FAST SUPER-RESOLUTION FOR RATIONAL MAGNIFICATION FACTORS

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ABSTRACT

Super-resolution (SR) is a technique that generates a high-resolution (HR) image from a set of low-resolution (LR) ones. Previous preconditioning methods for SR did not consider the case of rational magnification factors. In this paper, a method for preconditioning SR problems involving such factors is presented. We show that by reordering the pixels of the observed LR images, the structure of the linear problem to solve is modified in such a way that preconditioners based on circulant operators can be used. Simulations with magnification factors of practical interest demonstrate the effectiveness of our approach.

Index Terms— Super-resolution, preconditioning, image restoration, linear systems

1. INTRODUCTION

HR images are desired and often required in several applications. Since increasing resolution by employing a better camera can be costly and sometimes infeasible, SR [1][2] can constitute a good alternative. This technique synthesizes a HR image from a set of degraded and aliased LR ones by exploiting knowledge of the relative subpixel displacements of each LR image with respect to a reference frame.

Many SR algorithms boil down to solving a large structured and sparse system of linear equations. Iterative solution methods such as conjugate gradients (CG) [3] are often employed and can benefit from performance improvements due to preconditioning, which transforms a system into another having the same solution, but that can be solved either more accurately or faster [4]. Recently, Nguyen *et al* [5] showed how to accelerate least-square SR algorithms by reordering the pixels of the HR image in the formulation of the problem. Assuming that the magnification factor is an integer, the new coefficient matrix associated with the proposed reordering can be well approximated by a block matrix whose blocks are circulant matrices. Since a circulant matrix is unitarily similar to a diagonal matrix through the Fast Fourier Transform (FFT) [4], efficient preconditioners can be developed using such approximations. Later, Bose *et al* [6] presented a related technique where the approximating matrix is a block matrix whose blocks are block-circulant matrices with circulant blocks (BCCB) instead of simple circulant matrices. In their work, the authors implicitly assume that the magnification factor is an integer.

In this paper, we show how to apply preconditioners based on circulant matrices to SR problems involving a rational magnification factor in order to accelerate the solution process. More precisely, we describe a technique for reorganizing the coefficient matrix given the parameters of the SR problem to solve. This is motivated by recent work of Lin and Shum [7] suggesting that, under certain circumstances, optimal magnification factors for SR are non-integers.

2. MATHEMATICAL MODEL

The problem consists of recovering an N_x-by-N_y HR image $\mathcal{X}$ from a set of k registered M_x-by-M_y LR images $\mathcal{Y}_i$, where $i = 0 \dots k-1$. Suppose that q represents the desired magnification factor in both the horizontal and vertical directions; we then have $N_x = qM_x$ and $N_y = qM_y$. Note that q does not need to be an integer; one could have $N_x = 150$, $M_x = 100$ and $q = 1.5$ for instance. It is assumed that the LR images were generated by shifting and blurring the HR image, then decimating the result. The blur is assumed to be linear shift-invariant (LSI). Let $\mathbf{x}$ be a vector of length N_xN_y whose elements are the pixel values of $\mathcal{X}$ in a given lexicographic order. Similarly, $\mathbf{y}_i$ is a vector of length M_xM_y representing the LR image $\mathcal{Y}_i$. The imaging process yielding an observed image $\mathbf{y}_i$ can thus be expressed in matrix form as

$$\mathbf{y}_i = \mathbf{D}_i\mathbf{B}_i\mathbf{S}_i\mathbf{x} + \mathbf{n}_i, \qquad (1)$$

where $\mathbf{D}_i$ is an M_xM_y-by-N_xN_y decimation matrix, $\mathbf{B}_i$ is an N_xN_y-by-N_xN_y blurring matrix, $\mathbf{S}_i$ is an N_xN_y-by-N_xN_y shift matrix and $\mathbf{n}_i$ is an additive noise vector of length M_xM_y. By stacking these equations on top of one another and using $\mathbf{H}_i = \mathbf{D}_i\mathbf{B}_i\mathbf{S}_i$ to simplify notation, one gets:

$$\begin{bmatrix} \mathbf{y}_0 \\ \vdots \\ \mathbf{y}_{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \\ \vdots \\ \mathbf{H}_{k-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{n}_0 \\ \vdots \\ \mathbf{n}_{k-1} \end{bmatrix}$$

This work was generously supported by a Precarn scholarship.

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}. \quad (2)$$

The system (2) is generally ill-conditioned and cannot be solved directly. By using linear least-squares regression with regularization, we minimize the following cost function instead:

$$\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2^2 + \lambda \|\mathbf{R}\mathbf{x}\|_2^2 \right\}, \quad (3)$$

where $\mathbf{R}$ is a regularization operator such as the first-order finite-difference operator. The minimization of expression (3) above amounts to solving this system:

$$\left(\mathbf{H}^T\mathbf{H} + \lambda \mathbf{R}^T\mathbf{R}\right)\mathbf{x} = \mathbf{H}^T\mathbf{y}. \quad (4)$$

3. PRECONDITIONING APPROACH

We follow the preconditioning approach of Nguyen *et al* by first finding approximations $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{R}}$ to $\mathbf{H}$ and $\mathbf{R}$ respectively. An approximation of the coefficient matrix of equation (4) can then be formed as follows:

$$\tilde{\mathbf{H}}^T\tilde{\mathbf{H}} + \lambda \tilde{\mathbf{R}}^T\tilde{\mathbf{R}}. \quad (5)$$

To simplify the explanation that will follow, we give a few definitions. Let $\mathcal{F}_{q,[i,\ j]}(\mathcal{X})$ be a discrete linear operator that downsamples an image $\mathcal{X}$ by a factor of q along both axes, starting with pixel (i,j). For example, applying $\mathcal{F}_{2,[1,\ 0]}$ to a 4-by-4 image $\mathcal{X}$ would give

$$\mathcal{X}' = \begin{bmatrix} \mathcal{X}_{(1,0)} & \mathcal{X}_{(3,0)} \\ \mathcal{X}_{(1,2)} & \mathcal{X}_{(3,2)} \end{bmatrix},$$

where $\mathcal{X}_{(i,j)}$ refers to pixel $(i,\ j)$ of $\mathcal{X}$. The equivalent matrix operator is denoted by $\mathbf{F}_{q,[i,\ j]}^{(L_x,L_y)}$, which is compatible with a vector representing an L_x-by-L_y image. Finally, let $\mathbf{F}_q^{(L_x,L_y)}$ be the square matrix obtained by stacking on top of one another the downsampling matrices $\mathbf{F}_{q,[i,\ j]}^{(L_x,L_y)}$ associated with all possible combinations of horizontal and vertical integer shifts smaller than q. Note that $\mathbf{F}_q^{(L_x,L_y)}$ is a permutation matrix, since $\mathbf{F}_q^{(L_x,L_y)}\mathbf{x}$ has the effect of reordering the elements of $\mathbf{x}$ without modifying them.

3.1. Integer magnification factor q

Nguyen *et al*'s technique reorders the columns of $\mathbf{H}$ and the pixels of $\mathbf{x}$ in a way that simulates several downsampling operations. Using the previous definitions, their reordering scheme can be defined by the permutation matrix $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{F}_q^{(N_x,\ N_y)} \quad (6)$$

and the desired reordering is $\mathbf{H}\mathbf{P}^T$. The reordered matrix is a k-by-q^2 block matrix with blocks of size M_xM_y-by-M_xM_y. Consequently, the approximation (5) of the normal equations will be a q^2-by-q^2 block matrix with blocks of size M_xM_y-by-M_xM_y.

To illustrate their method, we consider the simple case where $N_x = N_y = 4$, $M_x = M_y = 2$ and $q = 2$. Assuming for simplicity that both $\mathbf{B}_0$ and $\mathbf{S}_0$ are the identity matrix, the transformation matrix $\mathbf{H}_0$ associated with $\mathcal{Y}_0$ is:

$$\mathbf{H}_0 = \frac{1}{4}\begin{bmatrix} 1100 & 1100 & 0000 & 0000 \\ 0011 & 0011 & 0000 & 0000 \\ 0000 & 0000 & 1100 & 1100 \\ 0000 & 0000 & 0011 & 0011 \end{bmatrix}. \quad (7)$$

The permutation matrix is

$$\mathbf{P} = \mathbf{F}_2^{(4,4)} = \begin{bmatrix} \mathbf{F}_{2,[0,\ 0]}^{(4,4)} \\ \mathbf{F}_{2,[1,\ 0]}^{(4,4)} \\ \mathbf{F}_{2,[0,\ 1]}^{(4,4)} \\ \mathbf{F}_{2,[1,\ 1]}^{(4,4)} \end{bmatrix}, \quad (8)$$

and the reordered matrix is

$$\mathbf{H}_0\mathbf{P}^T = \frac{1}{4}\begin{bmatrix} 1000 & 1000 & 1000 & 1000 \\ 0100 & 0100 & 0100 & 0100 \\ 0010 & 0010 & 0010 & 0010 \\ 0001 & 0001 & 0001 & 0001 \end{bmatrix}. \quad (9)$$

3.2. Rational magnification factor $q = \frac{a}{b}$

When q is not an integer, one cannot create a permutation matrix $\mathbf{P}$ using the previous method such that $\mathbf{H}\mathbf{P}^T$ has a structure suitable for preconditioning. However, if q is a rational number such that $q = \frac{a}{b}$, where a and b are integers, and both M_x and M_y are chosen to be multiples of b, then one can find two permutation matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{Q}\mathbf{H}\mathbf{P}^T$ has the desired structure; these permutation matrices are defined as

$$\mathbf{Q} = \mathbf{F}_b^{(M_x,\ M_y)} \text{ and } \mathbf{P} = \mathbf{F}_a^{(N_x,\ N_y)}. \quad (10)$$

The effect of matrix $\mathbf{Q}$ can be seen as reordering the rows of $\mathbf{H}$ and the pixels of the LR images correspondingly. The reordered matrix is a kb^2-by-a^2 block matrix with blocks of size $\frac{M_xM_y}{b^2}$-by-$\frac{M_xM_y}{b^2}$. Consequently, the approximation (5) of the normal equations will be an a^2-by-a^2 block matrix with blocks of size $\frac{M_xM_y}{b^2}$-by-$\frac{M_xM_y}{b^2}$.

To illustrate our reordering method, we consider the case where $N_x = 6$, $N_y = 3$, $M_x = 4$, $M_y = 2$ and $q = \frac{3}{2} = 1.5$. Using the same simplifying assumptions as before about $\mathbf{B}_0$ and $\mathbf{S}_0$, we have that

$$\mathbf{H}_0 = \frac{1}{9}\begin{bmatrix} 420000 & 210000 & 000000 \\ 024000 & 012000 & 000000 \\ 000420 & 000210 & 000000 \\ 000024 & 000012 & 000000 \\ 000000 & 210000 & 420000 \\ 000000 & 012000 & 024000 \\ 000000 & 000210 & 000420 \\ 000000 & 000012 & 000024 \end{bmatrix}. \quad (11)$$

The permutation matrices are

$$\mathbf{Q} = \begin{bmatrix} \mathbf{F}^{(4,\,2)}_{2,[0,\,0]} \\ \mathbf{F}^{(4,\,2)}_{2,[1,\,0]} \\ \mathbf{F}^{(4,\,2)}_{2,[0,\,1]} \\ \mathbf{F}^{(4,\,2)}_{2,[1,\,1]} \end{bmatrix} \text{ and } \mathbf{P} = \begin{bmatrix} \mathbf{F}^{(6,\,3)}_{3,[0,\,0]} \\ \mathbf{F}^{(6,\,3)}_{3,[1,\,0]} \\ \mathbf{F}^{(6,\,3)}_{3,[2,\,0]} \\ \mathbf{F}^{(6,\,3)}_{3,[0,\,1]} \\ \mathbf{F}^{(6,\,3)}_{3,[1,\,1]} \\ \mathbf{F}^{(6,\,3)}_{3,[2,\,1]} \\ \mathbf{F}^{(6,\,3)}_{3,[0,\,2]} \\ \mathbf{F}^{(6,\,3)}_{3,[1,\,2]} \\ \mathbf{F}^{(6,\,3)}_{3,[2,\,2]} \end{bmatrix}, \qquad (12)$$

and the reordered matrix is

$$\mathbf{Q}\mathbf{H}_0\mathbf{P}^T = \frac{1}{9} \begin{bmatrix} 40 & 20 & 00 & 20 & 10 & 00 & 00 & 00 & 00 \\ 04 & 02 & 00 & 02 & 01 & 00 & 00 & 00 & 00 \\ 00 & 20 & 40 & 00 & 10 & 20 & 00 & 00 & 00 \\ 00 & 02 & 04 & 00 & 01 & 02 & 00 & 00 & 00 \\ 00 & 00 & 00 & 20 & 10 & 00 & 40 & 20 & 00 \\ 00 & 00 & 00 & 02 & 01 & 00 & 04 & 02 & 00 \\ 00 & 00 & 00 & 00 & 10 & 20 & 00 & 20 & 40 \\ 00 & 00 & 00 & 00 & 01 & 02 & 00 & 02 & 04 \end{bmatrix}. \qquad (13)$$

We obtain a block matrix where similar values in each block are aligned diagonally, which allows the utilization of preconditioners based on block matrices with circulant blocks. These circulant blocks are derived from preconditioners exploiting properties of circulant matrices such as Strang's preconditioner [8] or Hanke-Nagy's inverse preconditioner [9]; see Nguyen *et al*'s work [5] for more details.

3.3. Computational complexity

The two tasks that are the more computationally intensive are the block-diagonalization of the approximation of the coefficient matrix and its inversion. Using the FFT, the block approximation of the normal equations is transformed into an a^2-by-a^2 block matrix with diagonal block matrices of size $\frac{M_x M_y}{b^2}$-by-$\frac{M_x M_y}{b^2}$. This step takes $\mathcal{O}\left(a^4 \frac{M_x M_y}{b^2} log(\frac{M_x M_y}{b^2})\right)$. Due to its special structure, the inversion of this matrix can be performed by solving $\frac{M_x M_y}{b^2}$ independent systems of linear equations, each of size a^2. The complexity of this task is thus $\mathcal{O}\left(a^6 \frac{M_x M_y}{b^2}\right)$. It is interesting to note that when a becomes large, the problem becomes more computationally intensive to solve, even if the actual magnification factor is low. For this reason, one must be careful when choosing the rational magnification factor.

4. RESULTS

If the noise removal and registration steps are not sufficiently reliable, a magnification factor of 1.6 is the practical limit of SR [7], and a magnification factor of 2.5 is suggested when a larger value is desired. We present simulation results for these two cases of practical interest in Figures (1) and (2). For this purpose, we generated a set of LR images by shifting, blurring and downsampling an ideal 240-by-240 HR image according to the imaging model described in Section 2. In the first experiment, we downsampled the HR image by a factor of $q = \frac{5}{2} = 2.5$ to produce a set of nine LR images. In the second experiment, we produced a set of four noisy LR images by downsampling the ideal HR image by a factor of $q = \frac{8}{5} = 1.6$ and adding Gaussian noise to the result. The CG and preconditioned conjugate gradient (PCG) methods were then used to reconstruct the HR image from the LR ones. Strang's preconditioner was used in the PCG case. The

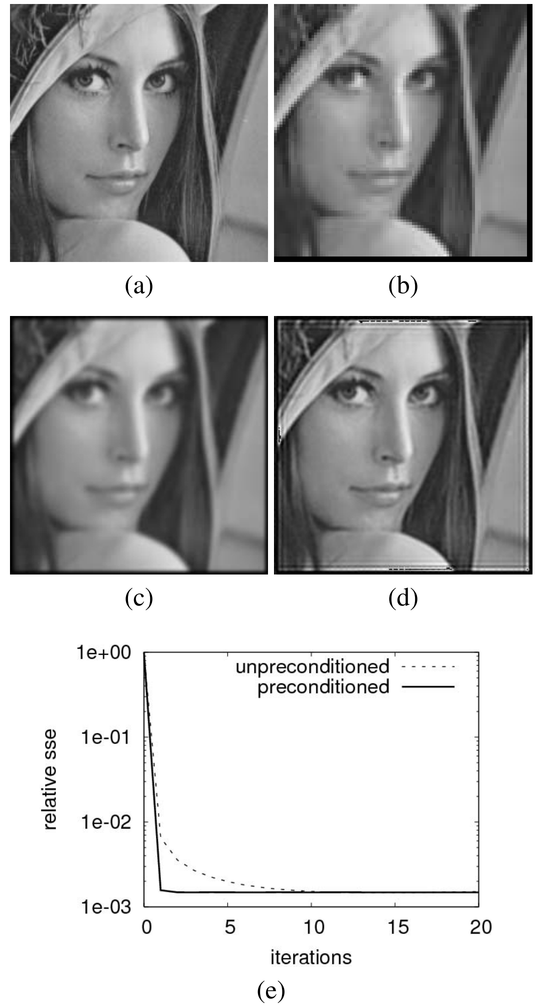
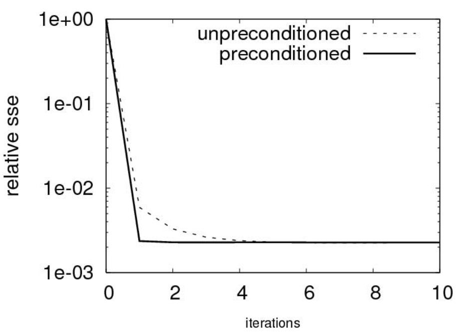


Fig. 1. SR with a magnification factor of 2.5: (a) the original image; (b) one of the 9 LR images (scaled); (c) the restored image after one CG iteration; (d) the restored image after one PCG iteration; (e) Relative SSE as a function of the number of iterations.

(a) (b)

(c) (d)



(e)

Fig. 2. SR with a magnification factor of 1.6: (a) the original image; (b) one of the 4 LR images (scaled); (c) the restored image after one CG iteration; (d) the restored image after one PCG iteration; (e) Relative SSE as a function of the number of iterations.

Laplacian was employed as a regularization term in both experiments and the value of λ was set to 0.001 and 0.005 in the first and second experiments respectively. The results of these experiments, shown in Figures 1 and 2, allow a comparison of the restored image after one iteration of either CG (c) or PCG (d), as well as the relative Sum-of-Squared Errors (SSE) between the ideal HR image (a) and the reconstructed HR image as a function of the number of iterations (e). One can see that the PCG method produces a better solution for the same amount of iterations. The actual computation time depends on the hardware used, the image size and the rational magnification factor, as discussed in section 3.3.

5. CONCLUSIONS

In this paper, we presented a technique for preconditioning SR problems involving a rational magnification factor. An interesting application of this work is the preconditioning of problems that employ the non-integer magnification factors advocated by Lin and Shum [7]. We also note that the proposed approach could easily be adapted to temporal super-resolution preconditioning [10] when a non-integer frame-rate improvement factor is desired.

6. REFERENCES

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