

P-LAPLACIAN DRIVEN IMAGE PROCESSING

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ABSTRACT

In this work, we take a novel line of approaches to evolve images. It is motivated by the total variation method, known for its denoising and edge-preserving effect. Our approach generalises the TV method by taking a general L_p norm of the gradients instead of the L_1 in the TV method. We generalise this method in a series of first and second order derivatives in terms of gauge coordinates. This method also incorporates the well-known blurring by a Gaussian filter and the balanced forward - backward diffusion.

The method and its properties are briefly discussed. The practical results are visualised on a real-life image, showing the expected behaviour. When a constraint is added that penalises the distance of the results to the input image, one can vary the desired amount of blurring and denoising.

Index Terms— Image processing, Partial differential equations, Differential geometry, Nonlinear differential equations, Image analysis

1. INTRODUCTION

Total variation is well-known for its edge preserving properties while smoothing the image (also known as cartooning) [1]. It is obtained by minimizing the L_1 norm of the norm of the gradient squared and approaching the minimum by a steepest decent method. When the L_2 is minimized, one obtains the heat equation (or Gaussian blurring [2]). In this work, instead of taking the gradient descent equation of the L_2 norm of the gradients, the L_p norm is used, thus obtaining so-called p-Laplacians. Firstly the concepts of gauge coordinates, variational derivatives, and p-Laplacians are discussed. Secondly, it will be shown that the p-Laplace evolution equation is a PDE that can be simplified using gauge coordinates and its properties are discussed in relation to image filtering. Thirdly, both noise-constrained and unconstrained evolution of this approach is visualized on the famous Lena.

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1.1. Gauge coordinates

An image can be thought of as a collection of curves with equal value, the isophotes. Most isophotes are not self-intersecting. At extrema an isophote reduces to a point, at saddle points the isophote is self-intersecting. At the non-critical points Gauge coordinates v, w can be chosen. Gauge coordinates are locally set such, that the v direction is tangent to the isophote and the w direction points in the direction of the gradient vector [3, 4]. Consequently: $L_v = 0, L_w = \sqrt{L_x^2 + L_y^2}$. Of special interest in the remainder are the second order invariant structures:

$$\begin{aligned} L_{vv} &= \frac{L_x^2 L_{yy} + L_y^2 L_{xx} - 2L_x L_y L_{xy}}{L_x^2 + L_y^2} \\ L_{ww} &= \frac{L_x^2 L_{xx} + L_y^2 L_{yy} + 2L_x L_y L_{xy}}{L_x^2 + L_y^2} \end{aligned}$$

1.2. Minimizing methods

Consider an image L on the domain Ω . The first variation [5] of the functional E at L in the direction η is defined by

$$\delta E(L, \eta) = \frac{\partial}{\partial \epsilon} E(L + \epsilon \eta) |_{\epsilon=0}.$$

The variational derivative $\delta E(L)$ of the functional E at L in the direction η is defined by

$$\delta E(L, \eta) = \int_{\Omega} \delta E(L) \cdot \eta \, d\Omega$$

with $\eta \in C_0^\infty(\Omega)$ a test function that is zero at the boundaries.

Minimizing L with appropriate boundary conditions gives the Euler-Lagrange equation $\delta E = 0$. A dynamical system is obtained by the steepest decent approach $L_t = -\delta E$. So to find the minimum of $E(L)$ given an image L_0 is to solve

$$\begin{aligned} L_t &= -\delta E(L) \\ L(t=0) &= L_0 \end{aligned}$$

2. P-LAPLACIANS

Consider in general the integral

$$E(L) = \int_{\Omega} \frac{1}{p} \|\nabla L\|^p \, d\Omega$$

It is well-known as the p -Dirichlet energy integral with accompanying p -Laplacian equation $\delta E = 0$, with

$$\delta E = -\nabla \cdot (\|\nabla L\|^{p-2} \nabla L)$$

Alternatively, the energy can be written as

$$E(L) = \frac{1}{p} \int_{\Omega} L_w^p d\Omega$$

Theorem: The variational derivative $\delta E(L)$ can be written as $= -L_w^{p-2} ((p-1)L_{ww} + L_{vv})$.

Proof

$$\begin{aligned} \delta E(L; \eta) &= \frac{1}{p} \int_{\Omega} \frac{\partial}{\partial \epsilon} \|\nabla L + \epsilon \nabla \eta\|^p |_{\epsilon=0} d\Omega \\ &= \int_{\Omega} \|\nabla L\|^{p-2} \nabla L \cdot \nabla \eta d\Omega \\ &= (\|\nabla L\|^{p-2} \nabla L) \cdot \eta |_{\partial\Omega} \\ &\quad - \int_{\Omega} (\nabla \cdot (\|\nabla L\|^{p-2} \nabla L)) \eta d\Omega \end{aligned}$$

Since $\eta = 0$ on the boundary, $(\|\nabla L\|^{2\alpha-2} \nabla L) \cdot \eta |_{\partial\Omega} \equiv 0$ and the Euler-Lagrange equation $\delta E(L) = 0$ equals

$$-\nabla \cdot (\|\nabla L\|^{2\alpha-2} \nabla L) = 0$$

The left hand side equals the well-known variational derivative of the p -Laplacian. An explicit expression is obtained by applying the divergence operator to both terms, where gauge coordinates are used:

$$\begin{aligned} &-\nabla \cdot (\|\nabla L\|^{p-2} \nabla L) \\ &= -\nabla (\|\nabla L\|^{p-2}) \cdot \nabla L - \|\nabla L\|^{p-2} (\nabla \cdot \nabla L) \\ &= -(\nabla \cdot L_w^{p-2}) \cdot \nabla L - L_w^{p-2} \Delta L \end{aligned}$$

For the first part we have

$$\begin{aligned} (\nabla \cdot L_w^{p-2}) \cdot \nabla L &= (p-2) L_w^{p-3} \nabla L_w \cdot \nabla L \\ &= (p-2) L_w^{p-3} (\nabla L \cdot H L_w^{-1}) \cdot \nabla L \end{aligned}$$

where H is the Hessian matrix. Recall $(\nabla L \cdot H) \cdot \nabla L \equiv L_w^2 L_{ww}$ as given before. Therefore

$$(p-2) L_w^{p-3} L_w^{-1} L_w^2 L_{ww} = (p-2) L_w^{p-2} L_{ww}$$

and consequently we have

$$\delta E(L) = -((p-2) L_w^{p-2} L_{ww} + L_w^{p-2} \Delta L).$$

Using the identity $\Delta L = L_{ww} + L_{vv}$ this gives $\delta E(L) = -L_w^{p-2} ((p-1) L_{ww} + L_{vv})$. QED

For $p = 2$ we have the heat equation:

$$L_w^{2-2} ((2-2) L_{ww} + \Delta L) = \Delta L$$

Next, $p = 1$ gives the Total Variation flow :

$$L_w^{1-2} ((1-1) L_{ww} + L_{vv}) = L_w^{-1} L_{vv} = \kappa$$

In general, it gives a recipe for PDE-driven flow:

$$L_t = L_w^{p-2} ((p-2) L_{ww} + \Delta L)$$

The case $p \rightarrow \infty$ is known as the infinite Laplacian, denoted by ΔL_{∞} . This term is defined as either L_{ww} [6, 7, 8] or $L_w^2 L_{ww}$ [9]. It can be applied to image inpainting [10] and shape metamorphism [11].



Fig. 1. Values of k . From left to right $k < 0$, $0 < k < 1$, $1 < k < 4/3$, all three in the complex plane, and $4/3 < k$ in the real plane.

3. SOLVING THE P-LAPLACIAN PDE

To solve the p -Laplacian, it is assumed that the solution is independent of direction, size, dimension, and orientation. Therefore, $t \propto (x^2 + y^2)^{\frac{p}{2}}$, and the dimensionless variable

$$\xi = \frac{(a(x^2 + y^2))^{p/2}}{t}$$

is used (cf. the case $p = 2$: a Gaussian filter). Second, an addition t dependency is assumed. This is inspired by the observation that the solution for $p = 2$ contains the factor t^{-1} . One can say that the Gaussian that denotes the ‘‘observation’’ is expressed in term of Lumen per square meter.

The source solution (or Barenblatt [12] solution [13]) for the p -Laplacian equation with $p > 2$ and $1 < p < 2$ can be found by considering functions of the type

$$L(x, y, t, c) = t^a \left(k \left(t^q \sqrt{x^2 + y^2} \right)^m + c \right)^n$$

with a, q, m, n constants to be identified. The constant c can be chosen such, that $\int L d\Omega = M$, where M is the total intensity of the image, $\int_{\Omega} L_0 d\Omega$.

The p -Laplacian equation for this function (omitted) give a polynomial with terms in t , $c - \xi$, and $2c - \xi$, with $\xi = k \left(t^q \sqrt{x^2 + y^2} \right)^m + c$, that equals zero. Collecting the powers of $2c - \xi$ terms gives $n = \frac{p-1}{p-2}$, the t terms yield $q = \frac{-pa+2a-1}{p}$. The $c - \xi$ terms result in $m = \frac{p}{p-1}$. The remaining terms in p -Laplacian equation require $k = \frac{(p-2)}{p} (-4 + 3p)^{\frac{1}{1-p}}$. For k the following limits hold: $\lim_{p \rightarrow -\infty} k = 1$, $\lim_{p \rightarrow 0} k = \infty$, $\lim_{p \rightarrow 1} |k| = e^3$, $\lim_{p \rightarrow 4/3} k = -\infty$, $\lim_{p \rightarrow 2} k = 0$, $\lim_{p \rightarrow \infty} k = 1$, Intermediate values of k are shown in Figure 1, where the first three plots are in the complex plane. One can see that for certain rational values of $p < 4/3$ real solutions can be obtained, e.g. $p = 5/4$ gives $k = -768/5$ and $p = 1/2$ gives $k = -75/4$.

With these values for a, q, m, n , and k , the p -Laplacian filter $L(x, y, t, c)$ equals

$$t^{\frac{2}{4-3p}} \left(c - \frac{(p-2)}{p} (3p-4)^{\frac{1}{1-p}} \left(t^{-\frac{1}{3p-4}} \sqrt{x^2 + y^2} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p-2}}$$

Real solutions depend on values of k , since $t > 0$ and c can be taken sufficiently large.

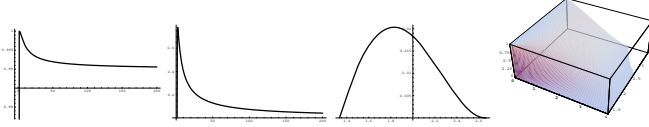


Fig. 2. p-Laplace filters for $t = 1, c = 1, y = 0$ and from left to right $x = .01, x = 1, x = 4$ as a function of p . Most right: filter in the x, t -plane. Note the part where data is missing due to complex values.

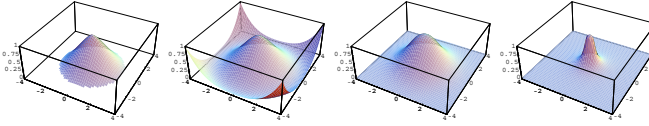


Fig. 3. p-Laplace filters for $t = 1, c = 1$ and from left to right $p = 4, p = 3, p = 11/6, p = 5/4$.

The filters L yield real solutions for $p > 2$ and $4/3 < p < 2$, as well as certain rational fractions. For $p > 2$, the solution decreases to zero with increasing radius and either increases again, or becomes complex. This is due to the fact that $k > 0$ for those values, so depending on the values of c , L becomes complex-valued, see Figure 2 and 3.

Only for $4/3 < p < 2$ the integral over L is real and bounded.

For $p = 2$, the standard Gaussian filter is obtained. Expressing $L_t = L_{xx} + L_{yy}$ using $L(x, y, t) = f(\xi)t^n$ gives

$$t^{n-1} (nf(\xi) - (4a + \xi)f'(\xi) - 4a\xi f''(\xi)) = 0$$

The second term has as solution

$$f(\xi) = e^{-\frac{\xi}{4a}} \left(c_1 \mathbf{U} \left(n + 1, 1, \frac{\xi}{4a} \right) + c_2 \mathbf{L}_{-n-1} \left(\frac{\xi}{4a} \right) \right).$$

Here $\mathbf{U}(a, b, z)$ is a confluent hypergeometric function and $\mathbf{L}_a(z)$ is the Laguerre polynomial expression [14]. Then the solution of the 2-Laplacian $L(\xi, n)$ with $\xi = \frac{x^2 + y^2}{4t}$ becomes

$$e^{-\xi} t^n (c_1 \mathbf{U}(n + 1, 1, \xi) + c_2 \mathbf{L}_{-(n+1)}(\xi))$$

For $n = -1, -2, \dots$ this reduces to

$$\begin{aligned} L(x, y, t, -1) &= \frac{c_1 + c_2}{t} e^{-\xi} \\ L(x, y, t, -2) &= \frac{(\xi - 1)(c_1 - c_2)}{t^2} e^{-\xi} \\ L(x, y, t, -3) &= \frac{(2 - 4\xi + \xi^2)(2c_1 + c_2)}{2t^3} e^{-\xi} \end{aligned}$$

These expressions are the Gaussian filter and its derivatives up to order t^{-1-n} . Only the zeroth order yields a positive filter.

3.1. Related work

Chen et al. [15] study the p-Laplacian for $1 \leq p \leq 2$ with p dependent on image (gradient) information. When consid-



Fig. 4. Lena image, noisy Lena, $\sigma = 20$

ering directional data, a p-Laplacian term enters when using theory of harmonic maps in liquid crystals [16].

In the case of non-linear diffusion, often the choice

$$L_t = \nabla \cdot (g(|\nabla L|)\nabla L)$$

is made. The function $g(\cdot)$ is chosen such, that it enhances the edges (where $|\nabla L|$ is large) and deblurs noisy (flat) regions (where $|\nabla L|$ is small). A classical example is the Perona-Malik equation [17], where $g(|\nabla L|^2) = \frac{1}{1 + |\nabla L|^2/\lambda^2}$. Note that by taking $g(\cdot) = 1$, the Gaussian scale space is obtained.

A general class of diffusivities [18] is obtained by $g(|\nabla L|) = \frac{1}{|\nabla L|^q}$. Then the diffusion becomes $L_t = \nabla \cdot (|\nabla L|^{-q}\nabla L)$, and by taking $q = -p + 2$ the p-Laplacian is obtained. For $q = 1$ one obtains Total Variation flow, while the case $q = 2$, known as balanced forward-backward diffusion, is also investigated [19]. Note that the latter case resembles $p = 0$, the case where the energy functional would simplify to a constant. The diffusion for $p = 0$ given above, is obtained by minimising $E(L) = \int_{\Omega} \log L_w \, d\Omega$.

Kim [20] studied the PDE $L_t = L_w^\alpha \nabla \cdot (L_w^{-\omega-1}\nabla L)$, together with a distance penalty $\beta(L - L_0)$. This equals the PDE $L_w^{\alpha-\omega-1} (-\omega L_{ww} + L_{vv})$. He called it the enhanced TV model for $\alpha = \omega + 1$, i.e. when the L_w term vanishes. This type of PDEs we discussed elsewhere.

4. RESULTS

An extra condition may occur in the presence of noise (assume zero mean, variance σ): $I = \int_{\Omega} \frac{1}{2}(f - f_0)^2 \, d\Omega = \sigma^2$, where f_0 is the initial image and f a denoised one.

The solution is obtained by the Euler Lagrange equation $\delta L + \lambda \delta I = 0$ with $\delta L = -\nabla \cdot (\|\nabla f\|^{p-2}\nabla f) = 0$, $\delta I = f - f_0$, and $\lambda = \frac{\langle \delta L, \delta I \rangle}{\langle \delta I, \delta I \rangle}$, where $\langle \delta I, \delta I \rangle = 2\sigma^2$. The solution can be reached by an evolution determined by a steepest decent evolution $f_t = -(\delta L + \lambda \delta I)$ When we set $\lambda = 0$, an unconstrained blurring process is obtained.

A forward Euler scheme is used to compute

$$\Delta L = CD_{i,j}^- (\|D_{i,j}^+ f\|^{p-2} D_{i,j}^+ f)$$

with $D_i^+ f = f(i + 1, j) - f(i, j)$, $D_i^- f = -f(i - 1, j) + f(i, j)$ and similar for j . The constant is chosen such, that the maximum update is less or equal to 10% of the maximal image intensity to stabilize numerical computation for large p

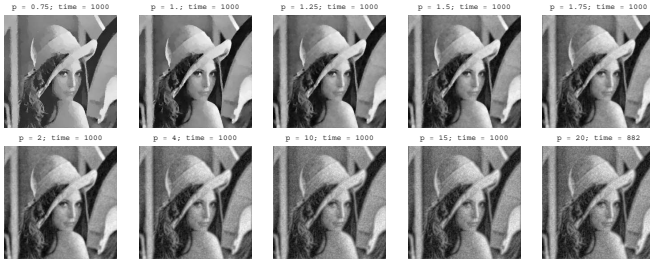


Fig. 5. Constrained Lena evolution

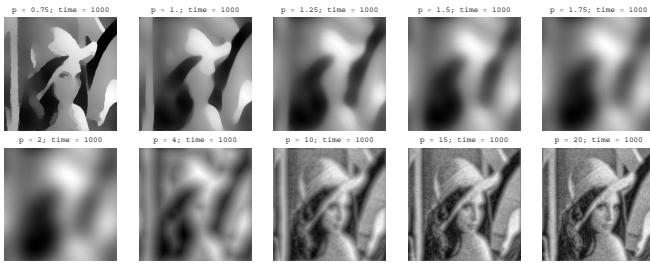


Fig. 6. Unstrained Lena evolution

values. The value for λ is computed in accordance with [1]. For the comparison of different values of p , 1000 iterations are computed. Note that p -Laplacians are defined for $1 < p < \infty$, but $p = .75$ - cartooning with extra deblurring - gives still stable results.

For the constrained evolution, convergence is obtained after approximately 400 iterations. Results are shown in Figure 5.

For the unstrained evolution convergence is obviously not obtained. Results are shown in Figure 6. For large values of p , updates are tempered due to locally large values in the norm of the gradient term. It appears that blurring is achieved without affecting the noise that much.

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