sparsity is imposed on the weights w [3], by assigning them a zero-mean Gaussian distribution:

$$p(w|\alpha) = N(w|0, A^{-1}), \quad (8)$$

with diagonal covariance matrix $A = diag\{\alpha\}$, where $\alpha = (\alpha_1, \ldots, \alpha_N)^T$. Therefore, each weight is assigned a separate precision parameter α_i, which is treated as a random variable that follows a Gamma distribution:

$$p(\alpha) = \prod_{i=1}^{N} Gamma(\alpha_i | a^\alpha, b^\alpha). \quad (9)$$

This two-level hierarchical prior is equivalent to a product of Student-t prior distributions. This can be realized by integrating out the parameters α_i to obtain the prior weight distribution $p(w)$:

$$p(w) = \int p(w|\alpha)p(\alpha)d\alpha = \prod_{i=1}^{N} St(w_i|0, \frac{b^\alpha}{a^\alpha}, 2a^\alpha), \quad (10)$$

where $St(w|0, \frac{b^\alpha}{a^\alpha}, 2a^\alpha)$ denotes a zero mean Student-t distribution with variance $\frac{b^\alpha}{a^\alpha}$ and $2a^\alpha$ degrees of freedom [4].

2.3. Image model

The image prior that we use is based on K filtered versions of the image: $\epsilon^k = Q^k f$, where Q^k are $N \times N$ convolutional operators of the filters ($k = 1, \ldots, K$). Equivalently, we use the $KN \times N$ operator $\tilde{Q} = ({Q^1}^T, \ldots, {Q^K}^T)^T$ that produces the $KN \times 1$ vector $\tilde{\epsilon} = ({\epsilon^1}^T, \ldots, {\epsilon^K}^T)^T$:

$$\tilde{\epsilon} = \tilde{Q} f = ((Q^1 f)^T, \ldots, (Q^K f)^T)^T. \quad (11)$$

We assume that ϵ_i^k is Gaussian distributed with distinct inverse variance γ_i^k:

$$p(\epsilon_i^k | \gamma_i^k) = N(\epsilon_i^k | 0, (\gamma_i^k)^{-1}). \quad (12)$$

Assuming the ϵ_i^k independent with respect to i induces a prior for the image, which is given by

$$p_k(f|\gamma^k) = N(f|0, ({Q^k}^T \Gamma^k Q^k)^{-1}), \quad (13)$$

with $\gamma^k = (\gamma_1^k \ldots \gamma_N^k)^T$ and $\Gamma^k = diag\{\gamma^k\}$. In order to combine the information that is available in priors p_k, we define a composite prior, which is the product of them:

$$p(f|\tilde{\gamma}) = \frac{1}{Z} \prod_{k=1}^{K} p_k(f|\gamma^k) = N(f|0, (\tilde{Q}^T \tilde{\Gamma} \tilde{Q})^{-1}), \quad (14)$$

where $\tilde{\gamma} = ({\gamma^1}^T, \ldots, {\gamma^K}^T)^T$ and $\tilde{\Gamma} = diag\{\tilde{\gamma}\}$. Unfortunately, we cannot analytically, compute the determinant $|\tilde{Q}^T \tilde{\Gamma} \tilde{Q}|$ that is required to estimate the normalization constant Z. Instead we use the following improper prior:

$$p(f|\tilde{\gamma}) \propto \prod_{k=1}^{K} \prod_{i=1}^{N} (\frac{\gamma_i^k}{2\pi})^{\frac{1}{2}} exp\{-\frac{1}{2} f^T \tilde{Q}^T \tilde{\Gamma} \tilde{Q} f\}. \quad (15)$$

The parameters γ_i^k are assumed to be independent identically distributed, Gamma random variables:

$$p(\tilde{\gamma}) = \prod_{k=1}^{K} \prod_{i=1}^{N} Gamma(\gamma_i^k | a^\gamma, b^\gamma). \quad (16)$$

Thus, the image prior for each ϵ^k is a Student-t pdf, similarly with (10).

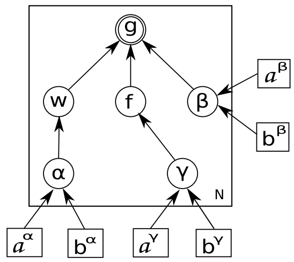

Fig. 1. Graphical model that describes the dependencies between the random variables of the proposed model.

2.4. Noise model

The noise n of the BID model (3) is assumed to be zero mean Gaussian distributed, given by:

$$p(n|\beta) = \prod_{i=1}^{N} N(n_i|0, \beta_i^{-1}) = N(n|0, B^{-1}), \tag{17}$$

with $\beta = (\beta_1, \dots, \beta_N)$ and $B = diag(\beta)$. The parameters β_i that define the variance of the noise at each pixel, are also assumed to be random variables with a Gamma distributed prior:

$$p(\beta) = \prod_{i=1}^{N} Gamma(\beta_i|a^\beta, b^\beta). \tag{18}$$

3. VARIATIONAL BAYESIAN INFERENCE

The dependencies among the random variables that define the proposed Bayesian model are shown in the graphical model of Fig. 1. Because of the complexity of this model, the posterior distribution of the parameters $p(\theta|D)$ cannot be computed and exact inference methods, such as maximum likelihood via the EM algorithm, can not be applied. Instead, we resort to approximate inference and specifically to the variational Bayesian methodology [5]. This is an approximate inference methodology, which assumes a family of approximate posterior distributions $q(\theta)$ and then seeks values for the parameters θ that best approximate the true posterior $p(\theta|D)$.

The approximate posterior distributions of the hidden variables are computed from

$$q(\theta^i) \propto \exp[\langle \ln p(D, \theta)\rangle_{q(\theta^{\setminus i})}], \tag{19}$$

where $\theta^{\setminus i}$ denotes the vector of all hidden variables except θ^i, and are the following:

$$\begin{aligned} q(w) &= N(w|\mu_w, \Sigma_w), && (20)\\ q(f) &= N(f|\mu_f, \Sigma_f), && (21)\\ q(\alpha) &= \prod_{i=1}^{N} Gamma(\alpha_i|\tilde{a}^\alpha, \tilde{b}_i^\alpha), && (22)\\ q(\beta) &= \prod_{i=1}^{N} Gamma(\beta_i|\tilde{a}^\beta, \tilde{b}_i^\beta), && (23)\\ q(\gamma) &= \prod_{k=1}^{K}\prod_{i=1}^{N} Gamma(\gamma_i^k|\tilde{a}_i^\gamma, \tilde{b}_i^{\gamma^k}), && (24) \end{aligned}$$

where

$$\begin{aligned} \mu_w &= \Sigma_w \Phi^T \langle F^T B\rangle g, && (25)\\ \Sigma_w &= \left(\Phi^T \langle F^T B F\rangle \Phi + \langle A\rangle\right)^{-1}, && (26)\\ \mu_f &= \Sigma_f \Phi^T \langle W^T B\rangle g, && (27)\\ \Sigma_f &= \left(\Phi^T \langle W^T B W\rangle \Phi + \tilde{Q}^T \langle \tilde{\Gamma}\rangle \tilde{Q}\right)^{-1}, && (28)\\ \tilde{a}^\alpha &= a^\alpha + 1/2, && (29)\\ \tilde{b}_i^\alpha &= b^\alpha + \frac{1}{2}\langle w_i^2\rangle, && (30)\\ \tilde{a}^\beta &= a^\beta + N/2, && (31)\\ \tilde{b}_i^\beta &= b^\beta + \frac{1}{2}\langle nn^T\rangle_{ii}, && (32)\\ \tilde{a}^\gamma &= a^\gamma + 1/2, && (33)\\ \tilde{b}_i^{\gamma^k} &= b^\gamma + \frac{1}{2}\langle (Q^k f)_{ii}^2\rangle. && (34) \end{aligned}$$

The required expected values can be computed as:

$$\begin{aligned} \langle w\rangle &= \mu_w, && (35)\\ \langle w_i^2\rangle &= \mu_{w_i}^2 + \Sigma_{w_{ii}}, && (36)\\ \langle f\rangle &= \mu_f, && (37)\\ \langle ff^T\rangle &= \mu_f \mu_f^T + \Sigma_f, && (38)\\ \langle \alpha_i\rangle &= \tilde{a}^\alpha / \tilde{b}_i^\alpha, && (39)\\ \langle \beta_i\rangle &= \tilde{a}^\beta / \tilde{b}_i^\beta, && (40)\\ \langle \gamma_i^k\rangle &= \tilde{a}^\gamma / \tilde{b}_i^{\gamma_k}, && (41) \end{aligned}$$

$$\langle nn^T\rangle = gg^T - 2\Phi\langle Fw\rangle g^T + \Phi\langle Fww^T F^T\rangle \Phi^T. \tag{42}$$

4. NUMERICAL EXPERIMENTS

Several experiments have been carried out, in order to demonstrate the properties of the proposed method and compare it with previous Bayesian BID formulations based on the Gaussian PSF.

Hereafter, we will refer to the proposed method as the StStSt method, to imply that three Student-t priors are used to model the PSF weights, the BID model errors and the image local differences. We also considered four simpler versions of this Bayesian model. The first one assumes a Gaussian distribution for the noise, $p(n) = N(n|0, \beta^{-1}I)$ and is denoted as StGSt to imply that only the noise is Gaussian. The second one assumes a Gaussian distribution only for the image local differences, in other words $p(f) = N(f|0, (\gamma Q^T Q)^{-1})$ and is denoted as StStG to imply that only the image prior is

(a) (b) StGG $2.17, 6.69$

(c) StStG $5.57, 10.91$ (d) StGSt $5.87, 8.12$

(e) StStSt $5.29, 9.44$ (f) GGG $0.70, -4.71$

Fig. 2. (a) Degraded image. Estimated images using the (b) StGG, (c) StStG (d) StGSt, (e) StStSt and (f) GGG. The numbers below each image are the ISNR values of the image ($ISNR_f$) and the corresponding PSF ($ISNR_h$).

assumed Gaussian. The next version considers Gaussian distributions for both the noise and image local differences and is denoted as StGG. The last version, assumes Gaussian priors for the PSF weights, imaging model errors and image local differences, and therefore it is denoted as GGG.

For the filters Q^k used to determine the image prior, we considered horizontal and vertical first order local differences, by defining Q^1 and Q^2 so that:

$$\epsilon^1(x,y) = f(x,y) - f(x+1,y), \quad (43)$$

$$\epsilon^2(x,y) = f(x,y) - f(x,y+1). \quad (44)$$

In the experiments, we compared all the methods using artificially degraded images. For the experiment shown here, we generated a degraded image by blurring the true image with a 7×7 square-shaped PSF h and then added Gaussian noise with variance $\sigma^2 = 10^{-6}$. The signal to noise ratio (SNR) of the observed image g was $SNR = 10\log_{10}\frac{\|f\|^2}{N\sigma^2} = 45dB$. We initialized the PSF as a Gaussian shaped function with variance $\sigma^2_{h_{in}} = 3$. The kernel function was set to a Gaussian with variance $\sigma^2_\phi = 0.1$ in order to be flexible enough to model the boundaries of the square. The estimated

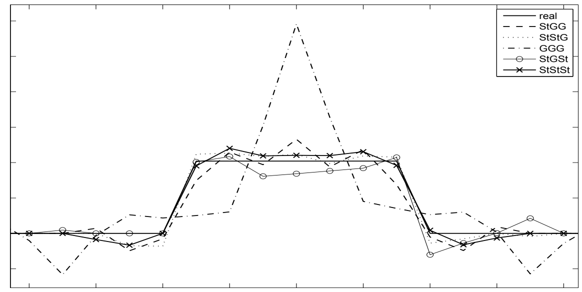

Fig. 3. One dimensional slice of true and estimated PSFs.

images for comparison of the algorithms are shown in fig. 2 and the PSFs in fig. 3. For each method we computed the improvement in signal to noise ratio of the image ($ISNR_f = 10\log\frac{\|f-g\|^2}{\|f-\hat{f}\|^2}$) and PSF ($ISNR_h = 10\log\frac{\|h-h_{in}\|^2}{\|h-\hat{h}\|^2}$). We have performed numerous other experiments, both with simulated and real image data, which cannot be included in this paper due to space constraints. These experiments also verify the superiority of the proposed model for BID as compared to models that use two or more Gaussians for the PSF, the imaging errors and image prior.

5. CONCLUSIONS

We presented a Bayesian model of the BID problem in which the PSF was modeled as a superposition of kernel functions. We used heavy tailed Student-t distributions to model the PSF weights, image local differences and BID errors in order to achieve PSF sparseness, reconstruvtion of edges and robustness to BID model errors.Because of the complexity of this model, the variational framework was used for inference. Experiments demonstrate the advantages of using a heavy tailed distribution for the PSF mode, the image model and the BID model errors.

6. REFERENCES

[1] A. Likas and N. P. Galatsanos, “A variational approach for Bayesian blind image deconvolution,” *IEEE Trans. on Signal Processing*, vol. 52, no. 8, pp. 2222–2233, 2004.

[2] R. Molina, J. Mateos, and A. Katsaggelos, “Blind deconvolution using a variational approach to parameter, image, and blur estimation,” *IEEE Trans. on Image Processing*, vol. 15, no. 12, pp. 3715–3727, 2006.

[3] M. E. Tipping, “Sparse Bayesian learning and the relevance vector machine,” *Journal of Machine Learning Research*, vol. 1, pp. 211–244, 2001.

[4] C. M. Bishop, *Pattern Recognition and Machine Learning*. Springer, August 2006.

[5] M. Jordan, Z. Gharamani, T. Jaakkola, and L. Saul, “An introduction to variational methods for graphical models,” in *Learning in Graphical Models*, M. Jordan, Ed. Kluwer, 1998, pp. 105–162.