HILBERT TRANSFORM PAIRS OF ORTHONORMAL SYMMETRIC WAVELET BASES USING ALLPASS FILTERS

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ABSTRACT

This paper proposes a new class of Hilbert transform pairs of orthonormal symmetric wavelet bases. The associated orthonormal filter banks with exactly linear phase responses are realized by using complex and real allpass filters, respectively. The filter characteristics with the maximally flat responses are investigated to show the effectiveness of the proposed pairs.

Index Terms— Allpass filter, orthonormality, symmetry, wavelet transform, Hilbert transform

1. INTRODUCTION

Hilbert transform pairs of wavelet bases have been proposed and found to be successful in many signal and image processing applications [1]∼[6]. It has been proven in [4], [7] and [8] that the half-sample delay condition between two scaling lowpass filters is the necessary and sufficient condition for the corresponding wavelet bases to form a Hilbert transform pair. The design procedures of Hilbert transform pairs of wavelet bases have been also described in [1]∼[6], where FIR filters (corresponding to the compactly supported wavelets) were mainly handled. It is well-known that there does not exist any nontrivial compactly supported orthonormal symmetric wavelets, except for the Haar wavelet. On the other hand, a class of orthonormal symmetric wavelets can be realized by using IIR allpass filters [9], [10]. A few work has addressed IIR solutions for Hilbert transform pairs of wavelet bases [5]. The IIR orthogonal solution proposed in [5] does not have linear phase responses.

In this paper, we propose a new class of Hilbert transform pairs of orthonormal symmetric wavelet bases. Two orthonormal filter banks associated with the wavelet bases are realized by using complex and real allpass filters, respectively, and thus have exactly linear phase responses. The maximally flat solutions for the allpass filters are given to obtain the maximum numbers of vanishing moments. Finally, the filter characteristics are investigated to demonstrate its effectiveness.

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2. HILBERT TRANSFORM PAIRS OF WAVELETS

It is well-known that orthonormal wavelet bases can be generated by two-band orthogonal filter banks $\{H_i(z),\ G_i(z)\}$, where $i = 1, 2$. We assume that $H_i(z)$ are lowpass filter, and $G_i(z)$ are highpass. The orthonormality condition of $H_i(z)$ and $G_i(z)$ is given by

$$\begin{cases} H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \\ G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2\ . \\ H_i(z)G_i(z^{-1}) + H_i(-z)G_i(-z^{-1}) = 0 \end{cases} \tag{1}$$

Let $\phi_i(t), \psi_i(t)$ be the scaling and wavelet functions, respectively. The dilation and wavelet equations give the scaling and wavelet functions;

$$\begin{cases} \phi_i(t) = \sqrt{2}\sum_n h_i(n)\phi_i(2t-n) \\ \psi_i(t) = \sqrt{2}\sum_n g_i(n)\phi_i(2t-n) \end{cases}, \tag{2}$$

where $h_i(n)$ and $g_i(n)$ are the impulse responses of $H_i(z)$ and $G_i(z)$, respectively. For $\phi_i(t)$ and $\psi_i(t)$ to be (anti-) symmetrical, $H_i(z)$ and $G_i(z)$ must have exactly linear phase responses.

It has been proven in [4], [7] and [8] that two wavelet functions $\psi_1(t)$ and $\psi_2(t)$ form a Hilbert transform pair;

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\}, \tag{3}$$

that is

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0) \end{cases}, \tag{4}$$

if and only if two scaling lowpass filters satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}}, \tag{5}$$

where $\Psi_i(\omega)$ are the Fourier transform of $\psi_i(t)$. This is the half-sample delay condition between two scaling lowpass filters. Equivalently, the scaling lowpass filters should be offset from one another by a half sample. Eq.(5) is the necessary and sufficient condition for two wavelet bases to form a Hilbert transform pair.

3. ORTHONORMAL SYMMETRIC SOLUTION

In this section, we propose a new class of Hilbert transform pairs of orthonormal symmetric wavelet bases. Firstly, we use a complex allpass filter $A_c(z)$ of order $2N_1$ to construct $H_1(z)$ and $G_1(z)$ as shown in [10];

$$\begin{cases} H_1(z) = \dfrac{1}{\sqrt{2}}[A_c(z) + \tilde{A}_c(z)] \\ G_1(z) = \dfrac{z^{-1}}{j\sqrt{2}}[A_c(z) - \tilde{A}_c(z)] \end{cases}, \tag{6}$$

where $A_c(z)$ is defined by

$$A_c(z) = e^{j\eta} \frac{\sum_{n=0}^{N_1} a_{2n}^c z^{-2n} + j \sum_{n=0}^{N_1-1} a_{2n+1}^c z^{-2n-1}}{\sum_{n=0}^{N_1} a_{2n}^c z^{-2n} - j \sum_{n=0}^{N_1-1} a_{2n+1}^c z^{-2n-1}}, \tag{7}$$

where $a_n^c = a_{2N_1-n}^c$ are real, and $\eta = \pm\pi/4$ for even N_1, $\eta = \pm 3\pi/4$ for odd N_1. $\tilde{A}_c(z)$ has a set of coefficients that are complex conjugate with ones of $A_c(z)$.

Let $\theta_c(\omega)$ be the phase response of $A_c(z)$. From Eq.(7),

$$\theta_c(\omega) = \eta + 2\varphi(\omega), \tag{8}$$

where when N_1 is even,

$$\varphi(\omega) = \tan^{-1} \frac{2 \sum_{n=0}^{N_1/2-1} a_{2n+1}^c \cos(N_1 - 2n - 1)\omega}{a_{N_1}^c + 2 \sum_{n=0}^{N_1/2-1} a_{2n}^c \cos(N_1 - 2n)\omega}, \tag{9}$$

and when N_1 is odd,

$$\varphi(\omega) = \tan^{-1} \frac{a_{N_1}^c + 2 \sum_{n=0}^{(N_1-3)/2} a_{2n+1}^c \cos(N_1 - 2n - 1)\omega}{2 \sum_{n=0}^{(N_1-1)/2} a_{2n}^c \cos(N_1 - 2n)\omega}. \tag{10}$$

Therefore we have

$$\begin{cases} H_1(e^{j\omega}) = \sqrt{2}\cos\theta_c(\omega) \\ G_1(e^{j\omega}) = \sqrt{2}\sin\theta_c(\omega) e^{-j\omega} \end{cases}. \tag{11}$$

It is clear that $H_1(z)$ and $G_1(z)$ have exactly linear phase responses and satisfy the orthonormality condition in Eq.(1).

The vanishing moments is also one of the desired properties for wavelets. To obtain the maximum numbers of vanishing moments, $H_1(z)$ and $G_1(z)$ should have the maximally flat magnitude responses. In [10], the closed-form formula for the maximally flat solution has been given by

$$a_n^c = \begin{cases} \dbinom{2N_1}{n} & (n : \text{even}) \\ -\dbinom{2N_1}{n} \tan\dfrac{\eta}{2} & (n : \text{odd}) \end{cases}, \tag{12}$$

for $n = 0, 1, \cdots, 2N_1$. It is clear that once N_1 is given, the filter coefficients a_n^c can be calculated. The maximally flat filter $H_1(z)$ has $2N_1$ zeros at $z = -1$ (i.e., $\omega = \pi$), thus we obtain the wavelet bases with $2N_1$ vanishing moments.

Next, we use a real allpass filter $A_r(z)$ of order N_2 to construct $H_2(z)$ and $G_2(z)$ as shown in [9];

$$\begin{cases} H_2(z) = \dfrac{1}{\sqrt{2}}[z^K A_r(z^2) + z^{-K-1} A_r(z^{-2})] \\ G_2(z) = \dfrac{1}{\sqrt{2}}[z^K A_r(z^2) - z^{-K-1} A_r(z^{-2})] \end{cases}, \tag{13}$$

where K is integer and $A_r(z)$ is defined by

$$A_r(z) = z^{-N_2} \frac{\sum_{n=0}^{N_2} a_n^r z^n}{\sum_{n=0}^{N_2} a_n^r z^{-n}}, \tag{14}$$

where a_n^r are real, and $a_0^r = 1$. Let $\theta_r(\omega)$ be the phase response of $A_r(z)$,

$$\theta_r(\omega) = -N_2\omega + 2\tan^{-1} \frac{\sum_{n=0}^{N_2} a_n^r \sin n\omega}{\sum_{n=0}^{N_2} a_n^r \cos n\omega}. \tag{15}$$

Therefore the frequency responses of $H_2(z)$ and $G_2(z)$ are given by

$$\begin{cases} H_2(e^{j\omega}) = \sqrt{2}\cos\{\theta_r(2\omega) + (K + \frac{1}{2})\omega\} e^{-j\frac{\omega}{2}} \\ G_2(e^{j\omega}) = \sqrt{2}\sin\{\theta_r(2\omega) + (K + \frac{1}{2})\omega\} e^{j(\frac{\pi}{2} - \frac{\omega}{2})} \end{cases}. \tag{16}$$

It can be seen that $H_2(z)$ and $G_2(z)$ have exactly linear phase responses and satisfy the orthonormality condition in Eq.(1).

Similarly, the closed-form formula for the maximally flat solution has been given in [9] by

$$a_n^r = \binom{N_2}{n} \prod_{i=1}^{n} \frac{N_2 - i - \frac{K}{2} + \frac{3}{4}}{i + \frac{K}{2} + \frac{1}{4}}, \tag{17}$$

for $n = 0, 1, \cdots, N_2$. It can be seen that the filter coefficients a_n^r are dependent on not only N_2 but also K. An inappropriate K will cause a bad magnitude response with an undesired zero and bump nearby $\omega = \pi/2$. Therefore, it has been pointed out in [9] that $K = 2(N_2 - 2k)$ or $K = 2(N_2 - 2k) - 1$

must be chosen to obtain a pair of reasonable lowpass and highpass filters, where $k = 0, 1, \cdots, N_2$. The maximally flat filter $H_2(z)$ has $2N_2 + 1$ zeros at $z = -1$ (i.e., $\omega = \pi$), thus the corresponding wavelet bases possess $2N_2 + 1$ vanishing moments.

To form a Hilbert transform pair of wavelet bases, two scaling lowpass filters $H_1(z)$ and $H_2(z)$ must satisfy the half-sample delay condition in Eq.(5). That is, their magnitude responses are the same, and the phase responses are different with a half sample. It can be seen in Eqs.(11) and (16) that the phase condition have been already satisfied, thus, we only need to consider the magnitude condition. It is known that $H_1(z)$ has $2N_1$ zeros at $z = -1$, while $H_2(z)$ has $2N_2 + 1$ zeros at $z = -1$. To ensure that $H_1(z)$ and $H_2(z)$ have a close number of zeros at $z = -1$ as possible, we should choose $N_1 = N_2$ or $N_1 = N_2 + 1$. Therefore, the resulting pairs of orthonormal symmetric wavelet bases have an almost same number of vanishing moments. For $H_1(z)$ and $H_2(z)$ to approximately satisfy the magnitude condition, we must investigate the magnitude responses and then appropriately choose K. See design example in detail.

4. DESIGN EXAMPLE

In this section, we will show a design example with the maximally flat frequency response and examine the filter characteristics. Firstly, we have designed $H_1(z)$ with $N_1 = 2$, and show its magnitude response in Fig.1 in the solid line. Note that the passband gain has been normalized to 1. Next, we set $N_2 = N_1 - 1 = 1$ and $N_2 = N_1 = 2$ to design $H_2(z)$. When $N_2 = 1$, $K = 1, 2$ and $K = -2, -3$ can be chosen, while when $N_2 = 2$, $K = 0, 3, 4$ and $K = -1, -4, -5$. It should be noted that two filters $H_2(z)$ with $K = K_1$ and $K = -K_1 - 1$ have the same magnitude response, where K_1 is non-negative integer. Therefore, we will investigate the magnitude responses of $H_2(z)$ with non-negative K only. The magnitude responses of $H_2(z)$ with $N_2 = 1, K = 1, 2$ and $N_2 = 2, K = 0, 3, 4$ are shown in Fig.1 also. It is seen in Fig.1 that $H_2(z)$ have the closer magnitude responses with $H_1(z)$ when $N_2 = 2$, compared with $N_2 = 1$. The difference of the magnitude responses between $H_1(z)$ and $H_2(z)$ with $N_2 = 2$ are shown in Fig.2. It is clear that when $K = 3$, two filters have the closest magnitude responses. Therefore, we choose $H_2(z)$ with $N_2 = 2, K = 3$ for $H_1(z)$ with $N_1 = 2$. For $H_1(z)$ with $N_1 = 1 \sim 4$, the best $H_2(z)$ are listed in Table 1. The scaling and wavelet functions obtained from $H_1(z)$ with $N_1 = 2$ and $H_2(z)$ with $N_2 = 2, K = 3$ are shown in Fig.3. $\phi_1(t)$ and $\phi_2(t)$ are symmetrical and satisfy the half-sample delay condition, while $\psi_1(t)$ and $\psi_2(t)$ are symmetrical and antisymmetrical, respectively. The spectrum of the obtained wavelet functions are shown in Fig.4, which are enlarged twice for comparison. The spectrum $\Psi_1(\omega) + j\Psi_2(\omega)$ of $\psi_1(t) + j\psi_2(t)$ shows that it approximates zero for $\omega < 0$, as expected if $\psi_1(t)$ and $\psi_2(t)$ form a Hilbert transform pair.

Table 1. Hilbert transform pair $\{H_1(z), H_2(z)\}$

$H_1(z)$	$H_2(z)$
$N_1 = 1$	$N_2 = 1,\ K = 2$
$N_1 = 2$	$N_2 = 2,\ K = 3$
$N_1 = 3$	$N_2 = 3,\ K = 2$
$N_1 = 4$	$N_2 = 4,\ K = 4$

5. CONCLUSION

In this paper, we have proposed a new pair of orthonormal symmetric wavelet bases that form the Hilbert transform. Two orthonormal filter banks associated with the orthonormal symmetric wavelet bases have been constructed by using complex and real allpass filters, and thus have exactly linear phase responses. The maximally flat solutions for the allpass filters have been given to obtain the maximum numbers of vanishing moments. Finally, the filter characteristics have been investigated to demonstrate the effectiveness of the proposed pairs.

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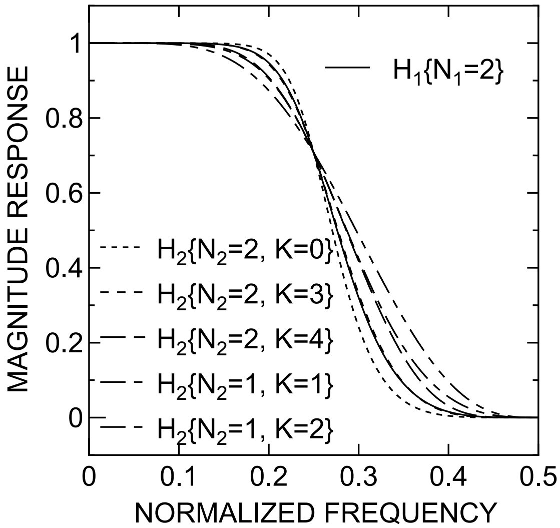

Fig. 1. Magnitude responses of the scaling lowpass filters.

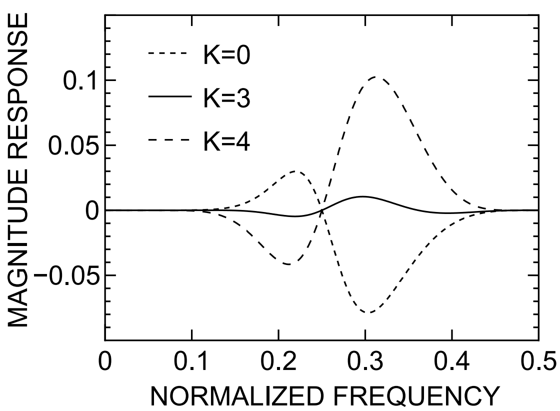

Fig. 2. Difference of magnitude responses between $H_1(z)$ and $H_2(z)$ with $N_1 = N_2 = 2$.

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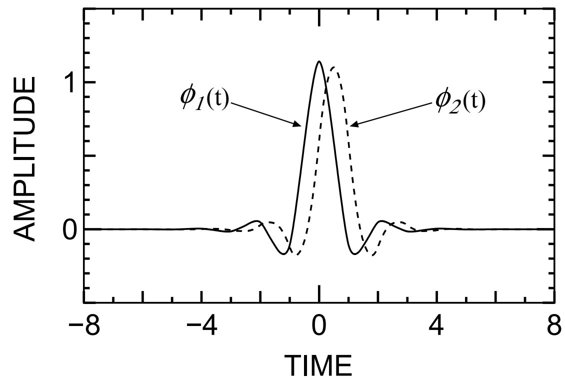
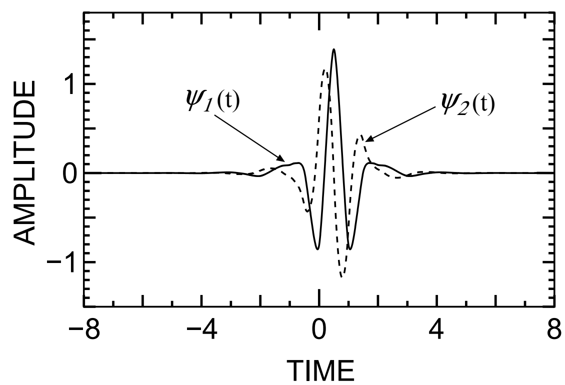


(a) Scaling functions



(b) Wavelet functions

Fig. 3. Scaling and wavelet functions with $N_1 = N_2 = 2$ and $K = 3$.

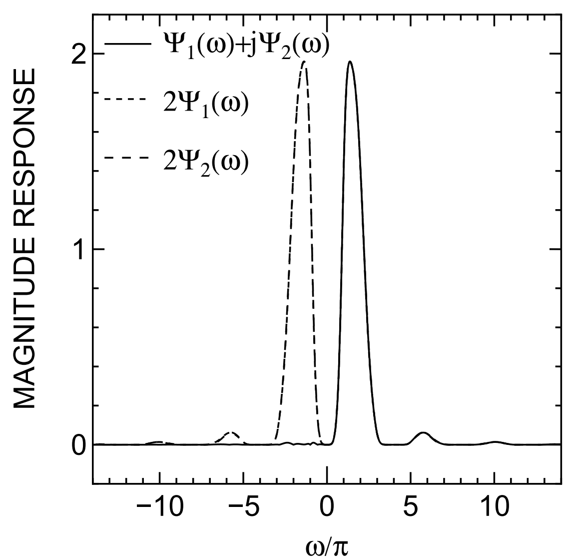


Fig. 4. Frequency spectrum of wavelet functions with $N_1 = N_2 = 2$ and $K = 3$.