FB ANALYSIS OF PMRI AND ITS APPLICATION TO $\mathcal{H}_\infty$ OPTIMAL SENSE RECONSTRUCTION

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ABSTRACT

This paper presents a filter bank (FB) analysis of parallel Magnetic Resonance Imaging (pMRI). The underlying image reconstruction strategies of the most widely used pMRI reconstruction methods are unified within the framework and their fundamental perfect reconstruction (PR) constraints are analyzed. Based on this analysis, an improved reconstruction method, called $\mathcal{H}_\infty$ optimal SENSE, is developed and its advantage is demonstrated by an example.

Index Terms— MRI, pMRI, FB, $\mathcal{H}_\infty$ optimization

1. INTRODUCTION

Magnetic Resonance Imaging (MRI) plays a major role in clinical diagnoses and medical research. The recent development in pMRI has attracted a great attention in many research disciplines. pMRI is a technique that uses multiple receiver coils to pick up the excited MR signal and then processes it digitally to reconstruct images. Hence, image reconstruction from the multiple-coil received signals is a major issue in pMRI and has been an active research topic in MRI field since early 1990's. A number of pMRI reconstruction methods have been developed and used in practice. Among these, SENSE, PILS, SMASH and GRAPPA [1, 2, 3, 4] are the most widely used ones in practice and have been examined in [5, 6, 7] for their intrinsic links and differences.

This paper gives a thorough FB analysis of pMRI reconstruction. Although such attempt is initially made in [8], the FB description there is confined in SENSE method and does not encompass the more complicated *k*-space methods, SMASH and GRAPPA. This paper casts all of the aforementioned methods into the FB framework, and more importantly, analyzed their fundamental PR constraints. These constraints explain the major source of reconstruction artifacts and lay a foundation for further improvements. Furthermore, this paper employs cyclic FB system instead of normal FB adopted in [8] which does not reflect the fact that real pMRI system can use only finite length DFT. Based on the analysis and $\mathcal{H}_\infty$ norm optimization theory, a new reconstruction method, $\mathcal{H}_\infty$ optimal SENSE, is proposed. The advantage of proposed method is demonstrated by an example.

2. THE FILTER BANK DESCRIPTION OF PARALLEL MRI

In order to obtain the FB analysis of pMRI, we begin with the intrinsic encoding process of an MRI with multiple receiver coils. For a Fourier encoded L-coil array, the *k*-space data received at each l-th receiver of the array is given by

$$s_l(\mathbf{k}) = \iint_{ROI} C_l(\mathbf{r})P(\mathbf{r})e^{j2\pi\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}. \tag{1}$$

In the above equation, ROI denotes the region of interests in 2D image domain, $P(\mathbf{r})$ is the excited spin density function with $\mathbf{r} = (x, y)$ denoting the 2D spatial position, $s_l(\mathbf{k})$ represents the encoded *k*-space with $\mathbf{k} = (k_x, k_y)$ denoting the 2D frequencies, $C_l(\mathbf{r})$ represents the spatial sensitivity function of the l-th coil, and $C_l(\mathbf{r})$, $P(\mathbf{r})$, $s_l(\mathbf{k})$ are all complex valued. In the case of Fourier Cartesian imaging with phase encoding along y direction, the above equation can be expressed as (see e.g. [6])

$$s_l(k) = \sum_{n=0}^{N-1} C_l(n)P(n)W_N^{kn}, 0 \le k \le N-1, \tag{2}$$

which is the discretized version of (1) with $W_N = e^{j\frac{2\pi}{N}}$. In (2), k is the *k*-space sampling index from 0 to $N-1$ along y direction, and n is the image domain sampling index from 0 to $N-1$ along y direction and covers the entire ROI. Note (2) repeats for each x due to the separability of the Cartesian sampling coordinates. The above equations have presented a single receiver in pMRI without accelerated imaging, where $s_l(k)$ are generated and sampled at each k. When accelerated imaging is employed, the system generates only

$$s_l(Mk) = \sum_{n=0}^{N-1} C_l(n)P(n)W_N^{Mkn}, 0 \le k \le \frac{N}{M} - 1, \tag{3}$$

which is acquired from each receiver coil. This is equivalent to an M fold downsampling of $s_l(k)$. The M here is called the accelerating factor and assumed to be a factor of N. The period of $s_l(Mk)$ is $K = \frac{N}{M}$. The above equation can be translated to image domain by taking K-point DFT of (3) to give

$$S_l(n) = \frac{1}{M}\sum_{m=0}^{M-1} C_l(n+mK)P(n+mK), 0 \le n \le K-1, \tag{4}$$

where $S_l(n)$ is the K-point DFT of $s_l(Mk)$ given by $S_l(n) = \frac{1}{K}\sum_{k=0}^{K-1} s_l(Mk)W_K^{-kn}$. Eq (3) can also be written as the cyclic convolution in *k*-space

$$s_l(Mk) = c_l(k) \circledast \rho(k) \downarrow_M, \tag{5}$$

where $\rho(k)$ and $c_l(k)$ are respectively the N-point inverse DFT, and $\circledast$ denotes the cyclic convolution with all arguments interpreted modulo K. According to [9], (4) and (5) define an L-channel cyclic FB system.

From above discussion, it can be seen that the multichannel subband signal $s_l(k)$s or $S_l(n)$s are generated by passing the excited MR signal through a bank of L filters consisting of coil sensitivity functions and then downsampled by a factor of M. In practice,

$L \geq M$, thus, it is an oversampled FB with uniform decimation. Apparently, the image reconstruction in pMRI is equivalent to the signal reconstruction in a cyclic FB. It is well known that the signal reconstruction in such FB [9] can be done either in image domain using the Aliasing Component (AC) matrices or in k-space using the polyphase matrices. According to (4), the analysis AC matrix is in the form, $\mathbf{C}(n) = [C_{ij}(n)] := [C_i(n + jK)],\ i = 0, 1, \cdots, L-1,\ j = 0, 1, \cdots, M-1$, and $\mathbf{F}(n)$ is the synthesis AC matrix of the form $\mathbf{F}(n) = [F_{ij}(n)] := [F_j(n + iK)],\ i = 0, 1, \cdots, M-1,\ j = 0, 1, \cdots, L-1$. Denote $\underline{\mathbf{E}}(n)$ and $\underline{\mathbf{H}}(n)$ as the analysis and synthesis polyphase matrices, respectively, with $\underline{\mathbf{E}}(n) = [E_{ij}(n)], i = 0, 1, \cdots, L-1, j = 0, 1, \cdots, M-1$ and $\underline{\mathbf{H}}(n) = [H_{ij}(n)], i = 0, 1, \cdots, M-1, j = 0, 1, \cdots, L-1$. Then, according to [9], $\underline{\mathbf{E}}(n)$ is given by

$$\underline{\mathbf{E}}(n) = \mathbf{C}(n)\mathbf{W}\Lambda, \tag{6}$$

where $\Lambda = diag\{1, W_N^n, \cdots, W_N^{(M-1)n}\}$ and $\mathbf{W}$ is the $M \times M$ DFT matrix .

The AC matrix approach, which is often called the frequency domain (image domain in pMRI case) method, reconstructs $\hat{P}(n)$ using

$$\widehat{\mathbf{P}}(n) = \mathbf{F}(n)\mathbf{S}(n), \quad 0 \leq n \leq K-1, \tag{7}$$

where $\hat{\mathbf{P}}(n) := [\hat{P}(n), \hat{P}(n+K), \cdots, \hat{P}(n + (M-1)K)]^T$ is blocked version of $\hat{P}(n)$ and $\mathbf{S}(n) := [S_0(n), S_1(n), \cdots, S_{L-1}(n)]^T$. The polyphase matrix approach, which is often called the time/spatial domain (k-space) method, constructs $\hat{\rho}(k)$ using

$$\widehat{\mathbf{p}}(Mk) = \underline{\mathbf{H}}(n)s(Mk), \quad 0 \leq k \leq K-1, \tag{8}$$

where $\hat{\mathbf{p}}(Mk) := [\hat{\rho}(Mk), \hat{\rho}(Mk-1), \cdots, \hat{\rho}(Mk-(M-1))]^T$, and $\mathbf{s}(Mk) := [s_1(Mk), s_2(Mk), \cdots, s_L(Mk)]^T$. Note that Eq. (8) has followed the convention in the FB literature [9] to represent $c_l(k) \circledast \rho(k)$ with $C_l(n)\rho(k)$.

The PR conditions for the polyphase and the AC matrix representations are respectively

$$\underline{\mathbf{H}}(n)\underline{\mathbf{E}}(n) = \mathbf{I}_M, \tag{9}$$

$$\mathbf{F}(n)\mathbf{C}(n) = \mathbf{I}_M. \tag{10}$$

The matrix form PR condition (10) can also be written equivalently as

$$\frac{1}{M}\sum_{l=0}^{L-1} F_l(n+iK)C_l(n+jK) = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}, \tag{11}$$

which gives explicitly the relationship of AC matrix elements to the reconstruction distortion and the aliasing.

The above PR conditions are in terms of either the polyphase matrices $\underline{\mathbf{H}}(n)$ and $\underline{\mathbf{E}}(n)$ or the AC matrix $\mathbf{F}(n)$ and $\mathbf{C}(n)$. Using (6) and (9), these conditions can also be represented in terms of the polyphase matrix and AC matrix as given below

$$\underline{\mathbf{H}}(n)[C_0(n), \cdots, C_{L-1}(n)]^T = [1, \cdots, W_N^{-(M-1)n}]^T \tag{12}$$

Aiming to reconstruct $\rho(k)$ from $s_l(k)$s or $P(n)$ from $S_l(n)$s, four widely used methods have been developed since 1997. As shown in next section, despite their different origins, all these methods can be unified within the filter bank framework.

3. FB ANALYSIS OF EXISTING RECONSTRUCTION METHODS

This section will cast the underlying image reconstruction strategies of SENSE, PILS, SMASH and GRAPPA into the problem of finding the synthesis FB to achieve PR under certain conditions. As can be seen, these methods are essentially obtained from different assumptions on the analysis or synthesis FBs.

3.1. SENSE and PILS methods

SENSE [2] is an AC matrix based method operating in the image domain. It obtains the synthesis FB by inverting the AC matrix $\mathbf{C}(n)$ which is called unfolding matrix in [2]. From (10), one may simply use the left pseudo (least squares) inverse on $\mathbf{C}(n)$, which yields

$$\mathbf{F}(n) = (\mathbf{C}^H(n)\mathbf{C}(n))^{-1}\mathbf{C}^H(n). \tag{13}$$

With the above $\mathbf{F}(n)$, the reconstructed output can be obtained by (7), ie, $\widehat{\mathbf{P}}(n) = \mathbf{F}(n)\mathbf{S}(n)$.

Eq. (13) is the core of SENSE method. Note that Eq. (13) is applicable only when $\mathbf{C}(n)$ has full column rank [10]. This condition normally holds for pMRI systems because $C_l(n)$s are designed to look at different regions of the object, which gives linearly independent $C_l(n)$s. Note also that the $\mathbf{F}(n)$ obtained from (13) is generally a noncausal and IIR transfer matrix even if $\mathbf{C}(n)$ is FIR, see [10] for details.

PILS method [3] is a special case of SENSE reconstruction. It assumes that each $C_l(n)$ has an ideal localized sensitivity function

$$C_l(n + l\frac{N}{L}) = \begin{cases} \text{non-zero}, 0 \leq n \leq \frac{N}{L} \\ 0, \text{otherwise} \end{cases}. \tag{14}$$

Here $\frac{N}{L}$ is assumed to be an integer and $\frac{N}{L} \leq K$ since $L \geq M$. Substituting such a $C_l(n)$ into (11), we can see that the aliasing free condition is automatically satisfied as long as $F_l(n)$ is also strictly bandlimited as $C_l(n)$. Therefore, $F_l(n)$ can be simply chosen as

$$F_l(n + l\frac{N}{L}) = \begin{cases} C_l(n + l\frac{N}{L})^{-1}, \ 0 \leq n \leq \frac{N}{L} - 1 \\ 0, \text{otherwise} \end{cases} \tag{15}$$

and the image reconstructed by PILS can be obtained by

$$\widehat{P}(n + l\frac{N}{L}) = F_l(n + l\frac{N}{L})S_l(n), \ 0 \leq n \leq \frac{N}{L} - 1. \tag{16}$$

PILS avoids the AC matrix inversion which can be very problematic when the condition number (known as the g-factor for SENSE method) of the AC matrix is poor in SENSE reconstruction. Note that because PILS does not allow any subband overlapping, it is essentially a critically sampled FB and cannot take the advantage of oversampled FB to improve SNR.

3.2. SMASH and GRAPPA methods

SMASH [1] is a polyphase matrix based method operating in the k-space. It reconstructs $\widehat{\rho}(k)$ using

$$\widehat{\rho}(Mk - m) = \sum_{l=0}^{L-1} h_{ml}s_l(Mk), \ \ m = 0, 1, \cdots, M-1, \tag{17}$$

where h_{ml} are the complex coefficients satisfying the constraint

$$\sum_{l=0}^{L-1} h_{ml}C_l(n) = W_N^{-mn}, \ \ m = 0, 1, \cdots, M-1. \tag{18}$$

Because $W_N = e^{j\frac{2\pi}{N}}$, (18) is often referred to as harmonic fitting in the MRI literature, with m being called the order of the harmonics and h_{ml} the fitting coefficients [1]. Define

$$\underline{\mathbf{H}}_0 := [h_{ml}], m = 0, 1, \cdots, M-1, \quad l = 0, 1, \cdots, L-1. \quad (19)$$

Then (17) and (18) can be written respectively in the matrix form

$$\widehat{\mathbf{p}}(Mk) = \underline{\mathbf{H}}_0 \mathbf{s}(Mk) \quad (20)$$

and

$$\underline{\mathbf{H}}_0 [C_0(n), \cdots, C_{L-1}(n)]^T = [1, \cdots W_N^{-(M-1)n}]^T. \quad (21)$$

Apparently, (20) and (21) are a special case of the polyphase based signal reconstruction formula (8) and the PR condition (12), where the synthesis polyphase matrix $\underline{\mathbf{H}}(n)$ is chosen as a constant complex matrix $\underline{\mathbf{H}}(n) = \underline{\mathbf{H}}_0$. In FB terms, SMASH assumes that there exists a constant synthesis matrix $\underline{\mathbf{H}}_0 = \underline{\mathbf{H}}(n)$ such that the PR condition (12) holds. It obtains this $\underline{\mathbf{H}}_0$ by solving (21) and uses it in (20) to reconstruct $\widehat{\mathbf{p}}(Mk)$.

Similar to SENSE and PILS, SMASH also requires the explicit knowledge of $C_l(n)$s in (21) for solving $\underline{\mathbf{H}}_0$. To avoid this requirement, the AUTO-SMASH [11] has been developed to allow direct estimation of $\underline{\mathbf{H}}_0$ during imaging process. AUTO-SMASH assumes that $\underline{\mathbf{H}}_0$ satisfies the constraint

$$h_{0l} = 1, 0 \le l \le L-1. \quad (22)$$

Under this constraint, the first equality in (17) for $m = 0$ becomes $\widehat{\rho}(Mk) = \sum_{l=0}^{L-1} s_l(Mk), 0 \le k \le K-1$. This equality holds for all $M = N/K$ and $k \in [0, K-1]$, therefore

$$\widehat{\rho}(k) = \sum_{l=0}^{L-1} s_l(k), 0 \le k \le N. \quad (23)$$

Physically (23) means that under the constraint (22), if the output of each coil is fully acquired without downsampling, then $\widehat{\rho}(k)$ at any $k \in [0, N]$ can be simply reconstructed by directly summing up the outputs of all coils acquired at the same k. This property has been used in AUTO SMASH to estimate the $\underline{\mathbf{H}}_0$ that satisfies the constraint (22).

GRAPPA [4] is a generalization of AUTO-SMASH methods, which also assumes the constraint (22). As seen from (23), under this constraint, the reconstruction of $\widehat{\rho}(k)$ using the downsampled $\widehat{s}_l(Mk), 0 \le l \le L-1$, is equivalent to the reconstruction of its L subcomponents $\widehat{s}_l(Mk-m), 1 \le m \le M-1, 0 \le l \le L-1$. Hence, instead of reconstructing $\widehat{\rho}(k)$ (or equivalently $\widehat{\rho}(Mk-m), 0 \le m \le M-1$) directly, GRAPPA reconstructs

$$\widehat{s}_i(Mk-m) = \sum_{l=0}^{L-1} \sum_{j=0}^{T-1} h_{ml}^i(Mj) s_l(M(k-j)), \quad (24)$$

where $m = 0, 1, \cdots, M-1$, $i = 0, 1, \cdots, L-1$, T is a fixed integer, and $h_{ml}^i(Mj)$ are complex coefficients with

$$h_{0l}^i(Mj) = \begin{cases} 1, l = i \text{ and } j = 0 \\ 0, \text{ otherwise} \end{cases}, \quad i = 0, 1, \cdots, L-1. \quad (25)$$

It then combines the reconstructed $\widehat{s}_i(Mk-m)$ using (23) or other methods such as the sum-of-squares, see [4] and the references therein. Define

$$H_{ml}^i(n) := \sum_{j=0}^{K-1} h_{ml}^i(Mj) W_K^{nj}, \quad \underline{\mathbf{H}}^i(n) := [H_{ml}^i(n)],$$

$$0 \le m \le M-1, 0 \le l \le L-1.$$

Then (24) can be written in the matrix form

$$\widehat{\mathbf{s}}_i(Mk) = \underline{\mathbf{H}}^i(n) \mathbf{s}(Mk), \quad i = 0, 1, \cdots, L-1, \quad (26)$$

where $\mathbf{s}(Mk)$ is as defined in the previous section and

$$\widehat{\mathbf{s}}_i(Mk) := [\widehat{s}_i(Mk), \ \cdots, \widehat{s}_i(Mk-(M-1))]^T. \quad (27)$$

From (26), it is clear that when (23) is used to obtain $\widehat{\rho}(k) = \sum_{l=0}^{L-1} \widehat{s}_l(k)$, the GRAPPA reconstruction can be written in the form (8), namely,

$$\widehat{\mathbf{p}}(Mk) = \sum_{i=0}^{L-1} \widehat{\mathbf{s}}_i(Mk) = \sum_{i=0}^{L-1} \underline{\mathbf{H}}^i(n) \mathbf{s}(Mk) = \underline{\mathbf{H}}(n) \mathbf{s}(Mk), \quad (28)$$

where $\underline{\mathbf{H}}(n) := \sum_{i=0}^{L-1} \underline{\mathbf{H}}^i(n)$. In FB terms, GRAPPA attempts to achieve the PR condition (12) using an FIR transfer matrix $\underline{\mathbf{H}}(n)$ with the order $T-1$ and the first row elements all being 1.

Remark: The analysis above is based on the assumption that the analysis AC matrix $\mathbf{C}(n)$ or the analysis polyphase matrix $\underline{\mathbf{E}}(n)$ is known. In practice, they are generally unknown and need to be estimated. In SENSE and PILS, $\mathbf{C}(n)$ is estimated by pre-imaging, while in AUTO-SMASH and GRAPPA, $\underline{\mathbf{H}}(n)$ (instead of $\underline{\mathbf{E}}(n)$) is estimated directly during real imaging process. Due to space limit, further discussion on this topic is impossible here. The reader is referred to [1, 2, 3, 4, 12] for the details of estimation methods.

4. $\mathcal{H}_\infty$ OPTIMAL SENSE RECONSTRUCTION

The results of previous sections have provided a unified framework for the analysis and improvement of the existing pMRI reconstruction methods and for the development of new methods. As a demonstration, we will use these results to improve SENSE method.

One of the major drawbacks of the original SENSE method is its sensitivity to the additive noises when the condition number of $\mathbf{C}(n)$ is poor [2]. In such case, the pseudo inverse $\mathbf{C}^\dagger(n)$ used in SENSE will have high gain, which will amplify the subband noises and significantly degrade the reconstructed image. This often happen when high acceleration is used for fast imaging.

From the above FB analysis of pMRI, it is easy to see that the pMRI reconstruction subject to noises is equivalent to the signal reconstruction in the FB subject to subband noises. As shown in [13], the solution to the latter problem can be obtained from

$$\min_{\mathbf{F}(n)} \{ \| [I - \mathbf{F}(n)\mathbf{C}(n)] \|_\infty^2 + \ \beta \|\mathbf{F}(n)\|_\infty^2 \}, \quad (29)$$

where β is a prescribed weighting factor. This is an H_∞ norm optimization problem, which can be solved by LMI optimization.

Let $\{A_C, B_C, C_C, D_C\}$ and $\{A_F, B_F, C_F, D_F\}$ be the state-space realizations of $\mathbf{C}(n)$ and $\mathbf{F}(n)$, respectively. Then it is routine to show that the state-space realization of the system $[I - \mathbf{F}(n)\mathbf{C}(n), \mathbf{F}(n)]$ is in the form $\left[\begin{array}{c|ccc} A & B_V & \vdots & B_N \\ \hline C & D_V & \vdots & D_N \end{array}\right] :=$ $\left[\begin{array}{cc|ccc} A_C & 0 & B_C & \vdots & 0 \\ B_F C_C & A_F & B_F D_C & \vdots & B_F \\ \hline -D_F C_C & -C_F & I - D_F D_C & \vdots & -D_F \end{array}\right]$. Following [13], it can be shown that the solution to (29) can be found by the following LMI optimization:

$$\min_{\gamma_a,\gamma_b,P,A_F,B_F,C_F,D_F} \{\gamma_a^2 + \beta\gamma_b^2\}, \tag{30}$$

subject to

$$\begin{bmatrix} A^T PA - P & A^T PB_N & C^T \\ B_N^T PA & B_N^T PB_N - I & D_N^T \\ C & D_N & -\gamma_b^2 I \end{bmatrix} < 0, \tag{31}$$

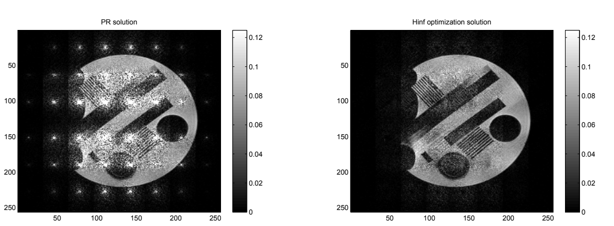
$$\begin{bmatrix} A^T PA - P & A^T PB_V & C^T \\ B_V^T PA & B_V^T PB_V - I & D_V^T \\ C & D_V & -\gamma_a^2 I \end{bmatrix} < 0, \; P = P^T > 0, \tag{32}$$

where γ_a and γ_b are the upper bounds on $\|[I - \mathbf{F}(n)\mathbf{C}(n)\|_\infty$ and $\|\mathbf{F}(n)]\|_\infty$, respectively. The $\mathcal{H}_\infty$ optimal SENSE obtained from the above optimization is compared with the original SENSE in the simulation described below.

An eight head coil array is simulated by numerical calculation of the Biot-Sarvart law to generate each coil's sensitivity function $C_l(n)$ [1]. The obtained $C_l(n)$ is then multiplied with an MR image and then Fourier transformed to produce the Nyquist sampled k-spaces. These k-spaces are further downsampled to generate $s_l(Mk)$s with $M = 8$. The white gaussian noise is added to $s_l(8k)$s with SNR=37dB. The above defined $\mathcal{H}_\infty$ optimization is performed to obtain the $\mathbf{F}(n)$, which is compared with the least squares PR solution of original SENSE algorithm. The reconstructed images of both methods are given in Fig 1. The table below lists the detailed simulation results where the relative reconstruction error $\epsilon = \sum_{i=1}^{256}\sum_{j=1}^{256} \Delta I_{i,j}^2 \,/\, \sum_{i=1}^{256}\sum_{j=1}^{256} I_{i,j}^2$, and $\Delta I_{i,j}$ and $I_{i,j}$ are respectively the absolute reconstruction error and the phantom input at the (i,j)-th pixel.

Table 1. Comparison of SENSE and $\mathcal{H}_\infty$ optimal SENSE

	SENSE	H_∞ SENSE
ϵ	81.48%	15.82%
$\|I - \mathbf{F}(n)\mathbf{C}(n)\|_\infty$	1.0247×10^{-9}	9.9867×10^{-1}
$\|\mathbf{F}(n)\|_\infty$	2.1665×10^{6}	3.9529×10^{4}



(a) Original SENSE (b) $\mathcal{H}_\infty$ optimal SENSE

Fig. 1. Comparison of original SENSE and $\mathcal{H}_\infty$ optimal SENSE

As can be seen from the simulation result, the $\mathcal{H}_\infty$ optimal SENSE provides better reconstruction than the original SENSE, resulting in significantly reduced reconstruction error.

5. CONCLUSIONS

An FB analysis of pMRI reconstruction has been presented. The underlying image reconstruction strategies of four most widely used reconstruction methods have been cast into achieving PR in FB under certain constraints. These results provide a unified theoretical framework for the analysis and further improvement of the existing pMRI reconstruction methods, and for the development of new methods. An improved reconstruction method is derived using the analysis results and $\mathcal{H}_\infty$ optimization, and its advantage is demonstrated by an simulated example. Application of this FB analysis to other pMRI reconstruction methods are give in [12].

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