SURFACE HARMONICS FOR SHAPE MODELING

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ABSTRACT

We present an approach for three dimensional (3D) shape modeling using Jacobi polynomials based surface harmonics. Because we construct a set of complete hemispherical harmonic basis functions on a hemisphere domain from the associated Jacobi polynomials, our shape modeling method work efficiently on the open hemisphere-like objects that often exist in medical anatomical structures (*e.g.*, ventricles, atriums, *etc.*). We demonstrate the effectiveness of our approach through theoretic and experimental exploration of a set of medical image applications.

Index Terms— Shape Modeling, Medical Image Computing

1. INTRODUCTION

Three dimensional (3D) shape modeling are now playing an important role in image processing with many applications, such as medical image processing, molecular biology, biochemistry, virtual reality, *etc.*, and the number of them is increasing greatly [1]. As one important application in image processing, medical image analysis urgently requires efficient and accurate shape modeling methods. In particular many diseases resulting in or from morphologic variations of structures (*e.g.*, heart, brain, *etc.*) shows the importance of analysis of shape variability for diagnostic classification and understanding of biomedical processes.

While there are multiple 3D modeling techniques, each has its own advantages and drawbacks depending on the application property. Mathematically the shape modeling methods include parametric models (such as harmonic functions [2], hyperquadrics [3], medial axis (skeleton) [4], distance distributions [5] and landmark theory based descriptors [6]. Because spherical harmonic descriptions are smooth, accurate fine-scale shape representations with a sufficiently small approximation error [7], they are widely studied and used in medical image analysis [8]. Chen *et al.* [9] presented their spherical harmonic model to analyze the left ventricular shape and motion. Similarly, Edvardson and Smedby [10] viewed a 3D object as a radial distance function on the unit sphere and tested their method on a data set from magnetic resonance imaging (MRI) of the brain. Because they used the radial surface function ($r(\theta,\phi)$) in all models, their methods are limited to represent only star-shape or convex objects without holes.

In [2], Brechbühler, Gerig and Kübler presented an extended spherical harmonic (SPHARM) method to model any simply connected 3D object. The key component of this method is the mapping of surfaces of volumetric objects to parametrized surfaces prior to expansion into harmonics. SPHARM method have been applied in many medical imaging applications, *e.g.*, shape analysis of brain structures [7].

In this paper, we propose the novel hemispherical harmonic shape modeling method to meet the requirement of surface reconstruction of hemisphere-like anatomical structures in medical image computing. Our novel global parametric shape modeling method is in the form of linear combination of hemispherical harmonic $H_l^m(\theta,\phi)$. The basis functions $H_l^m(\theta,\phi)$ derive from the shifted associated Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ with proof of their orthogonality property. The applications are vast in medical image computing such as structures in cardiac MR image sequences [11]. Our method performing on the left ventricular shape reconstruction is shown in section 3.

2. METHODS

In this section, we first propose a set of new surface harmonics based on Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ and demonstrate their orthogonal property. In order to more efficiently describe the hemisphere-like objects, we then present the other set of new hemispherical harmonics which also are based on the Jacobi polynomials. At last, we describe our surface parametrization method that generates the hemispherical harmonic shape modeling results.

2.1. Surface harmonics with Jacobi polynomials

The Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ are orthogonal on $(-1,1)$ with weight function $w(x)=(1-x)^\alpha(1+x)^\beta$. Their Rodrigues' formula is [12]

$$J_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{1}{(1-x)^\alpha}\frac{1}{(1+x)^\beta} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]. \quad (1)$$

Here, we only use the case of $\alpha = 0$ and $\beta = 1$:

$$\begin{aligned} J_n^{(0,1)}(x) &= \frac{(-1)^n}{2^n n!}\frac{1}{(1+x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x)^n(1+x)^{1+n}] \\ &= \frac{(-1)^n}{2^n n!}\frac{1}{(1+x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x^2)^n(1+x)]. \end{aligned} \quad (2)$$

As an important property, Jacobi polynomials are orthogonal polynomials on $(-1, 1)$ with weight function $w(x) = (1 + x)$ and satisfy [12]:

$$\int_{-1}^{1} J_s^{(0,1)}(x)J_t^{(0,1)}(x)(1+x)\mathrm{d}x = 2\delta_{st},$$

where δ_{st} is the Kronecker delta.

The associated Jacobi polynomials are defined as:

$$J_l^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}J_l^{(0,1)}(x). \quad (3)$$

Before we construct the surface harmonics by combination of $\{J_l^m(\cos\theta)\}$ $(-l \le m \le l)$ with $\{\cos(m\phi), \sin(m\phi)\}$, we must prove the orthogonality of associated Jacobi polynomials $J_l^m(x)$ for equal m and different l first. From Eq. (3), we know:

$$\int_{-1}^{1} J_l^m(x)J_{l'}^m(x)(1+x)\mathrm{d}x = \int_{-1}^{1}(1+x)(1-x^2)^m \frac{\mathrm{d}^m}{\mathrm{d}x^m}J_l^{(0,1)}(x)\frac{\mathrm{d}^m}{\mathrm{d}x^m}J_{l'}^{(0,1)}(x)\mathrm{d}x. \quad (4)$$

We can assume $m \le l < l'$. After defining

$$P_i = \frac{\mathrm{d}^i}{\mathrm{d}x^i}[(1+x)(1-x^2)^m\frac{\mathrm{d}^m}{\mathrm{d}x^m}J_l^{(0,1)}(x)], \quad (5)$$

$$Q_i = \frac{\mathrm{d}^{m-i}}{\mathrm{d}x^{m-i}}J_{l'}^{(0,1)}(x), \quad (6)$$

we can rewrite Eq. (3) as:

$$\int_{-1}^{1} J_l^m(x)J_{l'}^m(x)(1+x)\mathrm{d}x = \int_{-1}^{1} P_0Q_0\mathrm{d}x.$$

Combining the Eq. (2) with one step integration, we have:

$$\int_{-1}^{1} J_l^m(x)J_{l'}^m(x)(1+x)\mathrm{d}x = P_0Q_0|_{-1}^{1} - \int_{-1}^{1} P_1Q_1\mathrm{d}x. \quad (7)$$

From Eq. (5), we know $P_i = (1 - x^2)^{m-i}f(x)$, $f(x)$ is a polynomial with degree $l - m + i$. Thus, $P_1 = P_{-1} = 0$ when $i < m$. In Eq. 7, the integration can be continued m steps and the result is:

$$\begin{aligned} \int_{-1}^{1} J_l^m(x)J_{l'}^m(x)(1+x)\mathrm{d}x &= (-1)^m\int_{-1}^{1} P_mJ_{l'}^{(0,1)}(x)\mathrm{d}x \\ &= (-1)^m\int_{-1}^{1}(1+x)g(x)J_{l'}^{(0,1)}(x)\mathrm{d}x \end{aligned} \quad (8)$$

where $g(x)$ is a polynomial with degree l (we know that from Eq. (5)). Because the Jacobi polynomials $J_{l'}^{(0,1)}(x)$ is orthogonal to all polynomials $g(x)$ with $l < l'$ under the weight function $w(x) = 1 + x$ [13], thus

$$\int_{-1}^{1} J_l^m(x)J_{l'}^m(x)(1+x)\mathrm{d}x = 0 \quad (9)$$

with $l \neq l'$. Thus, associated Jacobi polynomials $J_l^m(x)$ are orthogonal for equal m and different l. We construct the surface harmonic basis functions $U_l^m(\theta, \phi)$ as:

$$\begin{cases} U_l^m(\theta,\phi) = J_l^m(\cos\theta)\sin(m\phi), m \in [1,l] \\ U_l^{-m}(\theta,\phi) = J_l^m(\cos\theta)\cos(m\phi), m \in [0,l] \end{cases} \quad (10)$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Similar to previous spherical harmonic shape modeling method [2, 7], we also can use these Jacobi polynomials based surface harmonic basis functions to reconstruct the surface of object as:

$$\mathbf{v}(\theta,\phi) \approx \sum_{l=0}^{L}\sum_{m=-l}^{l}\mathbf{c}_l^m U_l^m(\theta,\phi). \quad (11)$$

2.2. Hemispherical harmonics

When we define the hemispherical harmonics, we use the shifted Jacobi polynomials [12] that are a set of functions analogous to the Jacobi polynomials, but defined on the interval $[-1, 0]$. For Jacobi polynomials $J_l^{(0,1)}(x)$, we use the linear transformation $x = 2x' + 1$ to create the new polynomials:

$$\bar{J}_l^{(0,1)}(x') = J_l^{(0,1)}(2x'+1). \quad (12)$$

From [13], for orthogonal polynomials $J_l^{(0,1)}(x)$, the linear transformation $x = kx' + h$, $k \neq 0$, carries over the interval $[a, b]$ into an interval $[a', b']$ (or $[b', a']$), and the weight function $w(x)$ into $w(kx'+h)$. The polynomials $(\mathrm{sgn}\ k)^l|k|^{\frac{1}{2}}$ $J_l^{(0,1)}(kx' + h)$ are also orthogonal on the interval $[a', b']$ (or $[b', a']$) with the weight function $w(kx' + h)$. Because $x \in [-1, 1]$ in Jacobi polynomials $J_l^{(0,1)}(x)$, $\bar{J}_l^{(0,1)}(x')$ are also orthogonal on the interval $[-1, 0]$ with the weight function $w(x') = 2(1 + x')$.

For the associated polynomials $\bar{J}_l^m(x')$, using the linear transformation of x to $2x+1$, we get shifted associated Jacobi polynomials [12] over the interval $x' \in [-1, 1]$:

$$\bar{J}_l^m(x') = J_l^m(2x'+1), \quad (13)$$

and with respect to l,

$$\int_{-1}^{0} \bar{J}_l^m(x')\bar{J}_{l'}^m(x')dx' = \int_{-1}^{0} J_l^m(2x'+1)J_{l'}^m(2x'+1)dx'. \tag{14}$$

We get a set of orthogonal (combining Eq. (14) with Eq. (9)) associated polynomials $\bar{J}_l^m(\cos\theta)$ that are defined in the interval $\theta \in [\frac{\pi}{2}, \pi]$. Their relationship to associated Jacobi polynomials is:

$$\bar{J}_l(\cos\theta) = J_l(2\cos\theta + 1) \text{ on } \theta \in [\frac{\pi}{2}, \pi]. \tag{15}$$

Based on our shifted associated Jacobi polynomials $\bar{J}_l^m$ $(\cos\theta)$, we construct the Hemispherical harmonic basis functions $H_l^m(\theta,\phi)$ as:

$$\begin{cases} H_l^m(\theta,\phi) & = & \bar{J}_l^m(\cos\theta)\sin(m\phi), m \in [1,l] \\ H_l^{-m}(\theta,\phi) & = & \bar{J}_l^m(\cos\theta)\cos(m\phi), m \in [0,l] \end{cases} \tag{16}$$

with $\theta \in [\frac{\pi}{2}, \pi]$ and $\phi \in [0, 2\pi)$. Since the shifted associated Jacobi polynomials are orthogonal, $H_l^m(\theta,\phi)$ are also orthogonal over $[\frac{\pi}{2}, \pi] \times [0, 2\pi)$ with respect to both l and m.

2.3. Surface parametrization and description

In order to describe a voxel surface (figure 1(b)) using surface harmonics we first need to create a continuous and uniform mapping from the object surface (see figure 1(c)) to the surface of a half unit sphere (see figure 1(d)) so that each vertex on the object surface can be assigned a pair of spherical coordinates (θ,ϕ) in figure 1(a). This process is called *surface parameterization*, and the surface of the half unit sphere becomes our parameter space. Brechbühler *et al.* [2] proposed the spherical parameterization approach and we employ a hemispherical parameterization approach that is similar to their approach to exploit a square surface mesh.

The parameterization is constructed by creating a harmonic map from the object surface to the parameter surface. For colatitude θ two poles are selected in the surface mesh by finding the two vertices with the maximum (for the hemisphere-like object, it should be the center of top slice which includes many points with the same or close maximum z values, *e.g.*, our parametrization for left ventricular surface) and minimum z coordinate in object space. Then, a Laplace equation (Eq. (17)) with Dirichlet conditions (Eq. (18) and Eq. (19)) is solved for colatitude θ:

$$\nabla^2\theta = 0 \text{ (except at the poles)} \tag{17}$$

$$\theta_{north} = \frac{\pi}{2} \tag{18}$$

$$\theta_{south} = \pi \tag{19}$$

Since our case is discrete, we can approximate Eq. (17) by assuming that each vertex's colatitude (except at the poles') equals the average of its neighbours' colatitudes. Thus, after assigning $\theta_{north} = \frac{\pi}{2}$ to the north pole and $\theta_{south} = \pi$ to the south pole, we can form a system of linear equations by considering all the vertices and obtain the solution by solving this linear system. For longitude ϕ the same approach can be employed except that longitude is a cyclic parameter. To overcome this problem, a "date line" is introduced. When crossing the date line, longitude is incremented or decremented by 2π depending on the crossing direction. After slightly modifying the linear system according to the date line, the solution for longitude ϕ can also be achieved.

The parameterization result is a bijective mapping between each vertex $\mathbf{v} = (x, y, z)^T$ on a surface and a pair of spherical coordinates (θ, ϕ) $(\theta \in [\frac{\pi}{2}, \pi], \phi \in [0, 2\pi))$. We use $\mathbf{v}(\theta,\phi)$ to denote such a mapping, meaning that, according to the mapping, $\mathbf{v}$ is parameterized with the spherical coordinates (θ,ϕ). Taking into consideration the x, y, and z coordinates of $\mathbf{v}$ in object space, the mapping can be represented as $\mathbf{v}(\theta,\phi) = (x(\theta,\phi), y(\theta,\phi), z(\theta,\phi))^T$.

We use the surface net representation to expand the surface of object into our hemispherical harmonic basis functions with the coefficients $\mathbf{c}_l^m = (c_{lx}^m, c_{ly}^m, c_{lz}^m)^T$. Figure 1(f) shows the reconstructed surface result of voxel object in figure 1(b).

3. RESULTS

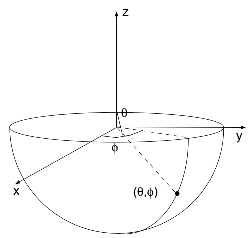
We apply our hemispherical harmonic surface modeling method to medical image analysis applications. Based on segmented image data of medical anatomical structures, we use the hemispherical harmonic method explained above for shape reconstruction and our novel shape description allows researchers to perform further shape analysis or classification and access more functional details.

During cardiac shape analysis study, since the ventricles and atriums are open objects, people only need the surface descriptions without the top parts. But the traditional spherical harmonics methods work only for closed surface. Thus, the closed shape descriptions introduce errors into shape analysis and classification for cardiac shape studies. In order to solve this problem, we apply our hemispherical harmonic surface modeling method to such objects. The surface of open objects are well reconstructed and the reconstruction result is shown in figure 1(f). Figure 1 also shows the surface modeling process, including surface parametrization, mapping, and reconstruction.

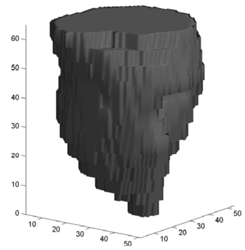
Without the top parts, these non-closed shape descriptors are more accurate in the left ventricular shape representation than the closed description method. They can provide more functional shape information for cardiac shape analysis.

4. CONCLUSIONS

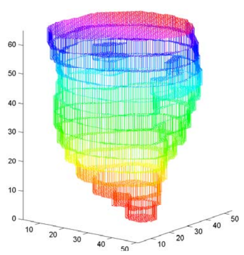
In this paper, we have presented a novel shape modeling method for the requirement of surface reconstruction for hemisphere-like anatomical structures. In order to propose the new shape descriptors, we use a set of new orthogonal associated Jacobi polynomials $J_l^m(x)$ to generate the orthogonal hemispherical harmonics $H_l^m(\theta, \phi)$. The hemispherical harmonics are defined on a hemisphere domain and we map the surfaces of volumetric objects to parametrized surfaces prior to expansion into hemispherical harmonics. The success of the algorithm is in its modeling to represent the open shape. The surface reconstruction results of using segmented cardiac MRI clearly demonstrate the effectiveness of our shape modeling method.



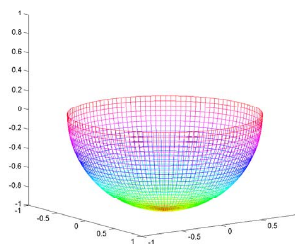
(a) Spherical coordinates



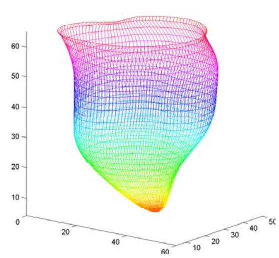
(b) Voxel object



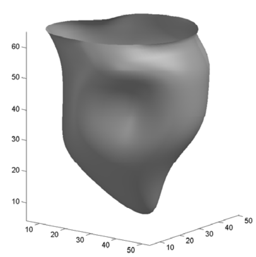
(c) Object surface with parametrization mapping



(d) Hemisphere used in parametrization mapping



(e) Reconstructed surface with parametrization mesh



(f) Final reconstructed surface

Fig. 1. Surface parametrization and reconstruction. (a) shows the spherical coordinates (θ, ϕ); (b) shows the voxel surface of left ventricle before surface reconstruction; (c) and (d) show the parametrization mapping from the object surface (c) to the surface of a half unit sphere (d); (e) shows the reconstructed surfaces of left ventricle with parametrization mesh; (f) shows the final reconstructed result.

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