MULTICHANNEL IMAGE DECOMPOSITION BY USING PSEUDO-LINEAR HAAR WAVELETS

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ABSTRACT

Recently it has been shown that in Image Processing, the usual sum and product of the reals are not the only operations that can be used. Several other operations provided by fuzzy logic perform well in this application. We continue this line of research and we study the possibility to use some pairs of pseudo-operations. We define in the present paper pseudo-linear Haar wavelets, and we perform multi-channel decomposition of images. We study some pairs of pseudo-operations determined by a continuous, strictly increasing generator instead of the classical sum and multiplication and Haar-type wavelets based on these operations. The results show us that pseudo–linear Haar wavelets can be used as an alternative of classical Haar wavelets since the perfect reconstruction property is conserved.

***Index Terms*—** pseudo-operations, Haar wavelets

1. INTRODUCTION

In classical Functional Analysis and classical Approximation Theory, the underlying algebraic structure is the linear space structure. The mathematical analysis using nonlinear mathematical structures is called idempotent analysis (see [6]) or pseudo-analysis (see [8], [7]) and it is shown to be a powerful tool in several applications.

Recently we have proposed the same problem in Approximation Theory i.e., is the linear structure the only one that can be used in the classical Approximation Theory? Moreover, are the addition and multiplication of the reals the only operations that can be used for defining approximation operators? All the approximation operators need to be linear? The answer to these questions is negative, and in this sense in [2] max-product Shepard approximation operators are studied. Also, in [3] Pseudo-Linear Approximation Operators are studied from the theoretical point of view.

In the present paper we will use pseudo-additions based on an additive generator and pseudo- multiplications based on the same (but multiplicative) generator. For such a pair of operations the distributivity property holds. Such pseudo-operations can be e.g. a uninorm and an absorbing norm ([9]). We will show that other pairs of pseudo-operations can be considered.

We will focus on the possibility to define and use in Image Processing a Haar-type wavelet based on pairs consisting of pseudo- operations. Also, we give in the present paper a multi-channel image decomposition based on pseudo-linear Haar wavelets. Wavelets are used usually for providing a decomposition of an original image and after the decomposition is performed, further processing is possible, depending on different application purposes. So, in the present paper we present a new image decomposition method based on pseudo-linear Haar wavelets. A decomposition is made correctly only if it has the perfect reconstruction property. In the present paper it is shown that the perfect reconstruction property holds. On the other hand the decompositions presented are different than the usual methods based on the classical Haar wavelets and the decomposition operators are non-linear (being pseudo-linear).

After a section in which we describe the algebraic structure, in Section 3 we present pseudo-linear Haar type wavelets. In Section 4 we present some numerical results which show the effectiveness of the proposed method. At the end of the paper some conclusions and topics for further research are given.

2. PSEUDO-OPERATIONS

Let $[a, b]$ be closed subinterval of $[-\infty, +\infty]$ (in some cases semiclosed subintervals will be considered) and let $\preceq$ be total order on $[a, b]$. Structure $([a, b], \oplus, \odot)$ is a *semiring* if the following hold:

- $\oplus$ is *pseudo-addition*, i.e., a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element, denoted by $\mathbf{0}$;
- $\odot$ is *pseudo-multiplication*, i.e., a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, associative and with a unit element denoted by $\mathbf{1}$;
- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

Semirings with continuous (up to some points) pseudo-operations are divided into three classes. The first class contains semirings with idempotent pseudo-addition and non idempotent pseudo- multiplication. Semirings with strict pseudo-operations defined by monotone and continuous generator function form the second class, and semirings with both idempotent operations belong to the third class.

2.1. Uninorms and absorbing norms

For the purpose of this construction we shall consider a semiring of the second class on the unit interval, i.e. $([0, 1], \oplus, \odot)$. As a pseudo-addition $\oplus : [0, 1]^2 \to [0, 1]$, the representable uninorm with neutral element $e \in (0, 1)$ will be used. In this case, for given $e \in (0, 1)$ and a strictly increasing continuous function $g : [0, 1] \to \overline{\mathbf{R}}$ such that $g(0) = -\infty$, $g(e) = 0$ and $g(1) = +\infty$, operation $\oplus$ is

$$x \oplus y = g^{-1}(g(x) + g(y)), \tag{1}$$

for all $(x, y) \in [0, 1]^2 \backslash \{(0, 1), (1, 0)\}$. If $(x, y) \in \{(0, 1), (1, 0)\}$, one of the following conventions will be accepted: either $0 \oplus 1 = 1 \oplus 0 = 0$ or $0 \oplus 1 = 1 \oplus 0 = 1$.

A uninorm $U : [0, 1]^2 \to [0, 1]$ is a commutative, associative and increasing binary operator with a neutral element $e \in [0, 1]$ (see [9]). Now, corresponding pseudo-multiplication $\odot$ is

$$x \odot y = g^{-1}(g(x)g(y)), \tag{2}$$

and, for previously described generating function g, it belongs to the class of so-called absorbing norms.

An absorbing norm $\odot : [0, 1]^2 \to [0, 1]$ is a commutative, associative and increasing binary operator with an absorbing element $a \in [0, 1]$, i.e. $(\forall x \in [0, 1])(x \odot a) = a)$. ([1])

Some of the basic properties of operations (1) and (2) are:

(i) $\odot$ is an absorbing norm with e as absorbing element;

(ii) $\mathbf{1} = g^{-1}(1)$;

(iii) for all $x \in (0, 1)$ there exists $\ominus x \in (0, 1)$ such that $x \oplus (\ominus x) = \mathbf{0}$.

Since $\ominus x = g^{-1}(-g(x))$, the pseudo-subtraction for all $(x, y) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$ can be given in the following form:

$$x \ominus y = x \oplus (\ominus y) = g^{-1}\left(g(x) - g(y)\right). \tag{3}$$

As generators for the uninorm and absorbing norm we can use e.g.,

$$g(x) = \ln \frac{x^a}{1 - x^\alpha}, \tag{4}$$

which generates a parametric family of operations. The neutral element of the uninorm in this case is $\frac{1}{2^{\frac{1}{\alpha}}}$. This parametric family, for $\alpha = 1$ contains the famous 3 PI operation.

2.2. Another class of generated pseudo-operations

In the present subsection we describe some pairs of generated pseudo-operations on the real line which can be used in the construction of pseudo-linear Haar wavelets.

Let us consider now the following simple family of strictly increasing generators

$$g : \mathbf{R} \to \mathbf{R},\ g(x) = sign(x)\, |x|^\alpha\,, \tag{5}$$

where $\alpha > 0$. The pseudo-addition determined by these generators is a Minkowski-type addition, while the pseudo- multiplication is the usual multiplication of te reals. It is interesting to remark that if we consider $\alpha \to \infty$, for $x, y > 0$ this operation coincides with the maximum, while for $x, y < 0$ this operation coincides with the minimum.

3. PSEUDO-LINEAR HAAR WAVELETS

3.1. One-dimensional case

We define in the present section a Haar-type wavelet based on a pair consisting of generated pseudo-operations.

Given an input signal $x : \mathbf{Z} \to \mathbf{R}$, we split it into $x_0(n)$ and $x_1(n)$ the odd and even samples. The classical two-channel Haar wavelet transform is given by the Analysis and Synthesis operations based on a perfect reconstruction condition.

3.1.1. Analysis:

The analysis operation gives the highpass and lowpass subband samples as:

$$\begin{pmatrix} x_0(k) \\ x_1(k) \end{pmatrix} \to \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix},$$

where

$$\begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{pmatrix} \frac{x_0(k)+x_1(k)}{\sqrt{2}} \\ \frac{x_0(k)-x_1(k)}{\sqrt{2}} \end{pmatrix}$$

3.1.2. Synthesis:

We apply the same operation as in analysis step to the signal y, and it is easy to check that we reobtain the original signal x.

Let us now propose the pseudo-linear version of the above described Haar wavelet.

3.1.3. Analysis:

The analysis operation gives the highpass and lowpass subband samples as a pseudo-linear transformation:

$$\begin{pmatrix} x_0(k) \\ x_1(k) \end{pmatrix} \rightarrow \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix},$$

where

$$\begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{pmatrix} (x_0(k) \oplus x_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ (x_0(k) \ominus x_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \end{pmatrix}$$

3.1.4. Synthesis:

We apply the same operation as in analysis step to the signal y, and it is easy to check that we reobtain the original signal x. Indeed, it is easy to see that

$$\begin{aligned} &(y_0(k) \oplus y_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= ((x_0(k) \oplus x_1(k)) \oplus (x_0(k) \ominus x_1(k))) \\ &\odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= (x_0(k) \oplus x_0(k)) \odot g^{-1}\left(\frac{1}{2}\right) = x_0(k). \end{aligned}$$

and that

$$\begin{aligned} &(y_0(k) \ominus y_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= ((x_0(k) \oplus x_1(k)) \ominus (x_0(k) \ominus x_1(k))) \odot g^{-1}\left(\frac{1}{2}\right) \\ &= (x_1(k) \oplus x_1(k)) \odot g^{-1}\left(\frac{1}{2}\right) = x_1(k). \end{aligned}$$

3.2. Bidimensional case

As the classical bidimensional linear Haar transform is obtained by applying two times the one dimensional transform on horizontal and vertical directions, the pseudo-linear case will be similar. We define in what follows the 4-channel pseudo-linear Haar wavelets in two dimensions.

3.2.1. Analysis

The analysis operation is given as follows:

$$\begin{array}{cccccc} x(2n) & x(2n^+) & \longrightarrow & \psi(n) & \omega_h(n) \\ x(2n_+) & x(2n_+^+) & \longleftarrow & \omega_v(n) & \omega_d(n) \end{array},$$

where

$$\psi(n) = (x(2n) \oplus x(2n^+) \oplus x(2n_+) \oplus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right)$$

$$\omega_h(n) = (x(2n) \oplus x(2n^+) \ominus x(2n_+) \ominus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right)$$

$$\omega_v(n) = (x(2n) \ominus x(2n^+) \oplus x(2n_+) \ominus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right)$$

$$\omega_d(n) = (x(2n) \ominus x(2n^+) \ominus x(2n_+) \oplus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right).$$

3.2.2. Synthesis

The synthesis operation is as in the classical case the same as the analysis operation, but applies to the result of the analysis operation. It is easy to check that the perfect reconstruction requirement is satisfied.

4. IMAGE PROCESSING EXPERIMENTS

We perform in this section multichannel image decomposition by using pseudo-linear Haar wavelets. In the first experiment we have considered a pair of a uninorm and corresponding absorbing norm, defined as above. These experiments show us the possible usefulness of the proposed method as an alternative to Haar wavelets.

In the following experiments image Lenna is decomposed until the second decomposition level by using the same generator (4) for different values of the parameter a (see Figs. 1, 2). The upper left corner corresponds to the component ψ, while the lower left, lower right and upper right correspond to $\omega_v, \omega_d, \omega_h$ respectively of the pseudo-linear Haar wavelet. Since the exact reconstruction of the original Lenna image is possible by using the given image decomposition results and the pseudo-linear Haar wavelet again, the effectiveness of the proposed method is shown.

It is interesting to observe that the basic color in the lower components increases together with the value of the parameter. This is due to the fact that the color in the lower component should be near to the null element of the operation.

In the second experiment the decomposition of the image Lenna is performed by using a pair of generated pseudo-operations as in (5). The image decomposition result is shown in Fig. 3.

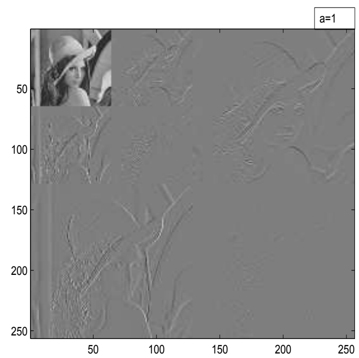

Fig. 1. Multichanel decomposition of the image Lenna based on the proposed method, parameter of the uninorm-absorbing norm $a = 1$

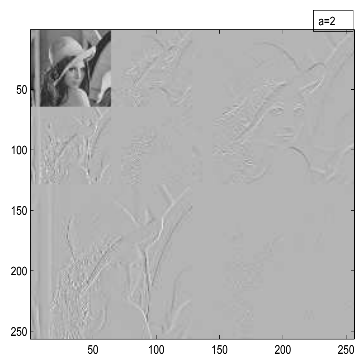

Fig. 2. Multichanel decomposition of the image Lenna based on the proposed method, parameter of the uninorm-absorbing norm $a = 2$

5. CONCLUSIONS

Several classes of pseudo-linear Haar-type wavelets were proposed. The theoretical study shows that the main properties such as the perfect reconstruction property, are conserved. The presented decomposition method is non-linear (being pseudo -linear) and it is different than the usual wavelets. The results show that a pair consisting of a pseudo-addition and a pseudo-multiplication can be used in Image processing as alternatives of sum and product operations.

As topics for future research let us mention the use of the proposed Haar-type wavelets for feature extraction, noise reduction or debluring of images. Surely, JPEG 2000 compression standard uses more sophisticated wavelets and their study is subject of future research together with the study of their application to image and video compression.

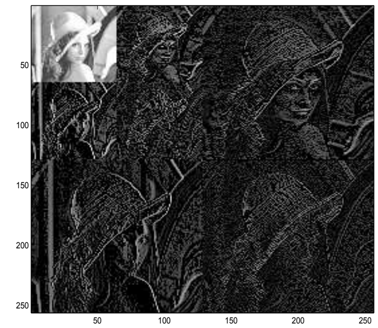

Fig. 3. Multichannel decomposition results by using the generator in (5), parameter $\alpha = 4$

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