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Kinematic Analysis of the Spherically Actuated Platform Manipulator

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***Abstract*—New methods for the inverse and forward kinematic analysis of the novel six Degrees of Freedom (6DOF) parallel manipulator which has only two legs are presented. The actuation of the new mechanism is through two base-mounted spherical actuators. In the inverse pose kinematic, active joint variables are directly calculated with no need for the evaluation of passive joint variables. In the forward pose kinematic, closed form solution adopting a new approach is presented. It is shown that the inverse and forward pose kinematic have sixteen and four different solutions, respectively. Moreover, closed form equations for the rate kinematic analysis are proposed. Finally, two different categories of the singularity points for the new mechanism with their geometrical interpretation are introduced. In one category the mechanism loses one or more DOF while in the other one it gains one or more DOF.**

I. Introduction

Due to their high stiffness, high speed, large load carrying ability and high precision positioning capability, parallel mechanisms have become very popular in the past decade. There are numerous parallel manipulators with different structure, different way of actuations and different number of the degree of freedom (DOF). The parallel manipulator proposed in [1] which is the subject of this paper as shown in Fig.1, has a different structure than the others. The specific feature of this manipulator is that with only two legs it has 6 DOF. Fewer legs leads to smaller required space for manipulator's installation, decreases the chance of the leg's collision during maneuver, and also means fewer moving part. However, although the manipulator has the above advantages, it suffers from the small load carrying capacity which is a direct consequence of its actuation. It is to be noted that, the manipulator has two base-mounted spherical actuators. An example of spherical actuator can be found in [2]. Since the prismatic joints are passive, the required torque for a specific maneuver might be large and the dynamic simulations presented in [3] justify this claim. Its authors' future work to find a remedy for this draw-back; for example by making the prismatic joints partially active. Here a complete solution for the kinematic analysis of the manipulator, considering passive prismatic joints which is the original manipulator introduced in [1], is given. New methods for the inverse and forward pose kinematic analysis different than the ones presented in [1] are proposed. Contrary to [1], in the inverse pose kinematic the active joint variables are evaluated with no need for the calculation of the passive ones. Different than the approach adopted in [1] for the forward pose analysis which does not lead to a closed form solution, here a closed form solution for the forward pose kinematic is introduced. The different possible numbers of solution for the inverse and forward pose kinematic analysis are given. A closed form solution for the rate kinematic is proposed. The two different categories of the singular points for the new mechanism with their geometrical interpretation are introduced. It is worth noting that in one category the mechanism loses one or more DOF while in the other one it gains one or more DOF.

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II. Mechanism Description

The parallel manipulator presented in this article consists of a moving platform connected to the base frame by two legs as shown in Fig. 1. Each leg is composed of the spherical (S), prismatic (P) and universal (U) joints, which is called a SPU leg. These joints in a serial manner construct each leg. Although, it consists of two legs, it has 6 DOF[1]. Moreover it should be mentioned that the actuators of the mechanism are at the spherical joints and are base-mounted.

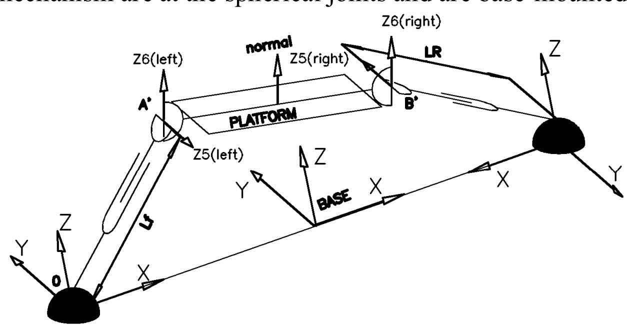

Fig. 1: Manipulator Configuration and Coordinate Description

III. Parameter and Coordinate Description

To specify the location and orientation of the moving platform with respect to (w.r.t) the base, a coordinate frame is attached to the moving platform in which its origin is at the center of the platform. The moving platform coordinate and the base coordinate are schematically shown in Fig. 1. The transformation matrix of the moving platform coordinate w.r.t the base frame coordinate using X-Y-Z Eulerian angles of Rotation is

$$ {}^{B}_{P}T = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma & x \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma & y \\ -S\beta & C\beta S\gamma & C\beta C\gamma & z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

For the inverse and forward pose kinematic analysis, the standard Denavit-Hartenberg parameters are required. The selected intermediate coordinates based on the Denavit-Hartenberg notation are shown in Figure 2. The spherical, prismatic and universal joints need three, one and two coordinate frames, respectively. The constant platform parameters as well as the Denavit-Hartenberg parameters for the left and right legs of the manipulator are:

ℓ_B: Distance between two spherical joints.

ℓ_P: Distance between two universal joints.

l_f, l_R: Length of the left and right legs, respectively.

$(\theta_{1f}, \theta_{2f}, \theta_{3f}), (\theta_{1R}, \theta_{2R}, \theta_{3R})$: Spherical joints variables for the left and right legs, respectively.

$(\phi_{1f}, \phi_{2f}), (\phi_{1R}, \phi_{2R})$: Universal joint variables for the left and right legs, respectively.

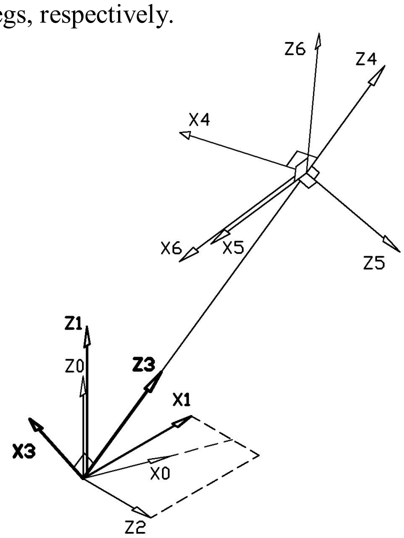

Fig. 2: Intermediate coordinates description for the left leg

The X and Z axes for each coordinate frame are presented in Fig. 2 and the Y axis can be found via the right-hand rule. The Denavit-Hartenberg Parameters for the selected intermediate coordinates are given in table1.

IV. Inverse Pose Kinematic

In the inverse pose kinematic the desired actuator variables which are the spherical joint variables, should be calculated having ${}^{B}_{P}T$. First the basic idea which leads to the calculation of the active joint variables with no need for the evaluation of the passive joint ones will be discussed and later its equivalent mathematical repression will be introduced.

Table 1.Denavit-Hartenberg parameters for the left leg

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90	0	0	$\theta_2 + 90$
3	90	0	0	θ_3
4	0	0	ℓ_f	0
5	-90	0	0	$90 - \phi_1$
6	90	0	0	ϕ_2

Having the platform's location and orientation in inverse pose process, location of the points A' and B' in Fig. 1 are known. Thus, the leg's direction is available. Therefore, two spherical variables θ_1, θ_2 which are related to the direction of the leg as shown in Fig. 3 can be evaluated.

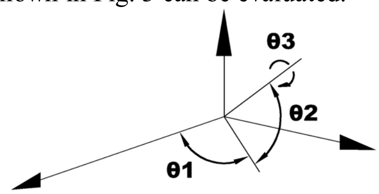

Fig. 3: Spherical joint's variables description

To have θ_3, rotation of the leg around itself has to be found. From Fig. 1, it is clear that the direction of z_5, solely depends on the rotation of the leg around itself. However, z_5 for each leg is perpendicular to its corresponding z_6 and leg. Since the direction of leg and z_6 is known, the direction of z_5 will be available by cross producting z_6 and the leg's direction. It is to be noted that, the direction of the leg is available having θ_1, θ_2. Also, the z_6 direction is the same as the direction of the normal vector of the manipulator which is known. In the following the mathematical equivalent of the above geometrical interpretation is provided.

The index *f* in the following formulation indicates that the related matrix evaluated for the left leg. The transformation matrix between coordinates (0) and (6) of the left leg, Fig. 2, using the Denavit-Hartenberg parameters given in table 1, is:

$$ ({}^{0}_{6}T)_f = ({}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T)_f \qquad (1) $$

This transformation is also:

$$ ({}^{0}_{6}T)_f = ({}^{B}_{0}T^{-1})_f\,{}^{B}_{P}T\,({}^{6}_{P}T^{-1})_f \qquad (2) $$

In which ${}^{B}_{0}T, {}^{6}_{P}T$ are:

$$ ({}^{0}_{B}T)_f = \begin{bmatrix} 1 & 0 & 0 & .5\ell_B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ({}^{6}_{P}T)_f = \begin{bmatrix} -1 & 0 & 0 & .5\ell_P \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Equaling Eqs. (1) and (2), by the fact that ${}^{B}_{P}T$ is completely known , we have:

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ 1 \end{bmatrix} = \overbrace{({}^{B}_{0}T^{-1})_f\,{}^{B}_{P}T({}^{6}_{P}T^{-1})_f \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}^{completely\ known,\ forth\ column\ of\ Eq.(2)} = \overbrace{\begin{bmatrix} \ell_f C\theta_{1f} C\theta_{2f} \\ \ell_f S\theta_{1f} C\theta_{2f} \\ \ell_f S\theta_{2f} \\ 1 \end{bmatrix}}^{forth\ column\ of\ Eq.(1)}$$

where $\begin{bmatrix} k_1 & k_2 & k_3 & 1 \end{bmatrix}^T$ is the forth column of Eq. (2) and completely known. Thus the spherical joint variables for the left are:

θ_{1f} is:

$$\theta_{1f} = \text{atan2}\,(k_2, k_1) \tag{3}$$

$$\theta_{2f} = \text{atan2}\,(k_3, k_1 / C\theta_{1f}) \quad \text{if} \quad \theta_{1f} = 0\ or\ \pi \tag{4}$$

$$\theta_{2f} = \text{atan2}\,(k_3, k_2 / S\theta_{1f}) \quad \text{if} \quad \theta_{1f} \neq 0\ and\ \pi \tag{5}$$

Having θ_{1f} and θ_{2f}, θ_{3f} can be calculated as follows. The direction of Z_{3f} is:

$$Z_{3f} = \begin{bmatrix} C\theta_{1f} C\theta_{2f} \\ S\theta_{1f} C\theta_{2f} \\ S\theta_{2f} \end{bmatrix} \tag{6a}$$

where θ_{1f}, θ_{2f} are already calculated. Also, the direction of Z_{6f} is:

$$Z_{6f} = \begin{bmatrix} (-C\theta_{1f} S\theta_{2f} C\theta_{3f} + S\theta_{1f} S\theta_{3f}) C\phi_{1f} + C\theta_{1f} C\theta_{2f} S\varphi_{1f} \\ (-S\theta_{1f} S\theta_{2f} C\theta_{3f} - C\theta_{1f} S\theta_{3f}) C\phi_{1f} + S\theta_{1f} C\theta_{2f} S\phi_{1f} \\ C\theta_{2f} C\theta_{3f} C\phi_{1f} + S\theta_{2f} S\phi_{1f} \end{bmatrix} \tag{6b}$$

Cross producting Z_{3f} and Z_{6f} and multiplying the result by ${}^{2}_{0}R = ({}^{0}_{1}R\,{}^{1}_{2}R)^{-1}$, one has:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = {}^{2}_{0}R\,(Z_{3f} \times Z_{6f}) = \begin{bmatrix} -S\theta_{3f} C\phi_{1f} \\ 0 \\ C\theta_{3f} C\phi_{1f} \end{bmatrix} \tag{7}$$

Since θ_{1f}, θ_{2f} and ${}^{B}_{P}T$ are available, the left had side of Eq. (7) is known. It is to be noted that Z_{3f} and ${}^{2}_{0}R$ are known because θ_{1f}, θ_{2f} are evaluated in previous step and Z_{6f} is identified since it is the third column of ${}^{B}_{P}T$. Therefore θ_{3f} is:

$$\theta_{3f} = \text{atan2}\,(-p_1, p_3) \tag{8}$$

Table.2: Multiple Answers for Inverse Solution of the Left Leg

First set	θ_{1f}	θ_{2f}	θ_{3f}	ϕ_{1f}
Second set	θ_{1f}	θ_{2f}	$\theta_{3f} + \pi$	$-\phi_{1f} + \pi$
Third set	$\theta_{1f} + \pi$	$\pi - \theta_{2f}$	θ_{3f}	$-\phi_{1f} + \pi$
Fourth set	$\theta_{1f} + \pi$	$\pi - \theta_{2f}$	$\theta_{3f} + \pi$	ϕ_{1f}

From Eqs. (3) and (8), it can be concluded that there are four possible answers for the inverse solution of the left leg which are given in table 2. The same procedure can be used to have the active variables of the right leg, except that ${}^{6}_{P}T, {}^{0}_{B}T$ for the right leg are:

$$({}^{0}_{B}T)_R = \begin{bmatrix} -1 & 0 & 0 & .5\ell_B \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ({}^{6}_{P}T)_R = \begin{bmatrix} 1 & 0 & 0 & -.5\ell_P \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since there are four possible solutions for each leg, there exist sixteen possible answers for the inverse problem of the introduced manipulator.

V. Forward Pose Kinematic

The forward pose problem for the majority of the parallel robots is solved through the numerical analysis and rarely a closed from solution is available. However, in this paper a closed form solution for the forward pose problem of the proposed manipulator is presented. Consider Fig.1 since the spherical joint's variables are given, the direction of $(Z_5)_{left}$ and $(Z_5)_{right}$ axis are available. Since ${}^{B}(Z_5)_{left}$ and ${}^{B}(Z_5)_{right}$ always lay on the moving platform surface, their cross product, m, gives the normal direction of the platform. Moreover, line $A'B'$ which connects point A' of the left leg to point B' of the right leg, placed on the moving platform surface. Thus:

$$m = {}^{B}(Z_5)_{left} \times {}^{B}(Z_5)_{right} \;\Rightarrow\; m \cdot \overrightarrow{B'A'} = 0 \tag{9}$$

or

$$m \cdot \overrightarrow{B'A'} = 0 \;\Rightarrow\; H\ell_f + E\ell_R + G = 0$$

$$\begin{cases} H = m_1[C\theta_{1f} C\theta_{2f}] + m_2[S\theta_{1f} C\theta_{2f}] + m_3 S\theta_{2f} \\ E = m_1[C\theta_{1R} C\theta_{2R}] + m_2[S\theta_{1R} C\theta_{2R}] - m_3 S\theta_{2R} \\ G = -\ell_b m_1 \end{cases}$$

where, the relation for the $\overrightarrow{B'A'}$ is given in Eq. (13). In addition, it is obvious that the distance between A' and B' is constant and is ℓ_p where ℓ_p is the length of the manipulator's platform thus:

$$|A'B'| = l_P \tag{10}$$

which lead to:

$$(\ell_f)^2 + (\ell_R)^2 - 2A\ell_f\ell_R - B\ell_R - C\ell_f = D$$

$$\begin{cases} A = -[C\theta_{2f} C\theta_{2R} C(\theta_{1f} - \theta_{1R}) - S\theta_{2f} S\theta_{2R}] \\ B = [2\ell_b C\theta_{1R} C_{2R}] \\ C = 2\ell_b C\theta_{1f} C\theta_{2f} \\ D = (\ell_p)^2 - (\ell_b)^2 \end{cases}$$

Eqs. (9) and (10) are two equations with two unknown which are ℓ_R and ℓ_f. From Eq. (9), ℓ_R can be found in terms of ℓ_f. Substituting the result ℓ_R in terms of ℓ_f in

Eq. (10), a polynomial of the second order in terms of ℓ_f will be obtained which can be solved easily. This second order polynomial is:

$$a(\ell_f)^2 + b\ell_f + c = 0$$

$$\begin{cases} a = 1 + I^2 - 2IA \\ b = 2FI - 2FA - BI - C \\ c = F^2 + BF - D \\ F = -G/E \\ I = -H/E \end{cases} \quad (11)$$

Solving Eq. (11) for ℓ_f, ℓ_R will be found from Eq. (9). To completely address the direct kinematics problem, ${}^B_PT$ has to be calculated which would be done in the remainder of this section.

${}^B(X_6)_{left}$ is equal to:

$${}^B(X_6)_{left} = \frac{B'A'}{|B'A'|} = \frac{B'A'}{\ell_P} \quad (12)$$

In which $B'A'$ is:

$$B'A' = \begin{bmatrix} x_{A'} - x_{B'} \\ y_{A'} - y_{B'} \\ z_{A'} - z_{B'} \end{bmatrix} = \begin{bmatrix} \ell_f C\theta_{1f} C\theta_{2f} - \ell_B + \ell_R C\theta_{1R} C\theta_{2R} \\ \ell_f S\theta_{1f} C\theta_{2f} + \ell_R S\theta_{1R} C\theta_{2R} \\ \ell_f S\theta_{2f} - \ell_R S\theta_{2R} \end{bmatrix} \quad (13)$$

expressing ${}^B(X_6)_{left}$ in the coordinate (3) (See Fig. 2):

$${}^3(X_6)_{left} = {}^3_BR\, {}^B(X_6)_{left} \quad (14)$$

Also ${}^3_6R = {}^3_4R\, {}^4_5R\, {}^5_6R$ is:

$${}^3_6R_{left} (= {}^3_4R\, {}^4_5R\, {}^5_6R)_{left} = \begin{bmatrix} S\phi_{1f} C\phi_{2f} & -S\phi_{1f} S\phi_{2f} & C\phi_{1f} \\ S\phi_{2f} & C\phi_{2f} & 0 \\ -C\phi_{1f} C\phi_{2f} & C\phi_{1f} S\phi_{2f} & S\phi_{1f} \end{bmatrix} \quad (15)$$

Equaling the first column of Eq. (15) to the Eq. (14), (since both represent ${}^3(X_6)_{left}$) the desired ϕ_{1f}, ϕ_{2f} could be determined as below:

$$n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \underbrace{{}^3(X_6)_{left} = {}^B(X_6)_{left}\, {}^3_BR}_{Completely\ known} = \begin{bmatrix} S\phi_{1f} C\phi_{2f} \\ S\phi_{2f} \\ -C\phi_{1f} C\phi_{2f} \end{bmatrix} \quad (16)$$

So:

$$\phi_{1f} = \operatorname{atan2}(n_1, -n_3) \quad (17)$$

and ϕ_{2f} is:

$$\phi_{2f} = \operatorname{atan2}(n_2, -n_3 / c\phi_{1f}) \quad \text{if } \phi_{1f} = 0 \ or\ \pi \quad (18)$$

$$\phi_{2f} = \operatorname{atan2}(n_2, n_1 / S\phi_{1f}) \quad \text{others} \quad (19)$$

Now ${}^B_PT$ can be computed as below:

$${}^B_PT = {}^B_0T\, {}^0_3T\, {}^3_4T\, {}^4_5T\, {}^5_6T\, {}^6_PT \quad (20)$$

${}^B_0T$: *Known(because the configuration is known)*

${}^0_3T$: *known(because inputs are known)*

${}^3_4T$: *known(because* ℓ_f *is calcualted)*

${}^4_5T, {}^5_6T$: *known(bacuase* ϕ_{1f}, ϕ_{2f} *are calcualeted)*

${}^6_PT$: *known(because the configuration is known)*

Finally, since Eq. (17) gives two possible answers for ϕ_{1f}, for each pair of (ℓ_f, ℓ_R) there are two possible answers for the forward pose solution. Moreover, Eq. (11) can lead to two answers for ℓ_f thus two pairs of (ℓ_f, ℓ_R) can be acceptable and as a result there exit four possible solutions for the forward pose problem.

VI. INVERSE RATE KINEMATICS

In inverse rate Kinematics, having the linear velocity of point G shown in Fig. 4 as well as the angular velocity of the moving platform, all $\dot{\theta}$ have to be evaluated.

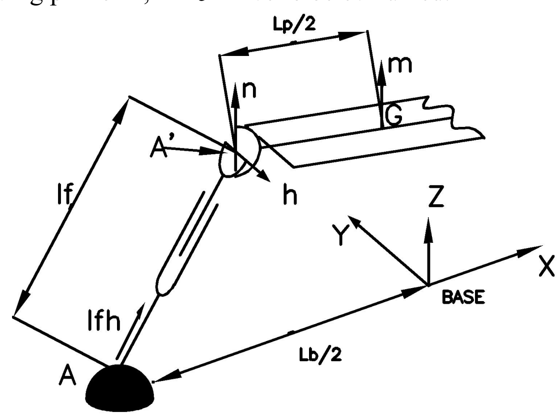

Fig. 4: The direction of h, n and m

Consider the following definitions:

$$\hat{z} = \hat{z}_3, \qquad S = l\ \hat{z} \qquad , q = \vec{R}_{A'/G}$$

where, $\hat{z}_3$ is shown in Fig. 2, l is the length of the link and $\vec{R}_{A'/G}$ is the position vector of point A' w.r.t point G. The velocity of point A' in Fig. 4 is:

$$\dot{S} = \dot{t} + \omega_P \times q \quad (21)$$

where $\dot{t}$ and ω_P are the linear velocity of point G and angular velocity of the moving platform, respectively and are known. Also $\dot{S}$ from the leg view is:

$$\dot{S} = \dot{l}\ \hat{s} + \overbrace{(\omega + \omega_Z)}^{\omega_L} \times S \quad (22a)$$

where ω_L is the angular velocity of the leg, ω_Z is the component of ω_L along z_3 and ω is composed of the components of ω_L which are in the plane that has z_3 as the normal vector. Since ω_Z and S have the same direction there cross product will be vanished. Therefore Eq. (22a) is:

$$\dot{S} = \dot{l}\hat{s} + \omega \times S \quad (22b)$$

Taking the dot and cross product of Eqs. (21) and (22b) with $\hat{s}$ respectively and equalling the results, one obtains:

$$\dot{l} = \hat{s} \cdot \dot{S} \quad , \quad \omega = \frac{\hat{s} \times \dot{S}}{l} \tag{23}$$

It must be noted that $\dot{S}$ is known from Eq. (21) and the calculated angular velocity of the leg in Eq. (23), dose not have its component along the leg, ω_Z. To have this component the following steps have to be taken. Equaling the angular velocity of the cross symbol of the universal joint, from the platform and leg's views, we have:

$$\omega_p + \omega_n = \overbrace{\omega + \omega_z}^{\omega_l} + \omega_h \tag{24}$$

In which ω_n is the relative angular velocity of the cross symbol of the universal joint w.r.t the platform and is in the direction of n; ω_h is the relative angular velocity of the cross symbol of the universal joint w.r.t the leg which is in the direction of h and at last ω_Z is the component of the leg's angular velocity in the leg's direction.

Defining k as:

$$k = n \times h \tag{25}$$

where n and h are shown in Fig. 4 and taking the dot product of the Eq. (24) with k leads to:

$$k \cdot \omega_p = k \cdot \omega + k \cdot \omega_z \tag{26}$$

Therefore ω_Z which is the component of the leg's angular velocity along itself is equal to:

$$\omega_z = \frac{k \cdot (\omega_p - \omega)}{k \cdot \hat{s}} \; \hat{s} \tag{27}$$

Thus the angular velocity of the leg is ($\omega_l = \omega_z + \omega$):

$$\omega_l = \left[\frac{k \cdot \omega_p}{k \cdot \hat{z}} + \left(k \cdot \left(\frac{\hat{s} \times \dot{S}}{l} \right) \frac{1}{k \cdot \hat{z}} \right) \right] \hat{s} + \frac{\hat{s} \times \dot{S}}{l} \tag{28}$$

Eq. (28) in the matrix format is:

$$[\omega_l] = [Fpv | Spv] \, [\dot{l} \quad \omega_p]^T \tag{29}$$

where

$$Fpv = \frac{\tilde{\hat{s}}}{l} - \frac{\hat{z}\hat{h}^T}{(\hat{h} \cdot \hat{z})\, l} \tilde{\hat{s}} \tag{30}$$

$$Fsv = \left(\frac{\hat{z}\hat{h}^T}{\hat{h} \cdot \hat{z}} \right) - \frac{\tilde{\hat{s}}\tilde{q}}{l} + \frac{\hat{z}\hat{h}^T}{(\hat{h} \cdot \hat{z})\, l} \left(\tilde{\hat{s}} \tilde{q} \right) \tag{31}$$

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The rule below was used in deriving Eqs. (29-31):

$$s \times m = \tilde{s} m \tag{32}$$

where $\tilde{s}$ is:

$$\tilde{s} = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \tag{33}$$

Utilizing Eqs. (29-30) for the left and right legs, we have:

$$\overbrace{\begin{bmatrix} \omega_{lf} \\ \omega_{lR} \end{bmatrix}}^{W} = \overbrace{\begin{bmatrix} (Fpv)_f \; (Spv)_f \\ (Fpv)_R \; (Spv)_R \end{bmatrix}}^{RM} \overbrace{\begin{bmatrix} \dot{t} \\ \omega_p \end{bmatrix}}^{\dot{P}} \tag{34}$$

where, ω_{lf}, ω_{lR} are the angular velocity of the left and right legs respectively. Also $(Fpv)_f$, $(Fpv)_R$, $(Spv)_f$ and $(Spv)_R$ are calculated from Eqs. (30) and (31) for the right and left legs, respectively. Thus having $\dot{t}, \omega_P$ the angular velocity of each leg is available. Moreover, the angular velocity of each leg in terms of $\dot{\theta}$ is: (as an example for the right leg)

$$(RV)^{-1} ({}^0_B R)_R \, \omega_{LR} = \dot{\theta}_R \tag{35}$$

where

$$RV = \begin{bmatrix} 0 & s\theta_{1R} & c\theta_{1R}c\theta_{2R} \\ 0 & -c\theta_{1R} & s\theta_{1R}c\theta_{2R} \\ 1 & 0 & s\theta_{2R} \end{bmatrix}, \dot{\theta}_R = \begin{bmatrix} \dot{\theta}_{1R} \\ \dot{\theta}_{2R} \\ \dot{\theta}_{3R} \end{bmatrix}, \omega_{LR} = \begin{bmatrix} \omega_{XR} \\ \omega_{YR} \\ \omega_{ZR} \end{bmatrix} \tag{36}$$

and $({}^0_B R)_R$ is the rotation matrix between coordinate 0 of the right leg and the base frame coordinate frame.

VII. Forward Rate Kinematics

In forward rate kinematics, having the active joints velocity, the linear velocity of point G and angular velocity of the moving platform has to be obtained. From Eq. (34) one has:

$$(RM)^{-1} W = \dot{P} \tag{37}$$

If the actuator rates are known, from Eq. (35) the angular velocity of each leg is (for example for the right leg):

$$\omega_{LR} = ((RV)^{-1} ({}^0_B R)_R)^{-1} \dot{\theta}_R$$

Thus W in Eq. (34) is available and Eq. (37) gives angular velocity of moving platform and the linear velocity of the point G. More details for the rate kinematics and also acceleration calculation are available in [4]

VIII. Singularity Analysis

Having the closed form solution of the rate kinematics, the singularity points can be found setting the determinant of the relation matrix equal to zero and infinity. Therefore, two different set of singularity points with different natures will be obtained [5] . However, this method requires the solution of a complicated set of nonlinear equations. Instead, here another approach with the same result but without any need to solve nonlinear equation will be presented.

1) The first kind of singularity occurs when the left or right legs reaches either a boundary of its work space or an internal boundary limiting different sub regions of the workspace. This kind of singularity includes points where different solutions can exist for the inverse kinematics problem. In such cases the mechanism loses one or more degree(s) of freedom(s). To find such points, the Jacobian Matrix for the left and right legs is obtained and then its determinant is set to zero. Thus

$$\det(J) = \ell^2 C\theta_2 C\phi_2 = 0 \tag{37}$$

which leads to:

$$\ell_f = 0 \quad or \quad \ell_R = 0 \tag{38a}$$

$$\theta_{2f} = \pm 90 \quad or \theta_{2R} = \pm 90 \tag{38b}$$

$$\phi_{1f} = \pm 90 \quad or \phi_{2f} = \pm 90 \tag{38c}$$

The schematics of the manipulator corresponding to Eqs. (38a-c) are given in figs. (5a-c), respectively.

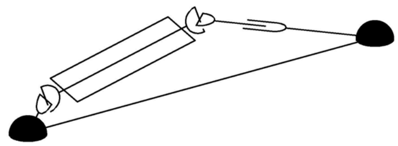

Fig. 5a: Schematic for Eq. (38a) when $\ell_f = 0$

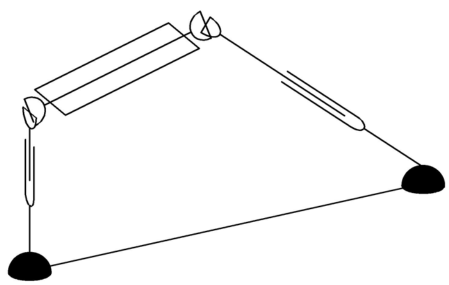

Fig. 5b: Schematic for Eq. (38b) when $\theta_{2f} = 0$

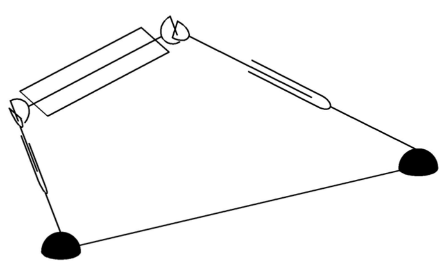

Fig. 5c: Schematic for Eq. (38c) when $\phi_{2f} = 0$

2) The second kind of singularity occurs when the moving platform is movable even when the actuators are locked. These kinds of singularity include points where different solution can exist for the forward kinematics problem. In such a case the mechanism gains one or more degree(s) of freedom(s). In the forward kinematics solution, the direction of the normal vector of the moving platform must be found by the cross product of ${}^{B}(Z_5)_{left}$ and ${}^{B}(Z_5)_{right}$ that is:

$$m = {}^{B}(Z_5)_{left} \times {}^{B}(Z_5)_{right} \tag{37}$$

If ${}^{B}(Z_5)_{left}$ and ${}^{B}(Z_5)_{right}$ are parallel then their cross product will be vanish. So Eq. (9) is always true and the only constraint equation for the two unknowns ℓ_f, ℓ_R is Eq. (10). Since the numbers of the unknowns are more than the number of equations there exist many solutions for ℓ_f, ℓ_R. In this case if the actuated joints locked that are direction of left and right legs are fixed, the platform could take different configuration. A schematic of the second category of singular point for the introduced mechanism is shown in Fig. 6.

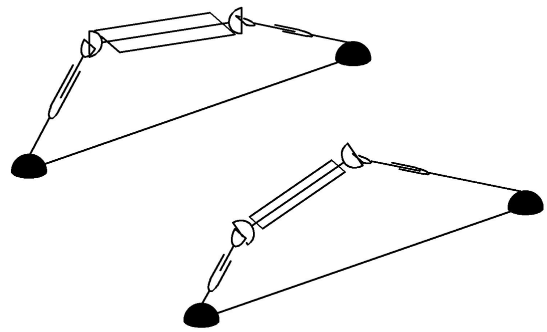

Fig. 6: Schematic of the manipulator correspond to the second category of the singular point

IX. CONCLUSION

In this article forward and inverse pose of the novel Spherically Actuated Manipulator introduced in [1], were presented. In the inverse pose solution, active joint variables were calculated with no need for the evaluation of the passive joint variables. It was shown that there are 16 possible answers for the inverse pose problem. A closed form solution for the forward pose of the mechanism was obtained. Also it was verified that the maximum number of the solutions for the forward problem are four. The rate kinematics problem was solved and a closed form relation was achieved.

Finally, two different sets of singularity points were given. Having the closed form rate kinematics solution, the singularity points could be found by setting the determinant of the Jabocian matrix equal to zero and infinity. By doing so a set of nonlinear equations should be solved. However, in our method the singularity points were derived form another way, which gives the same results, but with no need for solving any nonlinear algebraic equations.

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