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Optimal velocity planning for autonomous vehicles considering curvature constraints

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Abstract— **The paper addresses a velocity planning problem for autonomous vehicles. Appropriate bounds on velocities and accelerations have to be considered in order to avoid wheels skidding and actuators saturation. Planned profiles fulfill an assigned travelling time, while longitudinal jerk is minimized in order to increase the motion smoothness. In the paper it is shown that the velocity planning problem can be formulated as a nonlinear semi-infinite optimization to be solved in real time by means of an appositely devised algorithm characterized by a light computational burden.**

I. INTRODUCTION

The optimal velocity planning problem has been widely investigated in the past both in the ambit of industrial robotics [1]–[7] and mobile robotics [8]. The literature mainly addresses minimum-time velocity planning problems: a time optimal trajectory is planned subject to constraints on the maximum velocity, acceleration, and the acceleration derivative, i.e., the jerk. Sometimes, constraints deriving from dynamic solicitations, such as forces or torques, are also considered. On the contrary, the optimal velocity planning problem with assigned travelling time has been scarcely investigated. In this case, the target is the generation of a velocity profile which fulfills assigned kinematic and/or dynamic constraints and guarantees, at the same time, an exact travelling time and the minimization of an appropriate performance index. The problem is motivated by several applications where trajectory travelling time needs to be imposed. This is the case, e.g., of a robot which must intercept, or avoid, a moving object: any error in the time scheduling will lead to miss the appointment with the moving object or, in case of obstacle avoidance, to an undesired collision. The optimal, assigned-time, planning problem is clearly more complex than the minimum-time problem since it poses feasibility issues which do not appear in the latter one: if the imposed constraints are too restrictive, then the solution could even not exist or the feasible region could be hardly found.

The velocity planning problem considered in this paper can be collocated in such a framework and is motivated by the control strategy proposed in [9]. In that paper a smooth local planner was proposed to move an unicycle-like mobile robot between two arbitrary points assuming arbitrary interpolating conditions. A three steps procedure was proposed: first a smooth path was generated [10], then a smooth velocity profile was evaluated and, finally, the robot command signals were computed by means of an inversion based controller [9]. Path tracking errors, caused, e.g., by wheels skidding, actuator saturations and non modeled phenomena, were compensated by means of a periodical path replanning.

The smooth velocity planning problem has already been analyzed and partially solved in [11], [12]. In particular, in [12] a solution which fulfills assigned constraints on the maximum velocity and acceleration, while minimizing the maximum longitudinal jerk, has been proposed. The bounds on the maximum velocity and acceleration were supposed to be constant along the path. This is realistic only in the case of almost rectilinear paths, otherwise the maximum allowable velocity and acceleration must be correlated to the path curvature. In the paper it will be shown that it is possible to prevent the actuators saturation and the wheels skidding phenomenon by considering such relationship. Consequently, the fixed-time velocity planning problem will be solved by means of a semi-infinite optimization algorithm which minimizes the longitudinal jerk while considering the existence of dynamic constraints. Since the velocity function is periodically updated during the motion, such optimization problem must be particularly efficient in order to provide real-time solutions.

The paper is organized as follows. Section §2 reports some basic notations concerning the theory of planar curves, while the subsequent section proposes the kinematic model of an unicycle-like robot. The proposed notation is then used in §4 to formulate the semi-infinite optimization problem. In §5 kinematic and dynamic bounds are converted into semi-infinite constraints for the problem. A concise description of the optimization algorithm is proposed in §5, while §6 shows the results of a test case. The conclusions section ends the paper.

II. ESSENTIALS ON PLANAR CURVES

The requirements for the synthesis of an optimal velocity function $v(t)$ directly descend from the control technique proposed in [9]: a unicycle mobile robot must move along an assigned planar path of length s_e in a given time t_e. The path is represented by the image in the $\{x,y\}$-plane of a parametric curve $\mathbf{p}(u)$

$$\begin{array}{rcl}\mathbf{p}:[0,1] & \rightarrow & \mathbb{R}^2 \\ u & \rightarrow & [\gamma(u)\ \delta(u)]^T\ .\end{array} \qquad (1)$$

The curve is supposed to be *regular*, i.e., the derivative of $\mathbf{p}(u)$ with respect to u is piecewise continuous, $\dot{\mathbf{p}}(u) \in C_p([0,1])$, and $\dot{\mathbf{p}}(u) \neq \mathbf{0}$,$\forall u \in [0,1]$. If this condition holds,

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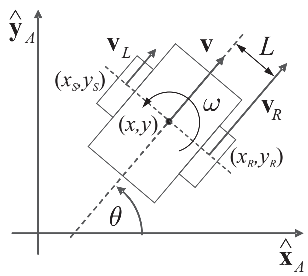

Fig. 1. State variables of an unicycle-like robot.

the curve length s, measured along $\mathbf{p}(u)$, can be expressed by means of a function s_u defined as follows

$$\begin{array}{lcc} s_u : [0,1] & \rightarrow & \mathbb{R}^+ \\ u & \rightarrow & s = \int_0^u \|\dot{\mathbf{p}}(\tau)\| \, d\tau \end{array} . \tag{2}$$

Evidently, for any regular curve $\mathbf{p}(u)$, the length function $s_u(u)$ is continuous over $[0,1]$, monotonically increasing, and, consequently, bijective. Associated with every point of a regular curve $\mathbf{p}(u)$ there is an orthonormal moving frame $\{\boldsymbol{\tau}(u), \boldsymbol{\nu}(u)\}$ that is congruent with the axes of the $\{x,y\}$-plane and where $\boldsymbol{\tau}(u) = \dot{\mathbf{p}}(u)/\|\dot{\mathbf{p}}(u)\|$ denotes the unit tangent vector of $\mathbf{p}(u)$. For any regular curve such that $\ddot{\mathbf{p}}(u) \in C_p([0,1])$, the scalar curvature κ, i.e., the reciprocal of the osculating circle radius, is well defined. It can be evaluated according to the Frenet formula $\frac{d\boldsymbol{\tau}}{ds}(u) = \kappa_u(u)\boldsymbol{\nu}(u)$ (see for example [13, p. 109]), where $\kappa_u(u)$ is the curvature function defined as follows

$$\begin{array}{lcc} \kappa_u : [0,1] & \rightarrow & \mathbb{R} \\ u & \rightarrow & \kappa = \kappa_u(u) \, . \end{array} \tag{3}$$

The scalar curvature can also be expressed as a function of the curve length s, i.e.,

$$\begin{array}{lcc} \kappa_s : [0,s_e] & \rightarrow & \mathbb{R} \\ s & \rightarrow & \kappa = \kappa_s(s) \, . \end{array} \tag{4}$$

Evidently, κ_s and κ_u are correlated since $\kappa_s(s) = \kappa_u[s_u^{-1}(s)]$, and, conversely, $\kappa_u(u) = \kappa_s[s_u(u)]$.

III. The vehicle model

When the wheels slipping phenomenon is avoided, the kinematic model of an unicycle-like vehicle can be represented as follows

$$\begin{array}{c} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega \end{array} \tag{5}$$

where x, y indicate the robot position with respect to a stationary frame, θ is its heading angle, v is the norm of its linear velocity and ω is its angular velocity. The robot linear velocity is perfectly aligned with its symmetry axis and is expressed by $\mathbf{v} = [\dot{x}\,\dot{y}]^T$. Obviously, $v := \|\mathbf{v}\|$. It can be easily proven that there is a direct relationship between v and ω and the wheels velocities norms v_L, v_R. Indeed, it is possible to write

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2L} & -\frac{1}{2L} \end{bmatrix} \begin{bmatrix} v_R \\ v_L \end{bmatrix} \tag{6}$$

where L is the distance between the robot midpoint and each one of its wheels. The transformation matrix is clearly nonsingular thus, evidently, the robot can be indifferently driven by means of v and ω or by means of v_R and v_L.

The curvature of the path generated by an unicycle-like vehicle can be evaluated as

$$\kappa = \frac{\omega}{v} \, . \tag{7}$$

Obviously, each one of the two wheels fulfills the unicycle model so that, e.g., for the right wheel we can correctly write

$$\begin{array}{c} \dot{x}_R = v_R\cos\theta \\ \dot{y}_R = v_R\sin\theta \\ \dot{\theta} = \omega \end{array} \tag{8}$$

where x_R, y_R are the coordinates of the wheel-ground contact point. Note that, necessarily, angular position θ and angular velocity ω are the same of the vehicle midpoint. Due to the analogy between (5) and (8), also for the right wheel it is possible to write

$$\kappa_R = \frac{\omega}{v_R} \tag{9}$$

Analogous expressions apply for the left wheel.

IV. Problem formulation

In [9] it has been pointed out that it is possible to design a smooth robot control by assuming that longitudinal velocity $v(t)$ fulfills some appropriate characteristics. First of all, it must be C^1, i.e., it must be continuously differentiable. Moreover, since $v(t)$ needs to be periodically replanned, the C^1 continuity must also be guaranteed at the replanning times. This result can be achieved by making it possible to arbitrarily impose velocities and accelerations at the boundaries of the planning intervals. Furthermore, for evident physical reasons, it is necessary to impose upper bounds on $v(t)$ and on its first derivative, i.e., the acceleration $a(t)$. Not only, the proposed control technique requires that $v(t) > 0$ for any $t \in (0,t_e)$, so that $v(t) = 0$ is only acceptable for $t = 0$ and $t = t_e$. As a consequence, it is possible to assert that for any $t \in [0,t_e]$, the distance s from the path beginning can also be expressed by means of a monotonic function s_t defined as follows

$$\begin{array}{lcc} s_t : [0,t_e] & \rightarrow & \mathbb{R}^+ \\ t & \rightarrow & s = \int_0^t v(\tau)d\tau \end{array} . \tag{10}$$

Owing to the reasons highlighted in the introduction, it is necessary to plan a $v(t)$ such that path length s_e is exactly travelled in t_e. Finally, since the proposed control technique aims to generate smooth movements, $v(t)$ is planned by minimizing the maximum longitudinal jerk $j(t)$. To this purpose, the following problem was formulated and solved in [12]

Problem 1: Let us assume that a continuously differentiable velocity function is parametrized by means of a vector $\mathbf{h} \in \mathcal{H} \subset \mathbb{R}^n$, so that it can be represented as $v(t;\mathbf{h}) \in C^1([0,t_e]), \forall \mathbf{h} \in \mathcal{H}$. Find the minimizer $\mathbf{h}^*$ which solves the following semi-infinite optimization problem

$$\min_{\mathbf{h}\in\mathcal{H}} \left\{ \max_{t\in[0,t_e]} |j(t;\mathbf{h})| \right\} \tag{11}$$

subject to

$$v(0,\mathbf{h}) = v_0; v(t_e,\mathbf{h}) = v_e; a(0,\mathbf{h}) = a_0; a(t_e,\mathbf{h}) = a_e; \quad (12)$$

$$s_e = \int_0^{t_e} v(\tau;\mathbf{h})d\tau\,; \quad (13)$$

$$0 < v(t;\mathbf{h}) \le \tilde{v}\,, \quad \forall t \in (0,t_e)\,; \quad (14)$$

$$-\tilde{a} \le a(t;\mathbf{h}) \le \tilde{a}\,, \quad \forall t \in [0,t_e]\,. \quad (15)$$

where $\tilde{v}$ and $\tilde{a}$ are known velocity and acceleration bounds.

In [12] it has been shown that, by means of an appropriate choice of the velocity function parametrization, it is possible to convert this problem into a standard optimization problem. In that paper, constraints $\tilde{a}$ and $\tilde{v}$ were supposed to be known and constant. Such hypothesis is acceptable only in the case of almost rectilinear paths, while it must be abandoned in the case of generic paths.

For this reason, the optimization problem is reformulated in this paper in order to obtain a more general solution. It will be shown that, given any regular curve $\mathbf{p}(u)$, the constraints on the maximum velocity and acceleration depend on the point u along the curve which is being considered, so that they will be indicated in the following as $\hat{v}(u)$ and $\hat{a}(u)$. Bearing in mind this hypothesis and the monotonicity of (10), it is possible to propose the following problem

Problem 2: Let us assume that a continuously differentiable velocity function is parametrized by means of a vector $\mathbf{h} \in \mathcal{H} \subset \mathbb{R}^n$, so that it can be represented as $v(t;\mathbf{h}) \in C^1([0,t_e]), \forall \mathbf{h} \in \mathcal{H}$. Find the minimizer $\mathbf{h}^*$ which solves the following semi-infinite optimization problem

$$\min_{\mathbf{h}\in\mathcal{H}} \left\{ \max_{u\in[0,1]} |j(u;\mathbf{h})| \right\} \quad (16)$$

subject to

$$v(0,\mathbf{h}) = v_0; v(t_e,\mathbf{h}) = v_e; a(0,\mathbf{h}) = a_0; a(t_e,\mathbf{h}) = a_e; \quad (17)$$

$$s_e = \int_0^{t_e} v(\tau;\mathbf{h})\,d\tau\,; \quad (18)$$

$$0 < v\left\{s_t^{-1}[s_u(u)];\mathbf{h}\right\} \le \hat{v}(u)\;, \forall u \in (0,1)\,; \quad (19)$$

$$-\hat{a}(u) \le a\left\{s_t^{-1}[s_u(u)];\mathbf{h}\right\} \le \hat{a}(u)\;, \forall u \in [0,1]. \quad (20)$$

The new problem cannot be reformulated as a finite dimensional optimization, so that an appropriate algorithm must be designed for its solution.

V. Velocity and acceleration constraints

It has been earlier pointed out that in (11)–(15) the constraints on the maximum velocities and accelerations were supposed to be known and constant. In the new formulation (16)–(20), velocity and acceleration constraints depend on the position along the curve, so that it is necessary to devise two appropriate functions $\hat{v}(u)$ and $\hat{a}(u)$ on the basis of the physical constraints which characterize the two driving wheels. In particular, it is necessary to guarantee that both wheels never exceed the maximum allowed velocity and acceleration and, furthermore, any slippage with the ground should be strictly avoided in order to improve the control accuracy.

The next three properties are devoted to this target. More precisely, the first one shows that the bounds on wheels velocities can be converted into velocity limits for the vehicle midpoint, while the second one evidences that this is also true for wheels accelerations. The last property considers the slipping avoidance problem.

In the proofs of the following properties the dependency of the involved functions on u will be dropped for conciseness, so that $\hat{v}(u)$ will be simply indicated by $\hat{v}$, $\kappa_u(u)$ by κ, and so on.

Proposition 1: Let us indicate with v_{max} the maximum allowed wheels velocity. Then, for an unicycle-like robot, longitudinal wheels velocities, $v_R(u)$ and $v_L(u)$, satisfy with certainty the two inequalities

$$|v_R(u)| \le v_{max}, \quad |v_L(u)| \le v_{max}\,,$$

if the following condition holds

$$|v(u)| \le \frac{v_{max}}{1+|\kappa(u)|L}\,. \quad (21)$$

Proof: Due to (6) and (7), it is possible to represent v_R and v_L in function of the robot midpoint velocity v and the scalar curvature κ of the path to be followed

$$v_R = v + L\omega = v(1+\kappa L)\,, \quad (22)$$

$$v_L = v - L\omega = v(1-\kappa L)\,. \quad (23)$$

If condition (21) holds, the velocity constraint is satisfied for the right wheel since from (22) it is possible to obtain

$$|v_R| \le |v|\,(1+|\kappa|L) \le \frac{v_{max}}{1+|\kappa|L}(1+|\kappa|L) \le v_{max}\,.$$

The same conclusion can be drawn for v_L by considering (23). ■

Proposition 2: Let us indicate with a_{max} the maximum wheels acceleration. Then, for an unicycle like robot, longitudinal wheels accelerations, $a_R(u)$ and $a_L(u)$, satisfy with certainty the two inequalities

$$|a_R(u)| = |\dot{v}_R(u)| \le a_{max} \quad (24)$$

$$|a_L(u)| = |\dot{v}_L(u)| \le a_{max} \quad (25)$$

if the following conditions hold

$$|v(u)| \le \alpha\sqrt{\frac{a_{max}}{|\kappa'(u)|}} \quad (26)$$

$$|a(u)| \le \frac{a_{max} - |\kappa'(u)|\,v^2(u)}{1+|\kappa(u)|L} \quad (27)$$

where $\alpha \in (0,1)$ and $\kappa'(u) = \frac{d\kappa}{ds}(u)L$.

Proof: In the following it will be proved that the acceleration constraint is satisfied if inequality (27) holds. Equation (26) guarantees that (27) is well posed. Indeed, it is easily possible to prove that, if (26) is fulfilled, then $a_{max} - |\kappa'|\,v^2 > 0$. Passages are omitted for brevity.

Let us now prove that (24) is satisfied owing to (27). By differentiating (22) it is possible to write

$$a_R = \dot{v}_R = \dot{v}(1+\kappa L) + v\dot{\kappa}L\,. \quad (28)$$

Bearing in mind that κ can be expressed, according to (4), as a function of s, its time derivative can be evaluated as follows

$$\dot{\kappa} = \frac{d\kappa}{dt} = \frac{d\kappa}{ds}\frac{ds}{dt} = \frac{d\kappa}{ds}v \quad (29)$$

so that a_R can be rewritten as

$$a_R = \dot{v}(1+\kappa L) + \frac{d\kappa}{ds} L v^2 = a(1+\kappa L) + \kappa' v^2 \,. \tag{30}$$

The last part of the demonstration is straightforward because, if (27) holds, from (30) we immediately get

$$|a_R| \leq |a|\,(1+|\kappa|\,L) + \left|\kappa'\right| v^2 \leq a_{max} \,. \tag{31}$$

With analogous reasonings it is also possible to demonstrate that (25) is fulfilled if (27) holds. Passages are omitted for brevity. ■

In order to avoid the wheel skidding phenomenon, i.e., in order to guarantee a pure rolling motion, it is necessary to verify that the modulus of the forces tangentially transmitted to the ground is smaller than gravity force

$$F_t \leq \mu\, mg \,. \tag{32}$$

Parameter μ represents the adherence coefficient between wheel and ground.

The tangential force applied to the ground is proportional to mass m and to the tangential acceleration. In the case of a pure rolling wheel moving along a path with curvature κ, it is possible to demonstrate that the tangential acceleration is made of two components: the first one is tangent to the path and corresponds to the longitudinal acceleration a, while the second one, which is due to centripetal acceleration, is orthogonal to the path. In the case of nonslipping wheels, this second component is equal to κv^2. As a consequence, the modulus of the total force tangentially applied to the ground can be expressed as

$$F_t = m\sqrt{a^2+\kappa^2 v^4} \,.$$

Owing to this considerations, wheel slipping is avoided if the following condition is satisfied

$$m\sqrt{a^2+\kappa^2 v^4} \leq m\mu g$$

or, equivalently, if

$$a^2+\kappa^2 v^4 \leq \mu^2 g^2 \,.$$

This inequality is essential to devise proper conditions in order to avoid the wheels skidding phenomenon in the case of unicycle like robots.

Proposition 3: The wheels slipping phenomenon is avoided for an unicycle like robot, i.e., the following two inequalities are simultaneously satisfied

$$a_R^2(u) + \kappa_R^2(u) v_R^4(u) \leq \mu^2 g^2 \,, \tag{33}$$

$$a_L^2(u) + \kappa_L^2(u) v_L^4(u) \leq \mu^2 g^2 \,, \tag{34}$$

if its longitudinal velocity $v(u)$ and acceleration $a(u)$ satisfy the following conditions

$$|v(u)| \leq \beta \frac{\sqrt{\mu g}}{\sqrt[4]{|\kappa'(u)|^2 + \kappa^2(u)(1+|\kappa(u)|\,L)^2}} \tag{35}$$

$$|a(u)| \leq \frac{\sqrt{\mu^2 g^2 - \kappa^2(u) v^4(u)(1+|\kappa(u)|\,L)^2} - v^2(u)\,|\kappa'(u)|}{1+|\kappa(u)|\,L} \tag{36}$$

where $\beta \in (0,1)$ and $\kappa'(u) = \frac{d\kappa}{ds}(u) L$.

Proof: Equation (35) ensures that the numerator of (36) is real and positive with certainty and, consequently, (36) is well posed. The obvious passages are omitted for brevity. Equation (36) guarantees that (33) and (34) are verified, so that wheels skidding is avoided.

Let now focus our attention on the acceleration constraint (33). Bearing in mind that

$$\kappa_R = \frac{\omega}{v_R} = \frac{\omega}{v+L\omega} = \frac{1}{\frac{v}{\omega}+L} = \frac{1}{\frac{1}{\kappa}+L} = \frac{\kappa}{1+\kappa L}$$

and taking into account (22) and (30), it is possible to write, for the right wheel,

$$\begin{aligned} a_R^2 + \kappa_R^2 v_R^4 &= [a(1+\kappa L) + v^2\kappa']^2 + \frac{\kappa^2}{(1+\kappa L)^2} v^4 (1+\kappa L)^4 \\ &\leq [|a|\,(1+|\kappa|\,L) + v^2\left|\kappa'\right|]^2 + \kappa^2 v^4 (1+|\kappa|\,L)^2 \,. \end{aligned}$$

Consequently, if (36) is satisfied, it is possible to write

$$|a|\,(1+|\kappa|\,L) \leq \sqrt{\mu^2 g^2 - \kappa^2 v^4 (1+|\kappa|\,L)^2} - v^2\left|\kappa'\right| \tag{37}$$

and immediately conclude, with a few algebraic manipulations, that condition (33) is fulfilled. Analogous considerations can be used to demonstrate that (34) holds. ■

Inequalities (21),(26),(27),(35), and (36) can be used to devise the constraint functions $\hat{v}(u)$ and $\hat{a}(u)$. In particular, $\hat{v}(u)$, $u \in [0,1]$ can be chosen as follows

$$\begin{aligned} \hat{v}(u) \quad := \quad \min\Bigg\{ & \frac{v_{max}}{1+|\kappa(u)|\,L}, \alpha\sqrt{\frac{a_{max}}{|\kappa'(u)|}}, \\ & \beta \frac{\sqrt{\mu g}}{\sqrt[4]{|\kappa'(u)|^2 + \kappa^2(u)(1+|\kappa(u)|\,L)^2}} \Bigg\} \,, \end{aligned} \tag{38}$$

while the following expression can be used for $\hat{a}(u)$, $u \in [0,1]$

$$\begin{aligned} \hat{a}(u) \quad := \quad \min\Bigg\{ & \frac{a_{max} - |\kappa'(u)|\,\hat{v}^2(u)}{1+|\kappa(u)|\,L}, \\ & \frac{\sqrt{\mu^2 g^2 - \kappa^2(u)\hat{v}^4(u)(1+|\kappa(u)|\,L)^2} - \hat{v}^2(u)\,|\kappa'(u)|}{1+|\kappa(u)|\,L} \Bigg\} \,. \end{aligned} \tag{39}$$

The two terms α and β have been introduced to avoid degenerate situations. Let us suppose that $\alpha\sqrt{\frac{a_{max}}{|\kappa'(u)|}}$ is the smallest term in (38). Then, owing to (38), it is possible to write

$$\alpha^2 a_{max} - \hat{v}^2(u)\left|\kappa'(u)\right| = 0 \,,$$

and, since $\alpha \in (0,1)$, it is possible to conclude that, with certainty,

$$a_{max} - \hat{v}^2(u)\left|\kappa'(u)\right| > 0 \,.$$

This implies that the first term in (39) is strictly greater than zero. The same conclusion can be drawn for β and the second term of (39). Practically, α and β are used to exclude the existence of path positions where the robot cannot accelerate because $\hat{a}(u) = 0$. Coefficients α and β represent a trade-off between velocity and acceleration constraints: the more α

and β will be small, the more stringent the velocity constraint and loose the acceleration constraint will be. The selection of α and β influences the optimization problem constraints and, consequently, has an impact on the problem minimizer. When the jerk minimization is not particularly critical, a constant value for α and β can be chosen in the range $[0.5, 0.7]$. A more sophisticated approach can be used in order to obtain better performance indexes, i.e., smaller jerks. It requires to augment the dimension of vector $\mathbf{h}$ by adding two more elements: α and β. In this way, α and β will be directly selected by the minimization program itself while minimizing the maximum jerk.

VI. The Optimization Algorithm

The optimization algorithm must basically satisfy two characteristics: it must be fast and it must guarantee that, at any iteration, a feasible solution is always available. It is executed in real-time, so that execution time must be moderate: this is the reason of the first requirement. The second requirement is also relevant because it guarantees that, even if the velocity updating time is reached before the algorithm has converged to the minimizer, a suboptimal, but feasible, solution is available, so that the robot control is not lost. For this reason, the algorithm immediately finds an initial feasible solution and, then, starts searching the minimizer by passing through a sequence of feasible solutions.

The used algorithm is derived from the one originally proposed in [12] to solve (11)–(15). It is based on an interior point method: first a generic feasible solution is found, then the performance index is improved by searching better solutions in the interior of the feasibility domain. During the first phase, the velocity and acceleration constraints are converted into constant, worst-case constraints by evaluating

$$\tilde{v} := \min_{u \in [0,1]} \{\hat{v}(u)\} \ , \tag{40}$$

$$\tilde{a} := \min_{u \in [0,1]} \{\hat{a}(u)\} \ . \tag{41}$$

In this way, the procedure proposed in [12] to find the initial feasible solution for (11)–(15) can also be used, without any adaptation, for problem (16)–(20).

The procedure used in the second phase has been appositely written for the new problem. The search strategy is the same already proposed in [11], but the routines used to check the solution feasibility have been rewritten in order to take into account that velocity and acceleration constraints are not constant along the path. In particular, as it is usually done in the case of semi-infinite optimization problems, both constraints have been discretized and evaluated for a finite number of path points. The algorithm convergence rate strongly depends on the number of selected points, while is scarcely affected by the path shape. The control technique that we are currently developing considers smooth, short-distance paths, so that the semi-infinite constraints need to be checked in a small number of points: normally ten points are sufficient. Nevertheless, in order to test the algorithm efficiency, this number has been raised to fifty in the example case proposed in the next section.

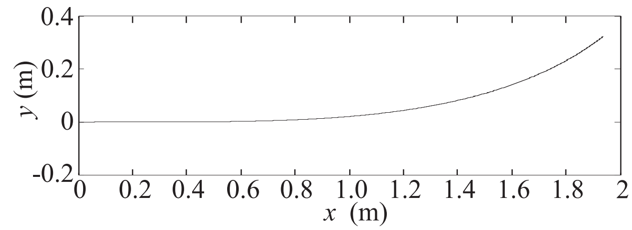

Fig. 2. The cubic spiral curve used for the example case.

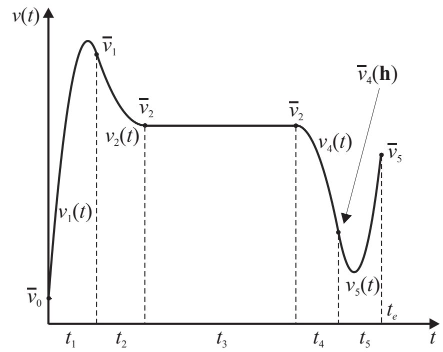

Fig. 4. A possible shape of the velocity profile

VII. An Example Case

Assume that a path has been planned between two points $\mathbf{p}_A := [x_A \ y_A]^T$ and $\mathbf{p}_B := [x_B \ y_B]^T$ in the cartesian space. Moreover, assume that in $\mathbf{p}_A$ and $\mathbf{p}_B$ the assigned interpolating conditions, i.e., tangent angles θ_A and θ_B, curvatures κ_A and κ_B, and curvatures derivatives κ'_A and κ'_B, are compatible with cubic spirals [14]. The path shown in Fig. 2, obtained by means of the interpolating conditions reported in Table I, has been used for the test case. Its total length is $s_e = 2$ m. As known, curvature κ has a parabolic trend along cubic spirals, thus maximum allowable velocity and acceleration cannot be constant along the path but, on the contrary, they must decrease as κ increases.

Velocity function $v(t)$ must guarantee that travelling time is exactly $t_e = 5$ s. Moreover, it must satisfy the following interpolating conditions: $v(0) = 0.4$ m/s, $v(t_e) = 0.2$ m/s, $a(0) = 0.08$ m/s^2, and $a(t_e) = 0.0$ m/s^2. The same parametrization used in [12] is adopted for $v(t)$. It is made of five parabolas planned such to guarantee that boundary conditions are fulfilled and $v(t) \in C^1([0, t_e])$. It is parametrized by means of vector $\mathbf{h} := [t_1 \, t_2 \, t_4 \, t_5 \, \bar{v}_1 \, \bar{v}_2 \, \bar{v}_3]^T$, where t_1, t_2, t_4, and t_5 are the partial travelling times, while $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$ are the velocities at the transit points (see also Fig. 4).

The algorithm first evaluates $\hat{v}(u)$ and $\hat{a}(u)$ by means of (38) and (39). To this purpose, it has been assumed that $v_{max} = 0.6$, $a_{max} = 0.4$, $L = 0.3$, $\alpha = 0.65$, $\beta = 0.65$, $\mu = 1.0$, and $g = 9.8$, while $\kappa(u)$ and $\kappa'(u)$ have been obtained using the cubic spirals equations. Successively, the algorithm evaluates $\tilde{v} = 0.4615$ and $\tilde{a} = 0.2585$ according to (40) and (41). As previously mentioned, these two values are used during the first phase in order to find an initial feasible solution by means of the procedure proposed in [12]. Fig. 3 shows $\tilde{v}$ and $\tilde{a}$ (dash-dotted lines), and the initial feasible

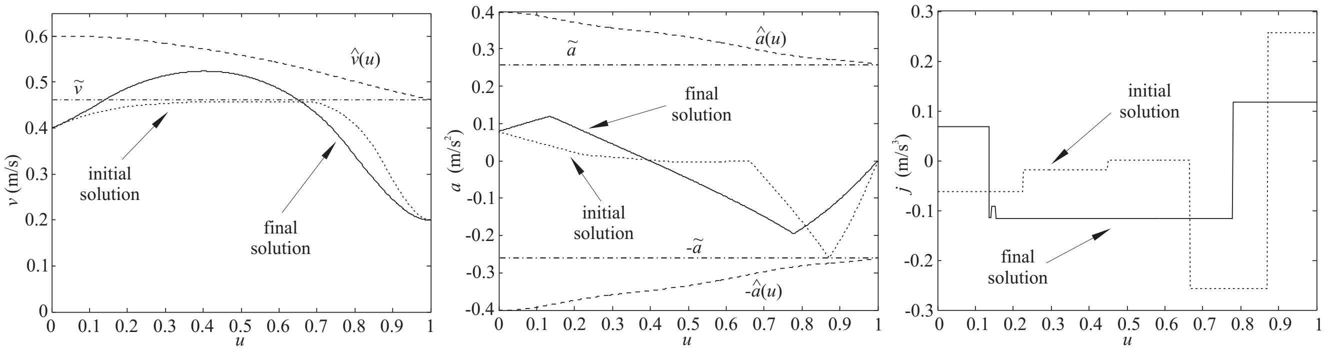

Fig. 3. Velocity, acceleration and jerk shapes for the initial and final solutions.

TABLE I

THE INTERPOLATING CONDITIONS FOR THE CUBIC SPLINE CURVE.

x_A	y_A	x_B	y_B	θ_A	θ_B	κ_A	κ_B	κ'_A	κ'_B
0.0	0.0	1.938	0.324	0.0	0.667	0.0	1.0	0.0	1.0

TABLE II

INITIAL GUESS $\hat{\mathbf{h}}$ AND FINAL MINIMIZER $\mathbf{h}^*$ OF THE OPTIMIZATION PROBLEM.

	t_1	t_2	t_4	t_5	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	j_{max}
$\hat{\mathbf{h}}$	1.0	1.0	1.0	1.0	0.449	0.458	0.458	0.2577
$\mathbf{h}^*$	0.6	0.03	2.65	1.67	0.460	0.464	0.470	0.1162

solution (dotted lines): note that, as required, $v(u) < \tilde{v}$ and $|a(u)| < \tilde{a}$, $\forall u \in [0,1]$. The maximum jerk is $j_{max} = 0.2577$. Then, the second phase starts. During this phase the actual velocity and acceleration constraints, evaluated by means of (38) and (39), are used while seeking the optimal (minimum jerk) solution.

The final result is shown in Fig. 3 by means of solid lines. The velocity $\hat{v}(u)$ and acceleration $\hat{a}(u)$ constraints have been fulfilled, while maximum jerk has been reduced to $j_{max} = 0.1162$. Vector $\hat{\mathbf{h}}$, corresponding to the first step solution, and $\mathbf{h}^*$, corresponding to the final minimizer, are reported in Tab. II.

The optimization algorithm has been executed on a PC Pentium 4, 3.2 GHz, Windows XP OS. It is worth noting that, in the case of the proposed example, the algorithm has converged in 2.625e-2 s. Similar convergence times have been obtained with other curves and interpolating conditions, thus proving the algorithm applicability in real-time frameworks.

VIII. CONCLUSIONS

The paper has shown how the velocity planning problem for mobile robots can be formulated and solved as a semi-infinite optimization problem. A realistic planning problem has been considered: planned velocity fulfills assigned kinematic and dynamic constraints which derive from the robot and the path characteristics, while longitudinal jerk is minimized. Differently from other approaches, travelling time is assigned: conditions are provided which guarantee the existence of a feasible solution. The proposed approach, owing to its moderate computational burden and its capability to reach the minimizer through a sequence of suboptimal, but feasible, solutions, is suited to be used in real time applications where the algorithm robustness represents a fundamental issue.

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