IMPLICIT EVOLUTION OF OPEN ENDED CURVES

Saurav Basu[1], Dipti Prasad Mukherjee[2], Scott T. Acton[1]

[1]Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, 22904 USA.
[2]Electronics and Communication Sciences Unit, Indian Statistical Institute, Kolkata, India 700108.

ABSTRACT

We introduce a theoretical framework for implicit evolution of an open ended curve in a two-dimensional image plane. This approach is particularly suitable for identifying thin filamentous structures present in 2D images. The open ended curve is represented as the *centerline* of the level set of a function (called the *curvature map*) defined on the curve. The iterative evolution of the curvature map is guided by a diffusion equation and constrained by the imaging force, such as the image intensity gradient. The centerline of the evolved curvature map provides the position of the curve in subsequent iterations. We have tested this new model on both synthetic and real images that contain structures including rivers/roads/arteries. Nine experiments show that our model is successful in identifying complex topological filaments with a low 9% RMSE pixel error, and that the technique withstands effects of shape irregularities such as kinks, bending, circularity and inconsistent edges.

***Index Terms*—** Image segmentation, open ended curve, diffusion

1. INTRODUCTION

The goal of this paper is to build a theoretical framework to represent and evolve *smooth* open ended curves implicitly in 2D images. The open ended curve is a special case of a manifold with boundaries and having *codimension*[1] greater than one. Most applications of, and existing mathematical theory for implicit models such as the level set method, focus on the evolution of hypersurfaces, i.e., codimension one motion. For example, level set methods are used in the segmentation and tracking of object boundaries in 2D medical or aerial images [6,10,11] which are instances of codimension one motion of a closed curve in 2D space. This traditional approach to the level set formulation of active contours needs to be extended to the evolution of open ended curves (of codimension 2) to identify/segment open ended filamentous structures in images such as thin blood vessels in MRI or road/river networks in satellite images.

The overall contribution of our work can be summarized as follows:

- Developing a novel implicit representation of open ended curves in 2D
- Establishing a mathematical model for the time evolution of the implicit representation that consequently evolves the open ended curves in 2D
- Applying the new model to track filamentous structures in 2D images by open ended curves

Recent work in differential geometry has laid the groundwork necessary to evolve manifolds in spaces of arbitrary dimension using level set methods [1]. These have been applied to the evolution of infinite (or closed) curves in 3D, as in the case of segmenting blood vessels in volumetric magnetic resonance angiography images [5]. The mathematical framework for motion of infinite (or closed) curves in 3D has also been developed independently by Burchard *et al.* [4]. A diffusion-generated motion by mean curvature also addresses the same problem by resorting to a modified form of the complex Ginzburg-Landau equation [8]. All of the existing extensions of the level set method to handle cases of arbitrary codimension work satisfactorily for evolving infinite (or closed) curves in $\Re^3$, but not for evolving open ended curves in 2D.

2. BACKGROUND

For an evolving parameterized closed curve $C(s,t):[0,1]\times t \to \Re^2$, where $C(0,t)=C(1,t)$, the level set method and in particular the motion by mean curvature of Osher and Sethian [10,12] defines a 2D Lipschitz level set function $\phi(x,y,t)$, $\phi:\Re^2\times[0,t)\to\Re$, over the plane of interest, such that at each instant $C(s,t)=\{(x,y)\mid\phi(x,y,t)=0\}$ defines the *zero-level set* of ϕ. The PDE describing the evolution of ϕ is given by $\phi_t=-v(s,t)|\nabla\phi|$, where $v(s,t)$ is the speed of zero level set of ϕ in the normal direction. A common choice for ϕ is a function that is defined positive (or negative) inside the

[1]The *codimension* of a manifold in the context of an embedding is defined as the difference between the dimension of the manifold and the dimension of the evolving space. For example, in the case of an open ended curve in a plane, the maximum codimension of the embedding is two, when we consider the end points of the curve having dimension zero being embedded in the plane with dimension two (with the resultant codimension being $2-0=2$).

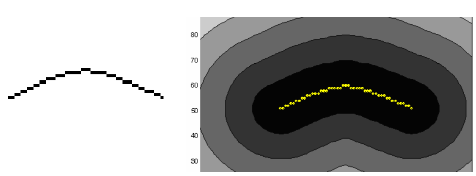
closed curve, and negative (or positive) outside it, so that the curve lies at the zero crossing (or zero level set) of the function.

The straightforward extension of this signed level set function to open ended curves in 2D is not possible due to the absence of the inside-outside dichotomy for open ended curves. There exist several problems with evolving a level set function ϕ for which the zero level set is not a zero crossing of the function. Numerical errors may accumulate, and it may be numerically infeasible to calculate the gradient at the zero level set due to singularities or to locate the zero magnitude points. For image processing applications, such instability may preclude segmentation. Also, the notion of mean curvature motion is undefined in the case of open curves as curvature itself is undefined at the endpoints of the open curve. Clearly, a new method of implicitly representing the open ended curve in a level set framework is necessitated.

3. PROPOSED MODEL

We define the *centerline* of a closed contour as the largest connected component remaining after removing the branches of the medial axis [2] corresponding to the closed contour.

The proposed model derives the open ended curve from the centerline of the level set contour of a function (named as the *curvature map*) defined on the curve. The level set is allowed to undergo a mean curvature motion, which approximately evolves the centerline of the level set according to the curvature between the end points and elongates the centerline at the end points in a tangential direction. The elongated centerline is taken as the updated position of the curve, and the calculations are repeated to determine the subsequent positions of the curve.



(a) (b)

Fig. 1: (a) An open ended curve; (b) Scalar distance function δ of curve in (a). Lighter intensity indicates greater distance.

Let us consider an open ended curve $C(s):[0,1]\rightarrow\Re^2$, where $C(0)\neq C(1)$ (see Fig. 1(a)). Let δ be the scalar distance function $(|\nabla\delta|=1)$ to the open curve $C(s)$ (e.g., Fig. 1(b)). We construct a regularized and smoothed gradient field of δ by employing δ as the *edge map* in a minimization functional and call it $\vec{G}$ (the minimization functional is similar to that used for computing gradient vector flow (GVF) in [12]) (see Fig. 2).

We now define a curvature map Ψ as the square of the divergence of $\vec{G}$ i.e., $\Psi=(\nabla\circ\vec{G})^2$. When the curve is not smooth or has sharp turns, $\nabla\delta$ and consequently $\vec{G}$ may have sharp discontinuities developing at some distance from the curve, thus giving high Ψ values at regions far from the curve. To mitigate this effect, we multiply Ψ with $H(\delta-\varepsilon)$, where H is the Heavyside function and ε is the distance from the open curve outside of which we want the curvature map to be zero. The product is the modified curvature map $\hat{\Psi}=H(\delta-\varepsilon)\Psi$ as shown in Fig. 3(a).

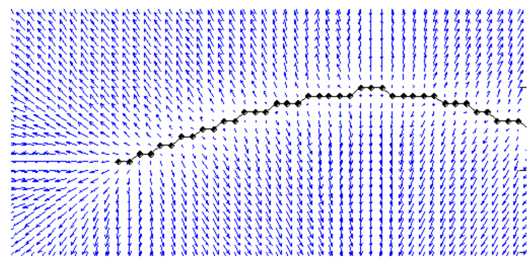
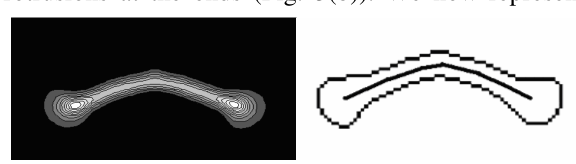


Fig. 2: The regularized gradient vector field $\vec{G}$ of the scalar distance function δ corresponding to Fig. 1(b) (only a portion of the open curve is magnified for visualization).

The absolute value of the divergence of the regularized gradient vector field $\vec{G}$ is low along the middle portions of the curve (since the curvature of the level sets of δ is relatively low there) but is high around the endpoints of the curve (since the curvature of the level sets of δ is much high there). Thus the level sets of the curvature map $\hat{\Psi}$ have a flat ribbon-shaped middle portion but dumbbell-shaped protrusions at the ends (Fig. 3(b)). We now represent our



(a) (b)

Fig. 3: (a) The (0-1) normalized curvature map $\hat{\Psi}$ of the regularized gradient vector field $\vec{G}$ of Fig. 2. Lighter shades indicate values closer to 1; (b) Centerline of the μ-level set of $\hat{\Psi}$ $(\mu=0.1)$ of (a).

original curve as the *centerline* of the μ- level set of $\hat{\Psi}$ (Fig. 3(b)). It is evident from the symmetry of the level sets of δ about the curve and the symmetry preserving regularization process used to construct $\vec{G}$ (and hence $\hat{\Psi}$) that the level sets of $\hat{\Psi}$ is symmetric about the original curve, thus the centerline of the level set of $\hat{\Psi}$ gives back the original curve (fig 1(e).). We initialize a characteristic function Γ of the μ-level set of $\hat{\Psi}$ such that $\Gamma(x,y)$ is 1 if

(x, y) is on or inside the μ − level set of $\hat{\Psi}$ and 0 if (x, y) is outside as in Fig. 4(a).

Our aim is to elongate the open ended curve, and we note that evolving the boundary of Γ according to the mean curvature would dilate the regions of low curvature of the μ-level set (this region is in the middle of the level set as shown in Fig. 4(a)) but would restrict dilation of the dumb-bell shaped ends that have a high curvature. Isotropic diffusion with intermediate re-initialization of Γ amounts to a mean curvature motion of the boundary of Γ along the normal direction to the boundary [9]. Therefore, the update for Γ is given by $\frac{\partial \Gamma}{\partial t} = \nabla^2 \Gamma$.

(a) (b) (c)

Fig. 4: (a) The characteristic function Γ of μ-level set of $\hat{\Psi}$ $(\mu = 0.1)$ of Fig. 3(a); (b) Updated Γ after convolving Γ of Fig. 3(b) with a Gaussian kernel; (c) The centerline of the boundary of the updated Γ of (b).

Also, since our open curve will be evolving in the 2D image, we incorporate image constraints. We may assume that the 1D structures have an average intensity that is in contrast to the average intensity of the background. Hence, we define a coefficient called the *coefficient of intensity difference* ℓ defined at every position (x, y) of the image I as

$$\ell(x, y) = \frac{1}{0.1 + \left| I_p(x, y) - \frac{\iint_{(x,y)\in I} \Gamma I_p(x, y)dxdy}{\iint_{(x,y)\in I} \Gamma dxdy} \right|} \qquad (1)$$

where the intensity value of the image I at location p is given by I_p. It can be observed from (1) that $\ell(x, y)$ takes a higher value when the intensity of the point (x, y) approaches the average intensity value inside the characteristic function. Similarly, the familiar anisotropic diffusion coefficient [7] $\wp$ can also be defined at every image point as $\wp(x, y) = \frac{1}{0.1 + |\nabla I_p(x, y)|}$.

If we initialize an open ended curve inside a 1D filamentous structure, the characteristic function Γ undergoes an isotropic diffusion by convolving it with a Gaussian kernel of standard deviation K [9]. We thus re-initialize the diffused Γ as

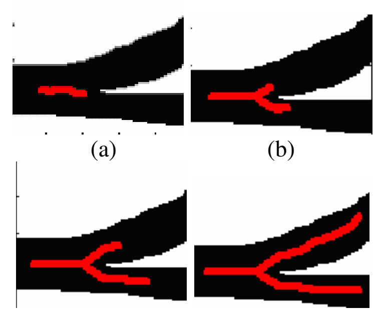
$$\Gamma_{re-initialized} = \begin{cases} 1 \text{ if } \wp\ell(\Gamma * K) \geq \lambda \\ 0 \text{ otherwise} \end{cases}, \qquad (2)$$

where λ is a user defined parameter and * denotes convolution. In (2), the product term $\wp\ell$ takes a very low value at the edges of the filament (since $\wp$ is low here) as well as areas that are just outside the boundary of the filament (ℓ is low here due to high difference between the average intensity inside Γ and the intensity of the area just outside the boundary of Γ). Hence, pre-multiplying the diffused Γ with $\wp\ell$ negates the effects of the diffusion in areas that are just at the edges or just outside the boundary of the filament. Thus the re-initialized Γ stays inside the filament that we are attempting to segment.

The boundary of Γ where the middle ribbon band joins the dumbbell shaped protrusions possesses high curvature. This region is smoothed more vigorously compared to the other locations in the boundary of Γ where curvature is relatively smaller (such as the middle ribbon portion or the rounded portions of the dumbbell). So in effect, the mean curvature motion of the boundary of Γ performs a differential flattening of the boundary depending on the curvature, consequently elongating its centerline. Fig. 4(b) shows the re-initialized Γ after applying (2), and Fig 4(c) shows the updated centerline of the re-initialized Γ.

4. RESULTS AND CONCLUSIONS

We present results of applying this model of implicit evolution of open ended curves to synthetic and real images. These images exhibit 1D filamentous structures for which traditional closed contour models are not practical. In each case an initial approximately linear segment is placed in the object to be tracked.



(a) (b)

(c) (d)

Fig. 5: (a)-(d) Results of topology-adaptive synthetic road tracking application using (2) after 1, 3, 7 and 14 iterations

The proposed approach is evaluated using a simulated road with a bifurcation. The result of evolution of the open curve after 1, 3, 7 and 14 iterations are shown in Fig. 5. Fig. 6 shows application of (2) to identify the thin black river in the satellite image taken from PAN/5.5m spatial resolution images of IRS 1C satellite. Application of (2) to identify an artery in a MRI of the human brain is shown in Fig. 7.

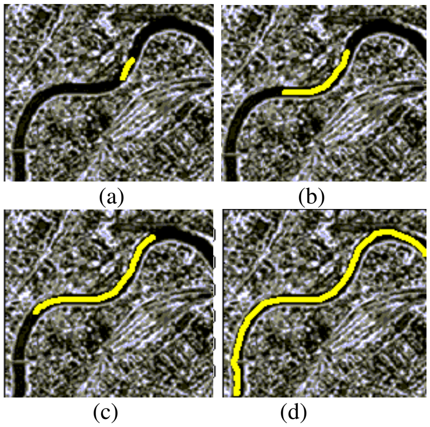


Fig. 6: (a)-(d) Result of applying (7) to a satellite image to track a river after 1, 8, 16, 32 iterations.

It should be noted from the examples that the proposed curve evolution model survives shape distortions or discontinuities of the filament edges. The evolution technique can also adapt to local topologies, as in the case of dividing and following two different branches of the simulated road in Fig. 5. It can also follow extreme bending/twisting of the 1D structures as is evident from the MR image example in Fig. 7.

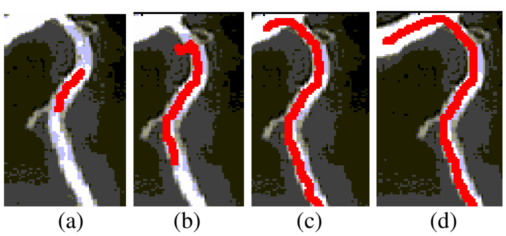


Fig. 7: (a)-(d) Result of applying (2) to an MRI image of an artery. The artery tracking is shown after 1, 6, 22 and 36 iterations.

Table I: Extent of elongation and bending of the open curve in tracking synthetic/real images.

Image set	% Elongation	Maximum Bending (degrees)	% RMSE pixel error
Fork road	525	30	3
River 1	1360	105	8
Artery 1	840	360	30
River 2	400	45	10
River 3	540	20	10
River 4	1360	105	11
River 5	675	175	6
River 6	600	77	4
Artery 2	350	89	2

Table I shows the numerical accuracy of the approach. We have compared the centerline (after manual segmentation and removing forks and H-junctions) for nine applications (both real and simulated images of roads/rivers/arteries). On average, the proposed segmentation agrees with the manual segmentation within a mean RMSE pixel error of 9%.

We have introduced a solution for evolving an open ended curve in a 2D image plane. This approach yields an implicit representation of open curves in a 2D plane and is particularly suitable for tracking or segmenting 1D filamentous structures in the 2D images.

5. REFERENCES

[1] L. Ambrosio and H. M. Soner, "Level Set Approach to Mean Curvature Flow in Arbitrary Codimension," *J. Diff. Geometry*, 43:693-737, 1996.

[2] C. Arcelli and G. Sanniti di Baja, "A width-independent fast thinning algorithm," *IEEE Trans. on PAMI*, 7:463-474, 1985.

[3] Brian Bradie, *A Friendly Introduction to Numerical Analysis*, Pearson International Education, 2005.

[4] P. Burchard, L. T. Cheng, B. Merriman, and S. Osher, "Motion of curves in three spatial dimensions using a level set approach," *J. Comput. Phys.*, 170:720-741, 2001.

[5] L.M. Lorigo, O. Faugeras, W.E.L. Grimson, R. Keriven, R. Kikinis, A. Nabavi, and C.F. Westin, "Codimension-Two Geodesic Active Contours for the Segmentation of Tubular Structure," in *Proceedings of the International Conference on Computer Vision and Pattern Recognition*, Hilton Head Island, South Carolina, pp. 444-451, 2000.

[6] D.P. Mukherjee, N. Ray, and S.T. Acton, "Level set analysis for cell detection and tracking," *IEEE Trans. Image Processing*, 13:562-572, 2004.

[7] P. Perona and J. Malik, "Scale-Space and Edge Detection Using Anisotropic Diffusion," *IEEE Trans. on PAMI*, 12:629-639, 1990.

[8] S. Ruuth, B. Merriman, J. Xin, and S. Osher, "Diffusion generated motion by mean curvature for filaments," *Technical Report 98-47, UCLA Computational and Applied Mathematics*, 1998.

[9] S. Ruuth and B. T. R. Wetton, "A Simple Scheme for Volume Preserving Flow by Mean Curvature," *Journal of Scientific Computing*, 19:373-384, 2003.

[10] G. Sapiro, *Geometric Partial Differential Equation and Image Analysis*, Cambridge University Press, Cambridge, UK, 2001.

[11] J.A. Sethian, *Level Set Methods and Fast Marching Methods*, Cambridge University Press, Cambridge, UK, 1999.

[12] Chenyang Xu and Jerry L. Prince, "Snakes, Shapes, and Gradient Vector Flow," *IEEE Trans. on Image Processing*, 7:359-369, 1998.