SEGMENTATION OF IMAGES ON POLAR COORDINATE MESHES

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ABSTRACT

The Chan–Vese level set algorithm has been successfully applied to segmentation of images on Cartesian coordinate meshes, including ordinary planar images. In this paper we present a Chan–Vese model for segmentation of images on polar coordinate meshes, such as topography and remote sensing images. The image segmentation is accomplished by formulating the associated evolution equation in the polar coordinate system and then numerically solving the partial differential equation on an overset grid system called the Yin–Yang grid, which is free from the problem of singularity at the poles. We include examples of segmentations of real earth data that demonstrate the performance of our method.

Index Terms— Chan–Vese segmentation model, level sets, polar coordinate mesh, Yin–Yang grid

1. INTRODUCTION

The Chan–Vese algorithm is a commonly used technique for the important task of image segmentation. The Chan–Vese segmentation algorithm was formulated and implemented within the level set framework by Chan and Vese [1].

In the image segmentation problem, the level set approach [2] was presented by Malladi et al. [3]. It uses the gradient of the image to let the evolving curve stop on the edges. Therefore, although it can handle well interfaces with sharp corners, cusps, and topological changes, the evolving curve may not stop on the desired boundary, especially in the case of noisy images or images with texture regions. To overcome this difficulty, the geodesic active contour model was developed by Caselles et al. [4]. However, troublesome preprocessing is required to remove the noise in the input image while preserving the boundary edges.

While all these segmentation techniques use the gradient to find the boundary, the Chan–Vese model automatically detects the interface between the interior and exterior regions based on the classical Mumford–Shah model [5], thus the evolving curve stops on the desired boundary even for noisy images.

Our primary goal is to provide the Chan–Vese segmentation model for images on polar coordinate meshes. Converting the Chan–Vese model from Cartesian to polar coordinate system enables the direct segmentation of the original image on the polar coordinate mesh, such as topography and remote sensing images, without the need of a previous preprocessing step. Since the evolution equation in the Chan–Vese level set segmentation formulation includes the mean curvature and error terms, the converted version also will necessitate the implementation of their counterparts.

Recently, the implementation of the geodesic active contour segmentation model for images on surfaces was introduced by Spira et al. [6] using a projection of the flows from the surface to its parametrization plane. The Chan–Vese segmentation model was extended for images on surfaces by using the global conformal parametrization [7]. Both these approaches assume that the input images are painted on the surfaces represented in the Cartesian coordinate system, which are important in applications to areas such as medical imaging and computer graphics. This is the main difference with our method, which performs a segmentation of the images on the polar coordinate meshes, focusing on applications such as analysis of earth data such as topography and remote sensing imagery.

While the finite difference methods are commonly used in our segmentation, there are numerical singularities associated with the poles. In recent years, several researchers in computational geoscience have developed and used overset grid systems for overcoming polar problems naturally. Kageyama and Sato [8] proposed an singularity free grid system called Yin-Yang grid for solving partial differential equations on the sphere. It is composed of two identical latitude/longitude orthogonal component grids that are combined to cover the sphere with partial overlap on their boundaries.

In this paper we present a Chan–Vese model for segmentation of images on polar coordinate meshes. The image segmentation is accomplished by formulating the associated level set equation in the polar coordinate system and then numerically solving the evolution equation using the Yin-Yang grid system.

The outline of the paper is as follows. We first give a brief review of the original Chan–Vese segmentation algorithm. In Section 3 we derive the evolution equation for segmentation of images on polar coordinate meshes. In Section 4 we present a numerical scheme for this differential equation. In Section 5 we present experimental results. Section 6 presents concluding remarks.

2. CHAN–VESE SEGMENTATION MODEL

In this section, we present the original Chan–Vese segmentation formulation (see [1, 9] for details).

Let Ω be a bounded open subset of $\mathbb{R}^2$, and let $u_0 : \Omega \to \mathbb{R}$ be a given image. Let c^+ and c^- be unknown constants, and ϕ be the level set function. Then the minimization problem is formulated as $\min_{\phi,c^+,c^-} F(\phi,\, c^+, c^-; u_0)$, where the energy functional F is defined as

$$F(\phi,\, c^+, c^-; u_0) = \mu \int_\Omega \delta(\phi)|\nabla\phi|dx$$
$$+\lambda^+\int_\Omega |u_0 - c^+|^2 H(\phi)dx + \lambda^-\int_\Omega |u_0 - c^-|^2(1 - H(\phi))dx \,, \tag{1}$$

where $\mu > 0$ and $\lambda^+, \lambda^- > 0$ are weights for the regularizing term and the fitting term, respectively, δ and H are the Dirac delta function and the Heaviside function.

If one regularizes the delta function and the Heaviside function by two suitable smooth functions δ_ϵ and H_ϵ, respectively, the Euler–Lagrange equation for ϕ can be written as

$$\phi_t = \delta_\epsilon\Big[\mu\nabla\cdot\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) - \lambda^+(u_0 - c^+)^2 + \lambda^-(u_0 - c^-)^2\Big] \tag{2}$$

in Ω, and with the boundary condition

$$\frac{\delta_\epsilon(\phi)}{|\nabla\phi|}\frac{\partial\phi}{\partial\vec{n}} = 0\,, \tag{3}$$

on $\partial\Omega$, where $\vec{n}$ is the unit normal at the boundary of Ω.

Minimizing the energy F with respect to c^+, c^-, we get

$$c^+(\phi) = \frac{\int_\Omega u_0(x)H_\epsilon(\phi(x))dx}{\int_\Omega H_\epsilon(\phi(x))dx}, \tag{4}$$

$$c^-(\phi) = \frac{\int_\Omega u_0(x)(1 - H_\epsilon(\phi(x)))dx}{\int_\Omega(1 - H_\epsilon(\phi(x)))dx}\,. \tag{5}$$

Using shape derivation principle, the above Euler–Lagrange equation can be written in the level set form [9] as

$$\phi_t = \Big[\mu\nabla\cdot\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) - \lambda^+(u_0 - c^+)^2 + \lambda^-(u_0 - c^-)^2\Big]|\nabla\phi|. \tag{6}$$

3. CHAN–VESE MODEL IN POLAR COORDINATE

In this section, we derive the level set evolution equation for curves on surfaces in polar coordinates and formulate the Chan–Vese model for image segmentation on polar coordinate meshes.

Let P be a point on a curve C constrained on a given surface M in $\mathbb{R}^3$. Then its position vector $\mathbf{r}$ is given by $\mathbf{r} = (r\sin\theta\cos\psi,\, r\sin\theta\sin\psi,\, r\cos\theta)$, where r is the distance from the origin O in $\mathbb{R}^3$, θ is the polar angle, and ψ is the azimuth angle. Let $\mathbf{r}$ and (r, θ, ψ) be the position vector and the polar coordinate variables parameterized by the time variable t, by assuming that the position of the point P is changed over time. Taking a derivative of $\mathbf{r}$ with respect to t gives

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\psi}\sin\theta\,\hat{\boldsymbol{\psi}}, \tag{7}$$

where $\dot{r}$, $\dot{\theta}$ and $\dot{\psi}$ denote the derivatives of $r(t)$, $\theta(t)$ and $\psi(t)$, respectively, with respect to t, and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\psi}}$ are the basis vectors of the local coordinate system with the origin O.

We project the point P on M onto a unit sphere S in $\mathbb{R}^3$ centered at the origin O. Let there be a line passing O to P. This line will intersect S at a point P'. In this way, every point P on M will have a corresponding point P' on S. Therefore, we project the equation for curves on M onto S and then derive this equation. Given the speed F of P on M,

$$F = \sqrt{r^2(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta) + \dot{r}^2}\,, \tag{8}$$

the speed F' of P' on S becomes

$$F' = \sqrt{\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta} = \frac{F\cos\alpha}{r}\,, \tag{9}$$

where α is the angle between the normal to the surface M at the point P and the normal to the surface S at the point P'. Assuming that the projected curve C' on S is represented as the zero level set of a three-variables function $\phi(\theta, \psi, t)$ (i.e., $\phi(\theta(t),\, \psi(t),\, t) = 0$), taking a derivative of this equation with respect to t gives

$$\phi_t + \dot{\theta}\,\phi_\theta + \dot{\psi}\,\phi_\psi = 0\,. \tag{10}$$

Equation (10) can be also written as

$$\phi_t + \Big(\phi_\theta,\, \frac{1}{\sin\theta}\phi_\psi\Big)\begin{pmatrix}\dot{\theta}\\ \dot{\psi}\sin\theta\end{pmatrix} = 0. \tag{11}$$

Then, from Equation (11), we get

$$\phi_t + \nabla_{\theta\psi}\phi\cdot\dot{\mathbf{x}}_\Sigma = 0, \tag{12}$$

where $\nabla_{\theta\psi} = (\frac{\partial}{\partial\theta},\, \frac{1}{\sin\theta}\frac{\partial}{\partial\psi})$ and $\dot{\mathbf{x}}_\Sigma = (\dot{\theta},\, \dot{\psi}\sin\theta)$ is the velocity of P'. Next, we define the unit normal vector of the space curve C' at the point P' as a unit outward vector which is orthogonal to the tangent vector to the curve C' and which lies on the plane spanned by the bases $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\psi}})$. To find the expression for the unit normal vector $\mathbf{n}_\Sigma$, we now introduce the level set function ϕ remaining unchanged over time while $\dot{\mathbf{x}}_\Sigma \neq \mathbf{0}$. Hence, $\phi_t = 0$, $\forall t$. Then, by Equation (12), we get $\nabla_{\theta\psi}\phi\cdot\dot{\mathbf{x}}_\Sigma = 0$. In particular, $\dot{\mathbf{x}}_\Sigma \neq \mathbf{0}$. Then, $\dot{\mathbf{x}}_\Sigma$ has the same direction as the tangent vector of C'. Hence, $\nabla_{\theta\psi}\phi$

is orthogonal to the tangent vector. Normalizing it the unit normal to C' is given by

$$\mathbf{n}_{\Sigma} = \frac{\nabla_{\theta\psi}\phi}{|\nabla_{\theta\psi}\phi|} = \frac{1}{\sqrt{\phi_\theta^2 + \frac{1}{\sin^2\theta}\phi_\psi^2}}\Big(\phi_\theta\,,\ \frac{\phi_\psi}{\sin\theta}\Big). \qquad (13)$$

Combining $F' = \dot{\mathbf{x}}_\Sigma \cdot \mathbf{n}_\Sigma$, Equations (9) and (12), we get the evolution equation

$$\phi_t + \frac{F\cos\alpha}{r}|\nabla_{\theta\psi}\phi| = 0\,. \qquad (14)$$

As mentioned in the previous section, the Chan–Vese level set segmentation formulation uses mean curvature. The mean curvature κ' of C' on S is easily obtained from the divergence of $\mathbf{n}_\Sigma$ as [2],

$$\begin{aligned}\kappa' &= \nabla_{\theta\psi}\cdot\frac{\nabla_{\theta\psi}\phi}{|\nabla_{\theta\psi}\phi|}\\ &= \frac{\phi_{\theta\theta}\phi_\psi^2 - 2\phi_\theta\phi_\psi\phi_{\theta\psi} + \phi_\theta\phi_\psi^2\cot\theta + \phi_{\psi\psi}\phi_\theta^2}{(\phi_\theta^2 + \frac{1}{\sin^2\theta}\phi_\psi^2)^{3/2}\sin^2\theta}. \qquad (15)\end{aligned}$$

Since κ on M should be $\kappa = \frac{r}{\cos\alpha}\kappa'$ by definition of the curvature, by comparing Equation (6) with Equations (14) and (15), the level set evolution equation on M can be written as

$$\begin{aligned}\phi_t = \Big[\mu\nabla_{\theta\psi}\cdot\Big(\frac{\nabla_{\theta\psi}\phi}{|\nabla_{\theta\psi}\phi|}\Big) - \lambda^+\frac{(u_0 - c^+)^2\cos\alpha}{r}\\ +\lambda^-\frac{(u_0 - c^-)^2\cos\alpha}{r}\Big]|\nabla_{\theta\psi}\phi|\,. \qquad (16)\end{aligned}$$

4. NUMERICAL SCHEME

We can use a finite differences scheme [1] to solve the evolution equation (16). However, one of the critical problems in the polar grid is that the Courant-Friedrichs-Lewy (CFL) condition is severe because of the high density of the grid points near the poles. To avoid this difficulty, we use the overset grid called Ying–Yang grid [8]. The generation of the surface of Yin grid is defined in the polar coordinates by

$$\Omega = \{\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\} \cap \{\frac{\pi}{4} \le \psi \le \frac{7\pi}{4}\} \qquad (17)$$

or Yang grid (see Figure 1).

Note that in this overset grid approach, the conversion of polar coordinates from one component grid to the other is necessary in setting the initial condition and calculating the derivatives for grid points on the overlapping region between the two component grids. Nevertheless, we show below that the value of $|\nabla_{\theta\psi}\phi|$ in Equation (16) is invariant under this coordinate transformation. Now, we have the relationship as follows:

$$\theta' = \arccos(\sin\theta\sin\psi),\ \psi' = \arctan(\tan\theta\cos\psi), \qquad (18)$$

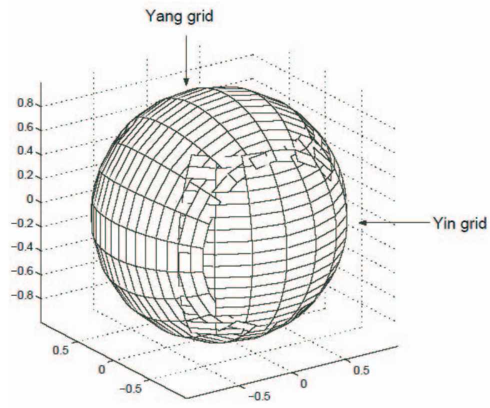

Fig. 1. Ying–Yang grid.

where (θ, ψ) and (θ', ψ') are the polar and azimuth angles in the polar coordinate system of Yin-grid and Yang-grid, respectively. Taking partial derivatives of Equation (18) with respect to θ and ψ, we get:

$$\theta'_\theta = -\frac{\cos\theta\sin\psi}{\sqrt{1-\sin^2\theta\sin^2\psi}},\ \theta'_\psi = -\frac{\sin\theta\cos\psi}{\sqrt{1-\sin^2\theta\sin^2\psi}}$$

$$\psi'_\theta = \frac{\cos\psi}{\cos^2\theta(1+\tan^2\theta\cos^2\psi)},\ \psi'_\psi = -\frac{\tan\theta\sin\psi}{1+\tan^2\theta\cos^2\psi}.$$

By the chain rule,

$$\phi_\theta = \theta'_\theta\phi_{\theta'} + \psi'_\theta\phi_{\psi'},\ \phi_\psi = \theta'_\psi\phi_{\theta'} + \psi'_\psi\phi_{\psi'}. \qquad (19)$$

Combining the above Equations, and substituting the resulting expressions for ϕ_θ and ϕ_ψ into the equation as

$$|\nabla_{\theta\psi}\phi|^2 = \phi_\theta^2 + \left(\frac{\phi_\psi}{\sin\theta}\right)^2, \qquad (20)$$

we get

$$\phi_\theta^2 + \left(\frac{\phi_\psi}{\sin\theta}\right)^2 = \phi_{\theta'}^2 + \left(\frac{\phi_{\psi'}}{\sin\theta'}\right)^2. \qquad (21)$$

Hence, we have $|\nabla_{\theta\psi}\phi| = |\nabla_{\theta'\psi'}\phi|$ as in Equation (21). This analysis indicates that in the Yin–Yang grid, the coordinate conversion of ϕ from one component grid to the other is not necessary.

5. EXPERIMENTAL RESULTS

Figs.2 and 3 show the performance of the Chan–Vese algorithm for earth images painted on a sphere and a human head model, respectively. The polar coordinate image in the NASA's earth image dataset is mapped onto the sphere and human head model in the figure. The initial contours consist of many small circles that are densely distributed throughout the model surface. They gradually merge and evolve to a number of larger contours, thus segmenting the land (sea) surface area from the mapped image pretty well.

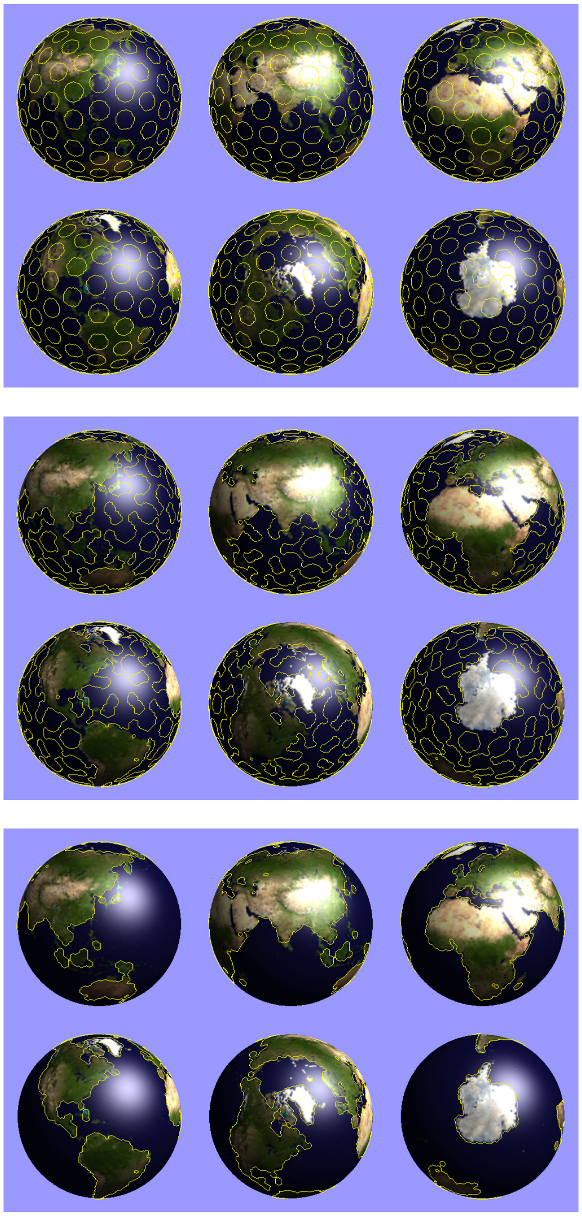

Fig. 2. Contour evolution for an earth image painted on the surface of the sphere (from top to bottom).

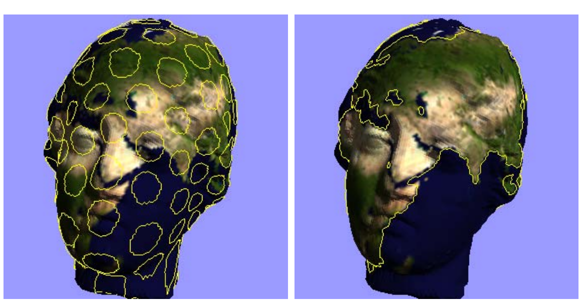

Fig. 3. Contour evolution for an earth image painted on a human head model (from left to right).

6. CONCLUSIONS

We have presented a method to segment images painted on polar coordinate meshes using the Chan–Vese level set segmentation model. This requires the formulation and implementation for Chan–Vese segmentation in a three-dimensional polar coordinate system. We have presented the explicit formulation and solution to this image segmentation problem and introduced the Ying–Yang grid system to overcome polar problems naturally. We have demonstrated the effectiveness of our approach by segmenting a remote sensing image painted on a polar coordinate mesh.

7. REFERENCES

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