2007 IEEE International Conference on
Robotics and Automation
Roma, Italy, 10-14 April 2007

Systematic Isotropy Analysis of Caster Wheeled Mobile Robots with Steering Link Offset Different from Wheel Radius

Sungbok Kim and Ilhwa Jeong

Abstract—**Previous isotropy analysis of a caster wheeled omnidirectional mobile robot (COMR) has been made under the assumption that the steering link offset is equal to the caster wheel radius. Nevertheless, many practical COMRs in use take advantage of the steering link offset different from the wheel radius. This paper presents a systematic isotropy analysis of a fully actuated COMR with variable steering link offset, which can be considered as the generalization of the previous analysis. First, with the characteristic length introduced, the kinematic model of a COMR is obtained based on the orthogonal decomposition of the wheel velocities. Second, the necessary and sufficient conditions for the isotropy of a COMR are examined to categorize three different groups, each of which can be dealt with in a similar way. Third, the isotropy conditions are further explored to identify four different sets of all possible isotropic configurations. Fourth, the expressions of the isotropic characteristic length required for the isotropy of a COMR are obtained.**

Index Terms—**Caster wheeled mobile robot, steering link offset, isotropy analysis, isotropic configuration.**

I. Introduction

There are a variety of mobile robot systems having different mobility structures: wheeled, legged, wheel-leg hybrid, tracked, and so on. Among them, wheels are widely accepted as a practical means due to the simplicity in design and control, specially for indoor applications. When a mobile robot is requested to navigate in an environment restricted in space and cluttered with obstacles, the omnidirectional mobility becomes a must. Several omnidirectional wheel mechanisms have been proposed, including universal wheels, Swedish wheels, orthogonal wheels, and ball wheels. Caster wheels were employed to develop an omnidirectional mobile robot [1], which was later commercialized as Nomadic Technologies XR4000. Since caster wheels operate without additional peripheral rollers or support structure, a caster wheeled omnidirectional mobile robot (COMR) can maintain good performance even though payload or ground condition changes.

There have been several works on the kinematics of a COMR. For a general form of wheeled mobile robots, a systematic procedure for kinematic modeling was presented [2], [3]. Regarding the minimal actuation set, it was shown that at least four joints out of two caster wheels should be actuated to avoid the singularity [4]. For a COMR under partial and full actuation, the isotropy analysis was made to identify all possible isotropic configurations [5]. The isotropy index of a COMR was defined and examined to determine the optimal design parameters [6], [7]. On the other hand, for an omnidirectional mobile robot employing Swedish wheels, the isotropy analysis was made but the results are incomplete and need further elaboration [8].

With nonzero steering link offset, the omnidirectional mobility of a COMR is always guaranteed independently of its wheel configurations. In contrast, a COMR may fall into the singularity or the isotropy depending on a given wheel configuration. At singular configurations, a COMR becomes instantaneously movable even when all the actuated joints are locked [9]. On the other hand, at isotropic configurations, the velocity transmission from the joint to the task spaces becomes uniform in all directions [10]. Obviously, it is desirable for better motion control to keep a COMR away from the singularity but close to the isotropy, as much as possible [8].

Previous isotropy analysis has been made only for a COMR in which the steering link offset is equal to the wheel radius [5]-[7]. It was found that such a restriction is necessary to have globally optimal isotropic characteristics of a COMR [6], [7]. Nevertheless, many practical COMR's in use take advantage of the steering link offset which is different from the wheel radius, mainly for improved tipover stability [11], [12]. The tipover stability becomes a critical issue when a COMR makes a rapid turn or external forces are applied to a COMR suddenly. The purpose of this paper is to present a systematic isotropy analysis of a fully actuated COMR with the steering link offset different from the wheel radius, which can be considered as the generalization of the previous analysis. The key of the systematic analysis is to deliberately incorporate the ratio of the steering link length to the wheel radius in the development of the isotropy conditions.

This paper is organized as follows. Section II obtains the kinematic model of a COMR based on the orthogonal decomposition of the wheel velocities. Sections III examines the necessary and sufficient conditions for the isotropy of a COMR to categorize three different groups, each of which can be treated in a similar way. Section IV explores the isotropy conditions to identify four different sets of all possible isotropic configurations. Section V obtains the expressions of the isotropic characteristic length required for the isotropy of a COMR. Finally, the conclusion is made in Section VI.

II. Kinematic Model

Consider a COMR with three identical caster wheels attached to a regular triangular platform moving on the

The authors are with the School of Electronics and Information Engineering, Hankuk University of Foreign Studies, Yongin-si, Kyungki-do 449-791, Korea (e-mail: {sbkim, ihgenius}@hufs.ac.kr).

xy-plane, as shown in Fig. 1. For each wheel, it is assumed that the steering link offset can be different from the wheel radius.

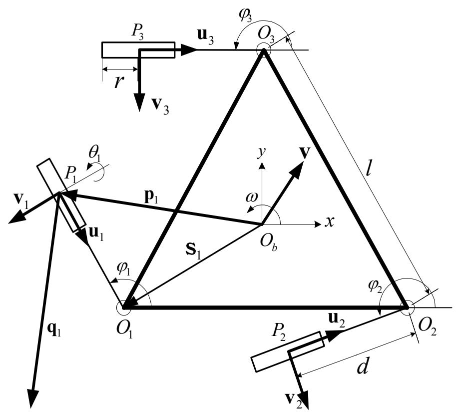

Fig. 1. A caster wheeled omnidirectional mobile robot.

For a COMR shown in Fig. 1, let l be the side length of the platform with the center denoted by O_b, and the vertices denoted by O_i, $i=1,2,3$. Without loss of generality, the side length is assumed to be unity, that is, $l=1.0$ $[m]$. For the i th caster wheel with the center denoted by P_i, $i=1,2,3$, we define the following: Let d (≥ 0) and r (>0) be the steering link offset and the wheel radius, respectively. And, let φ_i and θ_i be the steering and the rotating angles, respectively.

To describe the wheel configuration of a COMR, we define the following vector quantities:

$$\mathbf{u}_i=\begin{bmatrix}-\cos\varphi_i\\-\sin\varphi_i\end{bmatrix},\quad \mathbf{v}_i=\begin{bmatrix}-\sin\varphi_i\\\cos\varphi_i\end{bmatrix}=\mathbf{R}\,\mathbf{u}_i \tag{1}$$

$$\mathbf{s}_1=\frac{1}{\sqrt{3}}\begin{bmatrix}-\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{bmatrix},\quad \mathbf{s}_2=\frac{1}{\sqrt{3}}\begin{bmatrix}\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{bmatrix},\quad \mathbf{s}_3=\frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\end{bmatrix} \tag{2}$$

$$\mathbf{p}_i=\mathbf{s}_i-d\,\mathbf{u}_i,\quad \mathbf{q}_i=\mathbf{R}\,\mathbf{p}_i \tag{3}$$

where $\mathbf{R}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$. For later use, note that

$$\mathbf{u}_i\,\mathbf{u}_i^t+\mathbf{v}_i\,\mathbf{v}_i^t=\mathbf{I}_2 \tag{4}$$

$$\sum_1^3\mathbf{u}_i=0\ \Leftrightarrow\ \sum_1^3\mathbf{v}_i=0 \tag{5}$$

$$\sum_1^3\mathbf{s}_i=0 \tag{6}$$

$$\sum_1^3\mathbf{p}_i=0\ \Leftrightarrow\ \sum_1^3\mathbf{q}_i=0 \tag{7}$$

where $\mathbf{I}_n$ is the $n\times n$ identity matrix and 0 is the 2×1 zero vector.

Let $\mathbf{v}$ and ω be the linear and the angular velocities at O_b of the platform, respectively. For the i th caster wheel, $i=1,2,3$, the linear velocity at the point of contact with the ground can be expressed by

$$\mathbf{v}+\omega\,\mathbf{q}_i=r\,\dot{\theta}_i\,\mathbf{u}_i+d\,\dot{\varphi}_i\,\mathbf{v}_i. \tag{8}$$

Premultiplied by $\mathbf{u}_i^t$ and $\mathbf{v}_i^t$, respectively, we have

$$\mathbf{u}_i^t\,\mathbf{v}+\mathbf{u}_i^t\,\mathbf{q}_i\,\omega=r\,\dot{\theta}_i \tag{9}$$

$$\mathbf{v}_i^t\,\mathbf{v}+\mathbf{v}_i^t\,\mathbf{q}_i\,\omega=d\,\dot{\varphi}_i. \tag{10}$$

Notice that the instantaneous motion of the wheel is decomposed into two orthogonal components of the rotating and the steering joints. Using (3), the expressions of $\mathbf{u}_i^t\,\mathbf{q}_i$ and $\mathbf{v}_i^t\,\mathbf{q}_i$, $i=1,2,3$, can be written as

$$\mathbf{u}_i^t\,\mathbf{q}_i=\mathbf{v}_i^t\,\mathbf{p}_i=\mathbf{v}_i^t\,\mathbf{s}_i \tag{11}$$

$$\mathbf{v}_i^t\,\mathbf{q}_i=-\mathbf{u}_i^t\,\mathbf{p}_i=-\mathbf{u}_i^t\,\mathbf{s}_i+d. \tag{12}$$

With the introduction of the characteristic length [8], L (>0), the kinematic model of a COMR under full actuation is obtained by

$$\mathbf{A}\,\dot{\mathbf{x}}=\mathbf{B}\,\dot{\boldsymbol{\Theta}} \tag{13}$$

where $\dot{\mathbf{x}}=[\,\mathbf{v}\ \ L\omega\,]^t\in\mathbf{R}^{3\times 1}$ is the task velocity vector, and $\dot{\boldsymbol{\Theta}}=[\,\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3\ \dot{\varphi}_1\ \dot{\varphi}_2\ \dot{\varphi}_3\,]^t\in\mathbf{R}^{6\times 1}$ is the joint velocity vector, and

$$\mathbf{A}=\begin{bmatrix}\mathbf{u}_1^t & \frac{1}{L}\,\mathbf{u}_1^t\,\mathbf{q}_1\\ \mathbf{u}_2^t & \frac{1}{L}\,\mathbf{u}_2^t\,\mathbf{q}_2\\ \mathbf{u}_3^t & \frac{1}{L}\,\mathbf{u}_3^t\,\mathbf{q}_3\\ \mathbf{v}_1^t & \frac{1}{L}\,\mathbf{v}_1^t\,\mathbf{q}_1\\ \mathbf{v}_2^t & \frac{1}{L}\,\mathbf{v}_2^t\,\mathbf{q}_2\\ \mathbf{v}_3^t & \frac{1}{L}\,\mathbf{v}_3^t\,\mathbf{q}_3\end{bmatrix}\in\mathbf{R}^{6\times 3} \tag{14}$$

$$\mathbf{B}=\begin{bmatrix}r\,\mathbf{I}_3 & 0_3\\ 0_3 & d\,\mathbf{I}_3\end{bmatrix}\in\mathbf{R}^{6\times 6} \tag{15}$$

are the Jacobian matrices. Notice that the introduction of L makes all three columns of $\mathbf{A}$ to be consistent in physical unit.

III. Isotropy Condition

A. Three isotropy conditions

From (13), the inverse kinematics of a COMR is given by

$$\dot{\boldsymbol{\Theta}}=\mathbf{Z}\,\dot{\mathbf{x}} \tag{16}$$

where

$$\mathbf{Z}=\mathbf{B}^{-1}\,\mathbf{A}. \tag{17}$$

Based on (16), the necessary and sufficient condition for the isotropy of a COMR can be expressed as

$$\mathbf{Z}^t\,\mathbf{Z}=\sigma\,\mathbf{I}_3 \tag{18}$$

where

$$\sigma=\frac{3}{2}\left(\frac{1}{r^2}+\frac{1}{d^2}\right). \tag{19}$$

Using (14), (15), (17), and (19), from (18), the three isotropy conditions of a COMR can be obtained as follows:

$$\sum_{i=1}^{3}\left[\mu\,(\mathbf{u}_i\,\mathbf{u}_i^t)+(\mathbf{v}_i\,\mathbf{v}_i^t)\right]=\frac{3}{2}(\mu+1)\,\mathbf{I}_2 \tag{20}$$

$$\sum_{i=1}^{3}\left[\mu(\mathbf{u}_i^t\,\mathbf{q}_i)\,\mathbf{u}_i+(\mathbf{v}_i^t\,\mathbf{q}_i)\,\mathbf{v}_i\right]=0 \tag{21}$$

$$\frac{1}{L^2}\sum_{i=1}^{3}\left[\mu\,(\mathbf{u}_i^t\,\mathbf{q}_i)^2+(\mathbf{v}_i^t\,\mathbf{q}_i)^2\right]=\frac{3}{2}(\mu+1) \tag{22}$$

where

$$\mu = \left(\frac{d}{r}\right)^2 > 0 \tag{23}$$

Note that (23) represents the square of the ratio of the steering link offset d to the wheel radius r. The incorporation of μ is the key to the systematic isotropy analysis of a COMR with the steering link offset different from the wheel radius.

In general, the first and the second isotropy conditions, given by (20) and (21), are a function of the steering joint angles, $(\varphi_1, \varphi_2, \varphi_3)$, from which the isotropic configurations can be identified. With $(\varphi_1, \varphi_2, \varphi_3)$ known, the third isotropy condition, given by (22), determines the specific value of L, which is required for the isotropy, called the isotropic characteristic length, L_{iso}.

B. First isotropy condition

Using (1), the first isotropy condition (20) can be written as

$$\mu(c_1^2 + c_2^2 + c_3^2) + s_1^2 + s_2^2 + s_3^2 = \frac{3}{2}(\mu+1)$$
$$\mu(c_1 s_1 + c_2 s_2 + c_3 s_3) - (c_1 s_1 + c_2 s_2 + c_3 s_3) = 0 \tag{24}$$

where $c_i = \cos(\varphi_i)$ and $s_i = \sin(\varphi_i)$, $i = 1,2,3$. Using trigonometric function formulas, (24) becomes

$$(\mu-1)(1 + \cos 2\hat{\varphi}_2 + \cos 2\hat{\varphi}_3) = 0$$
$$(\mu-1)(\sin 2\hat{\varphi}_2 + \sin 2\hat{\varphi}_3) = 0 \tag{25}$$

where $\hat{\varphi}_2 = \varphi_2 - \varphi_1$ and $\hat{\varphi}_3 = \varphi_3 - \varphi_1$. There are three different groups of the solutions to (25), including $\mu = 1$ and two groups of $(\hat{\varphi}_2, \hat{\varphi}_3)$, as listed in Table I.

TABLE I
TWO GROUPS OF $(\hat{\varphi}_2, \hat{\varphi}_3)$

Group I	Group II	Group III
$\mu=1$	$(-\frac{\pi}{3}, \frac{\pi}{3})$	$(\frac{\pi}{3}, -\frac{\pi}{3})$
	$(-\frac{\pi}{3}, -\frac{2\pi}{3})$	$(-\frac{2\pi}{3}, -\frac{\pi}{3})$
	$(\frac{2\pi}{3}, \frac{\pi}{3})$	$(\frac{\pi}{3}, \frac{2\pi}{3})$
	$(\frac{2\pi}{3}, -\frac{2\pi}{3})$	$(-\frac{2\pi}{3}, \frac{2\pi}{3})$

C. Second isotropy condition

Using (11) and (12), the second isotropy condition (21) can be written as

$$\sum_{i=1}^{3}\left[\mu(\mathbf{v}_i^t \mathbf{s}_i)\mathbf{v}_i + (\mathbf{u}_i^t \mathbf{s}_i)\mathbf{u}_i - d\,\mathbf{u}_i\right] = 0 \tag{26}$$

which is

$$(\mu-1)\sum_{i=1}^{3}(\mathbf{v}_i^t \mathbf{s}_i)\mathbf{v}_i - d\sum_{i=1}^{3}\mathbf{u}_i = 0 \tag{27}$$

where

$$\sum_{i=1}^{3}(\mathbf{u}_i^t \mathbf{s}_i)\mathbf{u}_i + \sum_{i=1}^{3}(\mathbf{v}_i^t \mathbf{s}_i)\mathbf{v}_i = \sum_{i=1}^{3}\mathbf{s}_i = 0 \tag{28}$$

is used. In the next section, for three different groups of the solutions to (25), (27) will be further explored to identify all possible isotropic configurations of a COMR.

IV. ISOTROPIC CONFIGURATION

A. Isotropy analysis for Group I

With $\mu=1$, (27) reduces to

$$\sum_{i=0}^{3}\mathbf{u}_i = 0 \tag{29}$$

which yields

$$\varphi_2 = \varphi_1 + \frac{2\pi}{3}, \quad \varphi_3 = \varphi_1 - \frac{2\pi}{3}$$
$$\varphi_2 = \varphi_1 - \frac{2\pi}{3}, \quad \varphi_3 = \varphi_1 + \frac{2\pi}{3} \tag{30}$$

Equation (30) tells that there are two sets of infinitely many isotropic configurations: $(\varphi_1, \varphi_1 + \frac{2\pi}{3}, \varphi_1 - \frac{2\pi}{3})$ and $(\varphi_1, \varphi_1 - \frac{2\pi}{3}, \varphi_1 + \frac{2\pi}{3})$. Note that $\mu=1$ corresponds to the case of the steering link offset d equal to the wheel radius r [5].

B. Isotropy analysis for Group II

Let us consider the case of $(\hat{\varphi}_2, \hat{\varphi}_3) = (-\frac{\pi}{3}, \frac{\pi}{3})$, for which

$$\mathbf{v}_2^t \mathbf{s}_2 = \mathbf{v}_3^t \mathbf{s}_3 = -\mathbf{v}_1^t \mathbf{s}_1 \tag{31}$$

so that

$$\sum_{i=1}^{3}(\mathbf{v}_i^t \mathbf{s}_i)\mathbf{v}_1 = 0. \tag{32}$$

Plugging (32) into (27), we have

$$\sum_{i=1}^{3}\mathbf{u}_i = 0 \tag{33}$$

which cannot be satisfied. This implies that there does not exist any isotropic configuration when $(\hat{\varphi}_2, \hat{\varphi}_3) = (-\frac{\pi}{3}, \frac{\pi}{3})$. Similar analysis to the above can be made in the cases of $(\hat{\varphi}_2, \hat{\varphi}_3) = (-\frac{\pi}{3}, -\frac{2\pi}{3})$ and $(\frac{2\pi}{3}, \frac{\pi}{3})$, for which there exists no the isotropic configuration.

Finally, let us consider the case of $(\hat{\varphi}_2, \hat{\varphi}_3) = (\frac{2\pi}{3}, -\frac{2\pi}{3})$, for which

$$\mathbf{v}_1^t \mathbf{s}_1 = \mathbf{v}_2^t \mathbf{s}_2 = \mathbf{v}_3^t \mathbf{s}_3 \tag{34}$$

$$\sum_{i=1}^{3}\mathbf{u}_i = \sum_{i=1}^{3}\mathbf{v}_i = 0. \tag{35}$$

With (34) and (35) held, (27) is satisfied independently of the values of the steering link offset d and the ratio μ. This implies that there exist a single set of infinitely many isotropic configurations, $(\varphi_1, \varphi_1 + \frac{2\pi}{3}, \varphi_1 - \frac{2\pi}{3})$.

C. Isotropy analysis for Group III

Let us consider the case of $(\hat{\varphi}_2, \hat{\varphi}_3) = (\frac{\pi}{3}, -\frac{\pi}{3})$, for which

$$\begin{aligned} \mathbf{v}_1{}^t\,\mathbf{s}_1 &= -\frac{1}{2\sqrt{3}}c_1 + \frac{1}{2}s_1 \\ \mathbf{v}_2{}^t\,\mathbf{s}_2 &= -\frac{1}{\sqrt{3}}c_1 \\ \mathbf{v}_3{}^t\,\mathbf{s}_3 &= \frac{1}{2\sqrt{3}}c_1 + \frac{1}{2}s_1. \end{aligned} \tag{36}$$

Plugging (1) and (36) into (27), we obtain

$$(\mu-1)\begin{bmatrix} \frac{3}{4}c_1^2 + \frac{\sqrt{3}}{2}c_1 s_1 - \frac{3}{4}s_1^2 \\ -\frac{\sqrt{3}}{4}c_1^2 + \frac{3}{2}c_1 s_1 + \frac{\sqrt{3}}{4}s_1^2 \end{bmatrix} - d\begin{bmatrix} -2c_1 \\ -2s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{37}$$

which can be expressed in the form of

$$\begin{aligned} A\,(\mu-1) + B\,d &= 0 \\ C\,(\mu-1) + D\,d &= 0 \end{aligned} \tag{38}$$

where

$$\begin{aligned} A &= \frac{3}{4}c_1^2 + \frac{\sqrt{3}}{2}c_1 s_1 - \frac{3}{4}s_1^2, \qquad B = 2c_1 \\ C &= -\frac{\sqrt{3}}{4}c_1^2 + \frac{3}{2}c_1 s_1 + \frac{\sqrt{3}}{4}s_1^2, \qquad D = 2s_1. \end{aligned} \tag{39}$$

For the existence of the solution to (38), it should hold that

$$AD - BC = 0 \tag{40}$$

which is

$$c_1 - \sqrt{3}s_1 = 0 \tag{41}$$

hence

$$\varphi_1 = \frac{\pi}{6},\; -\frac{5\pi}{6}. \tag{42}$$

Plugging (42) into (37), we obtain

$$(\mu-1)c_1^2 + 2c_1 d = 0 \tag{43}$$

or

$$d = \frac{1}{2}(1-\mu)c_1 > 0. \tag{44}$$

From (42) and (44), it follows that

$$\begin{aligned} \varphi_1 &= \frac{\pi}{6}, \quad d = \frac{\sqrt{3}}{4}(1-\mu), \quad \text{if } 0<\mu<1 \\ \varphi_1 &= -\frac{5\pi}{6}, \quad d = \frac{\sqrt{3}}{4}(\mu-1), \quad \text{if } \mu>1 \end{aligned} \tag{45}$$

Equation (45) tells that there exists only a single isotropic configuration depending on the value of the ratio μ, and such an isotropic configuration can be found only for the specific value of the steering link offset d.

Similar analysis to the above can be made for the cases of $(\hat{\varphi}_2, \hat{\varphi}_3) = (-\frac{2\pi}{3}, -\frac{\pi}{3})$ and $(\frac{\pi}{3}, \frac{2\pi}{3})$. In the case of $(\hat{\varphi}_2, \hat{\varphi}_3) = (-\frac{2\pi}{3}, -\frac{\pi}{3})$, we obtain

$$\begin{aligned} \varphi_1 &= -\frac{\pi}{6}, \quad d = \frac{\sqrt{3}}{4}(1-\mu), \quad \text{if } 0<\mu<1 \\ \varphi_1 &= \frac{5\pi}{6}, \quad d = \frac{\sqrt{3}}{4}(\mu-1), \quad \text{if } \mu>1. \end{aligned} \tag{46}$$

And, in the case of $(\hat{\varphi}_2, \hat{\varphi}_3) = (\frac{\pi}{3}, \frac{2\pi}{3})$, we obtain

$$\begin{aligned} \varphi_1 &= \frac{\pi}{2}, \quad d = \frac{\sqrt{3}}{4}(1-\mu), \quad \text{if } 0<\mu<1 \\ \varphi_1 &= -\frac{\pi}{2}, \quad d = \frac{\sqrt{3}}{4}(\mu-1), \quad \text{if } \mu>1. \end{aligned} \tag{47}$$

Finally, let us consider the case of $(\hat{\varphi}_2, \hat{\varphi}_3) = (-\frac{2\pi}{3}, \frac{2\pi}{3})$, for which

$$\begin{aligned} \mathbf{v}_1{}^t\,\mathbf{s}_1 &= -\frac{1}{2\sqrt{3}}c_1 + \frac{1}{2}s_1 \\ \mathbf{v}_2{}^t\,\mathbf{s}_2 &= \frac{1}{\sqrt{3}}c_1 \\ \mathbf{v}_3{}^t\,\mathbf{s}_3 &= -\frac{1}{2\sqrt{3}}c_1 - \frac{1}{2}s_1 \end{aligned} \tag{48}$$

and

$$\sum_{i=1}^{3}\mathbf{u}_i = 0. \tag{49}$$

Plugging (1), (48), and (49) into (27), we have

$$(\mu-1)\begin{bmatrix} \frac{3}{4}c_1^2 + \frac{\sqrt{3}}{2}c_1 s_1 - \frac{3}{4}s_1^2 \\ -\frac{\sqrt{3}}{4}c_1^2 + \frac{3}{2}c_1 s_1 + \frac{\sqrt{3}}{4}s_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{50}$$

There does not exist φ_1 satisfying (50) unless $\mu = 1$, which implies that there is no isotropic configuration.

D. Summary

Seen from (30) and (45)-(47), the existence of the isotropic configurations of a COMR is dependent on the relationship of the ratio μ and the steering link offset d:

$$\mu = 1 \tag{51}$$

$$d = \frac{\sqrt{3}}{4}(1-\mu), \quad \text{if } 0<\mu<1 \tag{52}$$

$$d = \frac{\sqrt{3}}{4}(\mu-1), \quad \text{if } \mu>1. \tag{53}$$

Since μ is an auxiliary parameter introduced to expedite the systematic isotropy analysis, it is better to cast the relationships of μ and d into the relationships of the wheel radius r and d. For a given r, the specific value of d, which is required for the isotropy of a COMR, is called as the *isotropic steering link offset*, d_{iso}.

First, from (23), it is trivial that (51) is equivalent to

$$d_{\text{iso}} = r. \tag{54}$$

Next, plugging (23) into (52), we have

$$\sqrt{3}d^2 + 4r^2 d - \sqrt{3}r^2 = 0 \tag{55}$$

which yields

$$d_{\text{iso}} = \sqrt{r^2 + \left(\frac{2}{\sqrt{3}}r^2\right)^2} - \frac{2}{\sqrt{3}}r^2 < r \tag{56}$$

subject to

$$0 < d_{\text{iso}} < \frac{\sqrt{3}}{4}. \tag{57}$$

Similarly, it can be shown that (53) is equivalent to

$$d_{\text{iso}} = \sqrt{r^2 + \left(\frac{2}{\sqrt{3}}r^2\right)^2} + \frac{2}{\sqrt{3}}r^2 > r \tag{58}$$

subject to

$$d_{\text{iso}} > 0. \tag{59}$$

Note that (54), (56), and (58) are equivalent to (51), (52), and (53), respectively, and for a given wheel radius r, they represent the values of the isotropic steering link offset d_{iso}.

Summarizing the results obtained so far, all possible isotropic configurations, denoted by Θ_{iso}, of a COMR can be categorized into four different sets according to the relationships of the wheel radius r and the isotropic steering link offset d_{iso}. Attached at the end of this paper, Table II lists four different sets of Θ_{iso}, denoted by S1, S2, S3, and S4, and the corresponding value of d_{iso}. It should be noted that S1 places no restriction on d_{iso} unlike the other three sets, S2, S3, and S4.

V. Isotropic Characteristic Length

As discussed in Section III, the isotropy of a COMR can be achieved when three isotropy conditions, given by (20), (21), and (22), are all met. Once the isotropic configuration has been identified under the conditions of (20) and (21), the isotropic characteristic length L_{iso} can be determined under the condition of (22). From (22), we have

$$L_{\text{iso}} = \sqrt{\frac{2}{3}\frac{\sum_{i=1}^{3}\left[\mu(\mathbf{v}_i^t\mathbf{p}_i)^2+(\mathbf{u}_i^t\mathbf{p}_i)^2\right]}{\mu+1}}. \quad (60)$$

Using

$$\begin{aligned}(\mathbf{u}_i^t\mathbf{p}_i)^2 &= (\mathbf{u}_i^t\mathbf{s}_i)^2-2d(\mathbf{u}_i^t\mathbf{s}_i)+d^2\\ (\mathbf{v}_i^t\mathbf{p}_i)^2 &= (\mathbf{v}_i^t\mathbf{s}_i)^2\end{aligned} \quad (61)$$

the expression of (60) can be further elaborated for four different sets of the isotropic configurations listed in Table II. Note that the isotropy of a COMR cannot be achieved unless the characteristic length is chosen as the isotropic characteristic length, that is, $L = L_{\text{iso}}$.

First, consider the case of S1, for which $\Theta_{\text{iso}} = (\varphi_1, \varphi_1+\frac{2\pi}{3}, \varphi_1-\frac{2\pi}{3})$ so that

$$\begin{aligned}\mathbf{u}_i^t\mathbf{s}_i &= \frac{1}{2}c_1+\frac{1}{2\sqrt{3}}s_1\\ \mathbf{v}_i^t\mathbf{s}_i &= -\frac{1}{2\sqrt{3}}c_1+\frac{1}{2}s_1\end{aligned} \quad (62)$$

for $i=1,2,3$. Using (62), (60) can be expressed as

$$L_{\text{iso}} = \sqrt{\frac{2}{\mu+1}\left\{\left(\frac{1}{\sqrt{3}}\sin(\varphi_1-\frac{\pi}{6})\right)^2\mu+\left(\frac{1}{\sqrt{3}}\cos(\varphi_1-\frac{\pi}{6})-d\right)^2\right\}}. \quad (63)$$

Geometrically, (63), corresponds to a kind of weighted norm of the vector $\mathbf{p}_1$. Especially when $\mu=1$, (63) reduces to

$$L_{\text{iso}} = \sqrt{\left(\frac{1}{\sqrt{3}}\sin(\varphi_1-\frac{\pi}{6})\right)^2+\left(\frac{1}{\sqrt{3}}\cos(\varphi_1-\frac{\pi}{6})-d\right)^2} \quad (64)$$

or

$$L_{\text{iso}} = \sqrt{d^2-\frac{2}{\sqrt{3}}\cos(\varphi_1-\frac{\pi}{6})d+\left(\frac{1}{\sqrt{3}}\right)^2} \quad (65)$$

which is simply the Euclidean norm of $\mathbf{p}_1$. Note that

$$L_{\text{iso}} = (\|\mathbf{p}_1\|=\|\mathbf{p}_2\|=\|\mathbf{p}_3\|). \quad (66)$$

Next, consider the case of S2, for which $\mu=1$ and $\Theta_{\text{iso}} = (\varphi_1, \varphi_1-\frac{2\pi}{3}, \varphi_1+\frac{2\pi}{3})$, so that

$$L_{\text{iso}} = \sqrt{\frac{1}{3}\sum_{i=1}^{3}\|\mathbf{p}_i\|^2} \quad (67)$$

and

$$\|\mathbf{p}_i\|^2 = \|\mathbf{s}_i\|^2-2d(\mathbf{u}_i^t\mathbf{s}_i)+d^2 \quad (68)$$

$$\sum_{i=1}^{3}(\mathbf{u}_i^t\mathbf{s}_i) = 0. \quad (69)$$

Using (68) and (69), from (67), we obtain

$$L_{\text{iso}} = \sqrt{d_{\text{iso}}^2+\left(\frac{1}{\sqrt{3}}\right)^2}. \quad (70)$$

Next, consider the isotropic configuration of $\Theta_{\text{opt}} = (\frac{\pi}{6}, \frac{\pi}{2}, -\frac{\pi}{6})$ belonging to S3, for which

$$\begin{aligned}&\mathbf{u}_1^t\mathbf{s}_1=\frac{1}{\sqrt{3}}, \quad \mathbf{u}_2^t\mathbf{s}_2 = \mathbf{u}_3^t\mathbf{s}_3=\frac{1}{2\sqrt{3}}\\ &\mathbf{v}_1^t\mathbf{s}_1=0, \quad \mathbf{v}_2^t\mathbf{s}_2=-\frac{1}{2}, \quad \mathbf{v}_3^t\mathbf{s}_3=\frac{1}{2}.\end{aligned} \quad (71)$$

Using (61) and (71), (61) can be written as

$$L_{\text{iso}} = \sqrt{\frac{2}{\mu+1}\left\{\frac{1}{6}\mu+\left(d_{\text{iso}}^2-\frac{4}{3\sqrt{3}}d_{\text{iso}}+\frac{1}{6}\right)\right\}}. \quad (72)$$

Plugging (52) into (72), we obtain

$$L_{\text{iso}} = \sqrt{\frac{1}{1-\frac{2}{\sqrt{3}}d_{\text{iso}}}\left(d_{\text{iso}}^2-\frac{2}{\sqrt{3}}d_{\text{iso}}+\frac{1}{3}\right)} \quad (73)$$

or

$$L_{\text{iso}} = \frac{1}{\sqrt{1-\frac{2}{\sqrt{3}}d_{\text{iso}}}}\left(\frac{1}{\sqrt{3}}-d_{\text{iso}}\right) \quad (74)$$

subject to $0 < d_{\text{iso}} < \frac{\sqrt{3}}{4}$. It can be shown that the expression of (74) is also valid for the other two isotropic configurations belonging to S3, $\Theta_{\text{iso}} = (-\frac{\pi}{6}, -\frac{5\pi}{6} - \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{5\pi}{6}, -\frac{5\pi}{6})$.

Similar analysis to the above can be made for all three isotropic configurations belonging to S4, and the isotropic characteristic length L_{iso} is obtained by

$$L_{\text{iso}} = \frac{1}{\sqrt{1+\frac{2}{\sqrt{3}}d_{\text{iso}}}}\left(\frac{1}{\sqrt{3}}+d_{\text{iso}}\right) \quad (75)$$

subject to $d_{\text{iso}} > 0$.

It is interesting to investigate the possible relationships existing among four different sets listed in Table II. It is obvious that 1) S1 and S2 are disjoint, 2) S3 and S4 are disjoint, and 3) both S3 and S4 are disjoint with S2 unless $r=0$. To check the relationship between S3 and S1, let us impose the restriction of S3, given by (52), onto S1, given by (63):

$$L_{\text{iso}} = \sqrt{\frac{1}{1-\frac{2}{\sqrt{3}}d_{\text{iso}}}\left\{d_{\text{iso}}^2-\frac{2}{\sqrt{3}}\left(\frac{2}{3}\sin^2(\varphi_1-\frac{\pi}{6})+\cos(\varphi_1-\frac{\pi}{6})\right)d_{\text{iso}}+\frac{1}{3}\right\}}. \quad (76)$$

Comparing (76) with (73), the condition for which S1 and S3 have the same isotropic characteristic length should be

$$\frac{2}{3}\sin^2(\varphi_1-\frac{\pi}{6})+\cos(\varphi_1-\frac{\pi}{6}) = 1 \quad (77)$$

which yields

$$\varphi_1 = \frac{\pi}{6}, \frac{\pi}{2}, -\frac{\pi}{6} \quad (78)$$

which correspond to the values of φ_1 for the isotropic configurations belong to S3. It should be noticed that for a given the isotropic steering link offset d_{iso}, the same value of the isotropic characteristic length L_{iso}, given by (74), is valid for both S1 and S3.

Now, the relationship between S3 and S1 can be stated as follows: For a given wheel radius r, whose d_{iso} and L_{iso} are chosen as (56) and (74), respectively, there exist six isotropic configurations: three belonging to S3, $(\frac{\pi}{6}, \frac{\pi}{2}, -\frac{\pi}{6})$, $(-\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{2})$, and $(\frac{\pi}{2}, \frac{5\pi}{6}, -\frac{5\pi}{6})$, and three belong to S1, $(\frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2})$, $(-\frac{\pi}{6}, \frac{\pi}{2}, -\frac{5\pi}{6})$, and $(\frac{\pi}{2}, -\frac{5\pi}{6}, -\frac{\pi}{6})$.

Similar analysis to the above can be made for the relationship between S4 and S1: For a given wheel radius r, whose values of d_{iso} and L_{iso} are chosen as (58) and (75), respectively, there exist six isotropic configurations: three belonging to S4, $(-\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{5\pi}{6})$, $(\frac{5\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2})$, and $(-\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6})$, and three belong to S1, $(-\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{2})$, $(\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6})$, and $(-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6})$.

VI. Conclusion

In this paper, we presented the systematic isotropy analysis of a fully actuated caster wheeled omnidirectional mobile robot (COMR) with the steering link offset different from the wheel radius. First, with the characteristic length introduced, the kinematic model of a COMR was obtained based on the orthogonal decomposition of the wheel velocities. Second, the necessary and sufficient isotropy conditions of a COMR were examined to categorize three different groups, each of which can be handled in a similar way. Third, the isotropy conditions were further explored to identify four different sets of all possible isotropic configurations. Fourth, the expressions of the isotropic characteristic length required for the isotropy of a COMR were obtained. We hope that the results of this paper help for the optimal design and control of a COMR with variable steering link offset relative to wheel radius.

References

[1] R. Holmberg, "Design and Development for Powered-Caster Holonomic Mobile Robot," *Ph. D. Thesis*, Dept. of Mechanical Eng. Stanford University, 2000.

[2] P. F. Muir and C. P. Neuman, "Kinematic Modeling of Wheeled Mobile Robots," *Jour. of Robotic Systems*, vol. 4, no. 2, pp. 281-340, 1987.

[3] W. Kim, B. Yi, and D. Lim, "Kinematic Modeling of Mobile Robots by Transfer Method of Augmented Generalized Coordinates," *Jour. of Robotic Systems*, vol. 21, no. 6, pp. 301-322, 2004.

[4] G. Campion, G. Bastin, and B. D`Andrea Novel, "Structural Properties and Classification of Kinematic and Dynamic Models of Wheeled Mobile Robots," *IEEE Trans. on Robots and Automations*, vol. 12, no. 1, pp. 47-62, 1996.

[5] S. Kim and H. Kim, "Isotropy Analysis of Caster Wheeled Omnidirectional Mobile Robot," *Proc. of IEEE Int. Conf. on Robotics and Automation*, pp. 3093-3098, 2004.

[6] T. Park, J. Lee, B. Yi, W. Kim, B. You, and S. Oh, "Optimal Design and Actuator Sizing of Redundantly Actuated Omni-directional Mobile Robots," *Proc. of IEEE Int. Conf. on Robotics and Automation*, pp. 732-737, 2002.

[7] S. Kim and B. Moon, "Local and Global Isotropy of Caster Wheeled Omnidirectional Mobile Robot," *Proc. of IEEE Int. Conf. on Robotics and Automation*, pp. 3457-3462, 2005.

[8] S. K. Saha, J. Angeles, and J. Darcovich, "The Design of Kinematically Isotropic Rolling Robots with Omnidirectional Wheels," *Mechanism and Machine Theory*, vol. 30, no. 8, pp. 1127-1137, 1995.

[9] C. Gosselin and J. Angeles, "Singularity Analysis of Closed-Loop Kinematic Chains," *IEEE Trans. on Robots and Automations*, vol. 6, no. 3, pp. 281-290, 1996.

[10] T. Yoshikawa, "Analysis and Control of Robot Manipulators with Redundancy," *Robotics Research*, pp. 735-747, MIT Press, 1984.

[11] R. B. McGhee and A. A. Frank, "On the Stability of Quadruped Creeping Gaits," *Mathematical Biosciences*, vol. 3, no. 3, pp. 331-351, 1968.

[12] E. G. Papadopoulos and D. A. Rey, "A New Measure of Tipover Stability Margin for Mobile Manipulators," *Proc. of IEEE Int. Conf. on Robotics and Automation*, pp. 3111-3116, 1996.

TABLE II
FOUR DIFFERENT SETS OF ALL ISOTROPIC CONFIGURATIONS

Set	Θ_{iso}	d_{iso}	L_{iso}
S1	$(\varphi_1, \varphi_1+\frac{2\pi}{3}, \varphi_1-\frac{2\pi}{3})$	No restriction	$\sqrt{\frac{2}{\mu+1}\left\{\left(\frac{1}{\sqrt{3}}\sin(\varphi_1-\frac{\pi}{6})\right)^2\mu + \left(d-\frac{1}{\sqrt{3}}\cos(\varphi_1-\frac{\pi}{6})\right)^2\right\}}$
S2	$(\varphi_1, \varphi_1-\frac{2\pi}{3}, \varphi_1+\frac{2\pi}{3})$	r	$L_{\text{iso}} = \sqrt{d_{\text{iso}}^2+\frac{1}{3}}$
S3	$(\frac{\pi}{6}, \frac{\pi}{2}, -\frac{\pi}{6})$, $(-\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{5\pi}{6}, -\frac{5\pi}{6})$	$\sqrt{r^2+\frac{4}{3}r^4}-\frac{2}{\sqrt{3}}r^2$	$L_{\text{iso}} = \frac{1}{\sqrt{1-\frac{2}{\sqrt{3}}d_{\text{iso}}}}\left(\frac{1}{\sqrt{3}}-d_{\text{iso}}\right)$
S4	$(-\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{5\pi}{6})$, $(\frac{5\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2})$, $(-\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6})$	$\sqrt{r^2+\frac{4}{3}r^4}+\frac{2}{\sqrt{3}}r^2$	$L_{\text{iso}} = \frac{1}{\sqrt{1+\frac{2}{\sqrt{3}}d_{\text{iso}}}}\left(\frac{1}{\sqrt{3}}+d_{\text{iso}}\right)$