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Positive Span of Force and Torque Components of Four-Fingered Three-Dimensional Force-Closure Grasps

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***Abstract*— We consider a 3D grasping problem. We propose a test of $\mathbb{R}^3$-positive span of force cones and a test of $\mathbb{R}^3$-positive span of their torque set. Our methods consider directly the quadratic force cone without pyramidal linearization of a cone. The conditions can be computed in a constant time and they are used as a heuristic for 3D force closure test. We also show that our proposed heuristic greatly improve the performance of existing grasp testing algorithms.**

I. INTRODUCTION

For a robot to go completely autonomous, it should be able to grasp and manipulate any unknown object efficiently. A best method for such problem is yet to be found. A grasping problem has been studied for a long time (see [1] for a recent survey).A well known notion in the literature is *force closure*. Force closure indicates that a grasp is firm in the sense that any external disturbance to the grasped object can be countered by the grasping hand.

Identify a good grasp can be done using many approaches. One might choose to go completely systematic: defining the object by an algebraic model and then represent the problem as an optimization problem under various conditions using various objective function. This approach receives the most attention recently. For example, Liu *et al.* propose an enumerative method to search for a grasp on a discrete domain [2]. This method is guaranteed to identify the grasp. However, methods in this categories require elaborate computation technique which takes a relatively large amount of time.

Instead of being completely deterministic, one might employ a stochastic approach. For example, Brost *et al.* [3] proposes that a good quality grasp is common such that it is best to randomly generate a few grasps and pick the best one. Statistically, it is shown that one should be able to identify a suitable grasp from a few number of candidate solutions. In this approach, the important of a fast method to check whether the grasp is usable is most essential.

In the past few years, several force closure testing methods are proposed. For example, the ray shooting method of Liu [4], the Q-Distance method of Zhu and Wang [5] and its recent related method in [6], or the use of pseudodistance, such as GJK [7], as in [8]. However, all of these methods require considerable computation time. Introducing heuristic approach is one way to improve performance. For example, the work in [3] introduces a simple heuristic for 3D grasp.

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In this work, we investigate basic components of a 3D grasp: forces and torques that are exerted by contact points. Utilizing the characteristic of these components, we propose an efficient method to test whether forces and torques each separately satisfy necessary condition of force closure. Unlike many other works, our method directly consider the quadratic friction cone without pyramidal linearization. We also empirically show that, by integrating our heuristic into a force closure test, it could substantially reduce the overall time to test a set of grasps.

II. GRASPING BACKGROUND AND NOTATION

A grasp is defined by a set of contact points. Force closure is a property of a grasp, indicating that the grasp can counter any external disturbance to the grasped object. Interaction applying to the object is represented by forces and torques. To represent a force and a torque simultaneously, we coalesce a force $\boldsymbol{f} = (f_x, f_y, f_z)$ and a torque $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$ into a *wrench* $\boldsymbol{w} = (f_x, f_y, f_z, \tau_x, \tau_y, \tau_z)^T \in \mathbb{R}^6$.

We associate a contact point with a set of wrenches that the contact point could exert. The external disturbance is also represented by a single wrench. A grasp is said to achieve force closure when its contact points can produce every wrench in the wrench space ($\mathbb{R}^6$). Canonically, force closure property considers only directions of wrenches while their magnitudes are neglected. Hence, it is usually assumed that the contact can produce unlimited magnitude of force.

To check whether a grasp achieves force closure, we check the wrenches associated to the contact points of the grasp. Given two or more wrenches, we can produce other wrench that is a positive combination of the original wrenches. A set of wrenches achieve force closure when they can produce every wrench encompassing all directions in the space. The term $\mathbb{R}^n$*-positive span* is defined to represent such property.

Definition 2.1: A set of n wrenches $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ *positively spans* $\mathbb{R}^n$ if and only if, for any vector $\boldsymbol{v}$ in $\mathbb{R}^n$, there exists nonnegative constants $\alpha_1, \ldots, \alpha_n$ such that $\boldsymbol{v} = \alpha_1 \boldsymbol{w}_1 + \ldots + \alpha_n \boldsymbol{w}_n$.

Notice that this definition assumes that we can adjust the size of each wrench freely, indicating that we consider only the direction of the wrenches, not their particular size.

Definition 2.2: A wrench set $W = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ achieves force closure property when they positively span $\mathbb{R}^6$.

A set of wrenches that can be exerted by a contact point varies according to the model of the contact. This work assumes a hard contact with Coulomb friction model [9]. A hard contact cannot exert a pure torque. A torque from the contact must be the result of the applied force only. A contact point at $\boldsymbol{p}$ exerting a force $\boldsymbol{f}$ can be represented by a wrench $\boldsymbol{w} = (\boldsymbol{f}, \boldsymbol{p} \times \boldsymbol{f})$. Since the contact point is frictional, the contact point can exert some tangential force without slip. The maximum ratio between the magnitude of tangential force and the magnitude of the force in the normal direction is indicated by the frictional coefficient μ between the object and the contact point. This means that the set of exertable force is define by a *friction cone* where the half-angle θ of which is equal to $\tan^{-1}(\mu)$.

In this work, we make an extensive use of a plane containing the origin in $\mathbb{R}^3$. Let P be a plane in $\mathbb{R}^3$. We can represent P by its normal vector $\boldsymbol{n}$. Formally, $P = \{\boldsymbol{x} | \boldsymbol{x} \cdot \boldsymbol{n} = 0\}$. We say that a vector $\boldsymbol{x}$ is on the positive(negative) side of P when the sign of $\boldsymbol{x} \cdot \boldsymbol{n}$ is positive(negative). Two vectors are said to be on the different sides of P when one of them is on the positive side and the other of them is on the negative side.

III. THE HEURISTIC

The goal of our heuristic is to reduce the overall time to test several grasps. This could be beneficial when we are presented with a discrete set of grasps and to identify grasps that achieve force closure. The heuristic relies on a necessary but not sufficient condition. The heuristic itself is a test of force closure possible to report a fault positive but not a fault negative. The heuristic is required to be computational inexpensive to be useful: it has to perform much faster than the complete test of force closure.

From the definition of positive span, we can observe that if a set of wrenches W positively span $\mathbb{R}^n$, its projection on any subspace $\mathbb{R}^{k<n}$ must also positively span that subspace. However, this condition is only necessary but not sufficient. This observation is the basis of our heuristic which is "If the grasping wrenches do not positively span in any subspace, this grasp does not achieve force closure". Since we consider a wrench which consists of two distinct entities: a force and a torque, it is natural to consider the 3D force subspace and the 3D torque subspace.

An interesting thing about torque is that it varies according to the choice of the origin. However, force closure property is invariant to the choice of the origin. There also exists a special case when the origin coincides with the contact point $\boldsymbol{p}_i$. In this case, every wrench associated with the contact point $\boldsymbol{p}_i$ has zero torque and we can neglect one entire set of torque. In our heuristic, we check the positive spanning of the torque sets at four different origins, each of which is one of the four contact points of the grasp. If any of them renders the torques that do not positively span $\mathbb{R}^3$, we can conclude that the grasp does not positively span $\mathbb{R}^6$.

Our choice introduces five problems in 3D geometry. One is a test of $\mathbb{R}^3$-positive span of four force cones in 3D and the other four problems are tests of $\mathbb{R}^3$-positive span of three torque sets in 3D. As to be shown in Section IV and V, these five subproblems are simple to solve. This makes our heuristic very efficient. In Section VI, it will be empirically shown that introducing our heuristic is beneficial for force closure test of several grasps.

A. Positively Spanning in $\mathbb{R}^3$

Lemma 3.1: Four vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_4$ positively span $\mathbb{R}^3$ when the negative of any of these vectors lies strictly inside a pyramid formed by the other three vectors.

Ding *et al.* provided a proof of a more general version of Lemma 3.1 which can be found in [10].

Lemma 3.1 indicates that if there exists at least one vector that its negative lies inside the cone of the other wrenches, the wrenches positively span $\mathbb{R}^3$. It also indicates that if at least one vector has its negative *not* lies strictly inside the cone, we can immediately conclude that they do not positively span.

Lemma 3.2: Let W be a set of vectors. If there exists a plane P such that every vector in W lies either on P or on the same side of P, then W does not positively span.

Proof: If such plane exists, there is no vector that lies on the other side of P. Hence, a vector on the other side cannot be written as a positive combination of the vector in W. ■

In the upcoming section, we will encounter a need to describe a set of all positive combinations of vectors. We call such a set a *positive span* and denote by $\Psi(W)$ the positive span of W which is defined by $\Psi(W) = \{\sum \alpha_i \boldsymbol{v}_i | \alpha_i \geq 0, \boldsymbol{v}_i \in W\}$. When W does not positively span $\mathbb{R}^n$, $\Psi(W)$ is a convex subspace. We can also represent $\Psi(W)$ by the intersection of several half spaces defined by planes which are tangent to $\Psi(W)$. We call these planes *bounding planes*. We will restrict normal vectors of bounding planes to point inward such that $\boldsymbol{x} \cdot \boldsymbol{n} \geq 0$ for all normal vectors $\boldsymbol{n}$ of bounding planes of $\Psi(W)$.

IV. POSITIVE SPAN OF FORCE COMPONENTS

In this section, we concentrate on the question whether four 3D force cones positively span $\mathbb{R}^3$. Since positively spanning property does not consider the size of vectors, our method deliberately considers forces by their direction only.

The key concept of this method is to maintain a set of force cone and iteratively add another cone. At each step, we check whether the new cone, together with the positive span of the previous cones, positively span $\mathbb{R}^3$. The computation done in each step requires only a manipulation of a direction of vector in 3D. It will be shown that this step is complete and correct.

Since we concern only a direction of a vector, it is sufficient to represent a cone by its axis vector and its half angle. The axis of a force cone is the inward normal vector of the contact point and the half-angle of the cone is computed from the frictional coefficient. Without lost of generality, we assume that half-angle is the same for all cones. However, our method is not limited to such assumption, it can be modified for various half angles.

We define F_i as a force cone associated with the contact point $\boldsymbol{p}_i$ whose unit inward normal is $\boldsymbol{n}_i$. Formally, F_i is equal to $\{\boldsymbol{f} | (\boldsymbol{f} \cdot \boldsymbol{n}_i)/|\boldsymbol{f}| \geq \cos\theta\}$. We also define a negative cone $-F_i = \{-\boldsymbol{f} | \boldsymbol{f} \in F_i\}$ as the cone consisting of the negative member of F_i.

We say that a cone has non-boundary intersection with another object when the intersection is not on the boundary of the cone. The boundary of a force cone is the set $\{\boldsymbol{f} | (\boldsymbol{f} \cdot \boldsymbol{n}_i)/|\boldsymbol{f}| = \cos\theta\}$. We present a necessary and sufficient condition for several force cones to positively span $\mathbb{R}^3$ as follows. The condition is basically an extension of the Lemma 3.1.

Lemma 4.1: Let $F_1, \ldots, F_n$ be force cones of contact point at $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$. These force cones positively span $\mathbb{R}^3$ when there exists a non-boundary intersection between the negative of any cone and the positive span of the other cones.

Proof: Let $-F_1$ be the negative cone having non-boundary intersection with the positive span of $F_2, \ldots, F_n$. Let $\boldsymbol{v}$ be a vector in the non-boundary intersections. When $\boldsymbol{v}$ exists, we can always find three non-coplanar vectors in the positive span set such that $\boldsymbol{v}$ lies strictly inside the pyramid formed by the three vectors. Obviously, $-\boldsymbol{v}$ is a vector in F_1. Hence $\boldsymbol{v}$ is a negative vector lying strictly inside a pyramid of the other three vectors being a member of the positive span set. From Lemma 3.1, these cones positively span $\mathbb{R}^3$. The condition is sufficient.

Next, we consider the necessity of the condition. Let $\mathcal{W} = \Psi(\bigcup_{i=2}^{n} F_i)$. We will show that if $-F_1$ does not contain any wrench that lies strictly inside $\mathcal{W}$, then no vector in $-\mathcal{W}$ can be written as a positive combination of members of $F_1, \ldots, F_n$. Hence, they do not positively span $\mathbb{R}^3$.

Assume that there exists a subset A of F_1 and a subset B of $\mathcal{W}$ such that the positive combination of vectors in A and B is a vector in $-\mathcal{W}$. Since F_1 and $\mathcal{W}$ is a positive span, a positive combination of vectors in A and a positive combination of vectors of B must be a vector in F_1 and a vector in $\mathcal{W}$, respectively. Hence, it is sufficient to show that, there does not exists a vector $\boldsymbol{a} \in F_1$ that its positive combination with any vector in $\mathcal{W}$ is a vector in $-\mathcal{W}$.

Since $-\boldsymbol{a}$ does not lies in $\mathcal{W}$, there exists a bounding plane P of $\mathcal{W}$ such that $-\boldsymbol{a}$ is on the negative side of P. In other words, $\boldsymbol{a}$ lies on the positive side of P. Obviously, any vector in $-\mathcal{W}$ is on the negative side of P. Hence, it is not possible to write any vector in $-\mathcal{W}$ by a positive combination of $\boldsymbol{a}$ and some vector in $\mathcal{W}$. ■

To test four force cones, we pick two cones arbitrarily, says F_i and F_j, and then check whether these two cones positively span $\mathbb{R}^3$. If they do not, we pick another cone, says F_k. From Lemma 4.1, the only possibility that these three cones do positively span $\mathbb{R}^3$ is that $-F_k$ must have non-boundary intersection with $\Psi(F_i \cup F_j)$. Thus, we check for such intersection. If none such intersection exist, then, we take the last cone into account. Again, by Lemma 4.1, we know that these four cones positively span $\mathbb{R}^3$ only when the negative of the last cone has non-boundary intersection with $\Psi(F_i \cup F_j \cup F_k)$. The checking whether a negative cone intersect with positive span of one cone, two cones and three cones are described in Section IV-A, IV-B and IV-C, respectively.

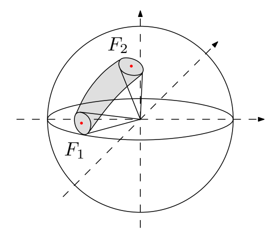


Fig. 1. Two force cones intersect with a unit sphere. The shaded area represents the intersection between the positive span of two cones and the sphere.

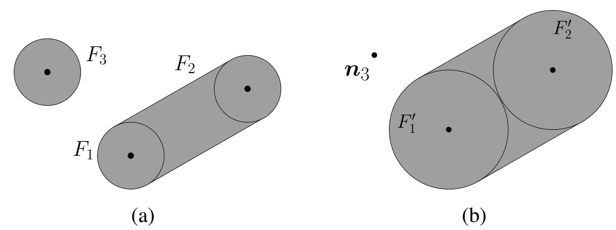


Fig. 2. Force cones as seen on the surface of the unit sphere. The picture does not preserve the linearity. It rather illustrates the are of interest. (a) F_3 and $\Psi(F_1 \cup F_2)$. (b) Enlarging of F_1 and F_2 by the half angle of F_3, i.e., the Minkowski's sum of each cone.

A. $\mathbb{R}^3$-*Positive Span of Two Force Cones*

Lemma 4.2: Two force cones with half-angles θ_1 and θ_2, respectively, positively span $\mathbb{R}^3$ if and only if the angle between the axis of one cone and the negative of the axis of the other cone is smaller than $\theta_1 + \theta_2$.

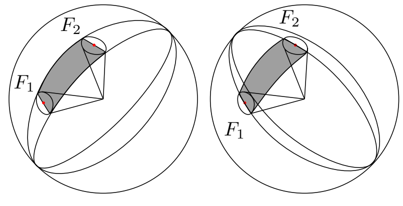
Proof: From the definition of a cone, when the angle between the axis is smaller than $\theta_1 + \theta_2$, we can find a vector $\boldsymbol{x}$ that strictly inside both cones, and vice versa. Since we consider the axis of the negative cone, it means that there exists a vector being a member of one cone and a member of the negative of the other cone concurrently. Hence, these two cones has non-boundary intersection if and only if the condition is satisfied. From Lemma 4.1, The proof is completed. ■

B. $\mathbb{R}^3$-*Positive Span of Three Force Cones*

Given three force cones $F_1, \ldots, F_3$, two of them, namely F_1 and F_2 are known to not positively span $\mathbb{R}^3$. Assume that their half angle are $\theta_1, \ldots, \theta_3$, respectively. Denote by $\boldsymbol{n}_i$ the axis of F_i. The question is whether $-F_3$ has non-boundary intersection with $\Psi(F_1 \cup F_2)$. Observed that the question is equivalent to the question whether a vector $\boldsymbol{n}_3$ lies strictly inside $\Psi(F_1' \cup F_2')$, where F_i' is a the cone F_i the half angle of which is increased by θ_3, the half angle of F_3.

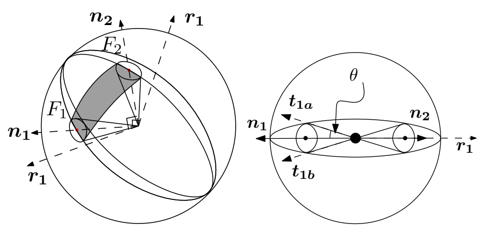
Since we concern only the direction of a force cone, let us represent a force cone by its intersection with a unit sphere. When we look at the surface of the sphere, the intersection of $\Psi(F_1 \cup F_2)$ resembles a racetrack (see Fig. 1). Fig. 2 displays the transformation.

From Fig. 2, a vector $\boldsymbol{n}_3$ is strictly inside a $\Psi(F_1' \cup F_2')$ only when $\boldsymbol{n}_3$ lies strictly inside F_1', F_2', or the area in between, which is represented by a shaded region in the figure. Checking whether a vector is inside a circular cone is merely checking whether angle between the vector and



(a) (b)

Fig. 3. The area where a vector can lies inside $\Psi(F_1' \cup F_2')$. (a) the great circles that passing through the double tangents, i.e., the upper and the lower plane. (b) the great circles that joining the double tangent of the same cone, i.e., the left and the right plane.



(a) (b)

Fig. 4. (a) The normal vectors of the left and right plane. (b) This figure show the cones as viewed along $\boldsymbol{n}_1 + \boldsymbol{n}_2$. The vectors that define the double tangent of F_1 and F_2. The vector t_{1a} and t_{1b} are the results of rotating $\boldsymbol{n}_1$ around $\boldsymbol{r}_1$ by θ and $-\theta$, respectively.

the cone's axis is smaller than the half angle of the cone. The remaining problem is to check whether the vector lies strictly inside the area in between.

The area in between can be represented by the intersection of four half spaces, each of which has its normal vector pointing inward to $\Psi(F_1' \cup F_2')$. Two planes are the planes containing the great circles that are the double tangents of F_1' and F_2' (see Fig. 3(a)). We call these planes the upper and the lower plane. The other two planes are the planes joining the double tangents of the same force cone, we call these planes the left plane and the right plane (also see Fig. 3(b)). Let P be the plane containing $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. Let $\boldsymbol{r}_i \in P$ be the vector that is perpendicular to $\boldsymbol{n}_i$. We restrict $\boldsymbol{r}_1$ to point toward $\boldsymbol{n}_2$ and $\boldsymbol{r}_2$ to point toward $\boldsymbol{n}_1$, i.e., $\boldsymbol{r}_1 \cdot \boldsymbol{n}_2 > 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{n}_1 > 0$. Obviously, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the normal vectors of the left and the right plane, respectively. The vectors on F_1 that define the double tangents to F_2 can be calculated by rotating $\boldsymbol{n}_1$ around $\boldsymbol{r}_1$ by θ_1 and $-\theta_1$. Also, The vectors on F_2 that define the double tangents to F_1 can be calculated by rotating $\boldsymbol{n}_2$ around $\boldsymbol{r}_2$ by θ_2 and $-\theta_2$ (see Fig. 4).

C. $\mathbb{R}^3$-Positive Span of Four Force Cones

This question is just an extension of the previous question. Let $\mathcal{W} = \Psi(F_1 \cup F_2 \cup F_3)$. First, let us consider the intersection of $\mathcal{W}$ with the unit sphere. Again, we extend each cone by the half-angle of F_4, and check whether the axis of F_4 lies inside $\mathcal{W}' = \Psi(F_1' \cup F_2' \cup F_3')$ where F_i' is defined in the same way as in Section IV-B. Fig. 5 illustrates $\mathcal{W}'$ as seen on the surface of the sphere. Obviously, a vector $\boldsymbol{n}_4$ is inside $\mathcal{W}'$ when either it is inside $\Psi(F_1 \cup F_2)$, inside $\Psi(F_2 \cup F_3)$, or inside $\Psi(F_3 \cup F_1)$, or, finally, inside the pyramid defined by the axis of the three cones. The last area

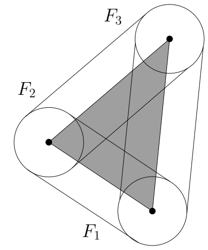


Fig. 5. $\mathcal{W}'$ as seen on the surface of the sphere. The shaded region is the area need to be checked in addition to the other checking that can be done by the previous method.

is illustrated as a shaded region in Fig. 5. The first three zones ($\Psi(F_i \cup F_j)$) can be checked directly by the method described in IV-B. The remaining problem is whether $\boldsymbol{n}_4$ lies strictly inside the pyramid of $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_3$.

Since we know that the three cones do not positively span, we know that $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_3$ inevitably forms a pyramid. We have to calculate the inward normal vector of the three bounding facets of the pyramid. The normal vector of the three faces of the pyramid is an ordered pairwise cross product of $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$. However, we need to know the correct order of the axis. There can be only two distinct orders, namely $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ or $(\boldsymbol{n}_2, \boldsymbol{n}_1, \boldsymbol{n}_3)$. This can be obtained directly by considering the facet that contain $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

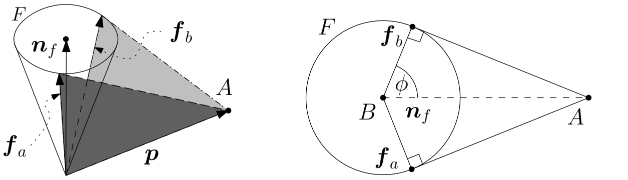
Obviously, the normal vector of the facet that contain $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ is either $\boldsymbol{n}_1 \times \boldsymbol{n}_2$ or $\boldsymbol{n}_2 \times \boldsymbol{n}_1$. The correct one the one that its dot product against $\boldsymbol{n}_3$ is positive, since we know that $\boldsymbol{n}_3$ must be on the inward side.

V. $\mathbb{R}^3$-POSITIVE SPAN OF TORQUE COMPONENTS

Let F be the force cone at the contact point $\boldsymbol{p}$. We denote by T the correspondent torque set of F. Formally, $T = \{\boldsymbol{p} \times \boldsymbol{f} | \boldsymbol{f} \in F\}$. As in the case of force cones, we concern only the direction of the torques. Since $\boldsymbol{p} \times \boldsymbol{f}$ must lies on the plane perpendicular to $\boldsymbol{p}$, T must also lies on the plane. Let us denote such plane by P_p. Given a particular force vector $\boldsymbol{f}$, observe that for all vectors $\boldsymbol{f}'$ lying on the plane containing $\boldsymbol{p}$ and $\boldsymbol{f}$, all the cross products $\boldsymbol{p} \times \boldsymbol{v}'$ lie on the same line. These facts are crucial because it implies that the rank of the torque set is less than 3.

Any plane containing $\boldsymbol{p}$ and intersecting F generates torques which all lie on the same line and no other plane can generate a torque. Let l be the line containing $\boldsymbol{p}$. If l intersects F, every plane containing $\boldsymbol{p}$ always intersects F, hence, T is exactly P_p.

Let us consider the remaining case where l does not intersect F. Let P_f be the plane containing the force $\boldsymbol{f}$ and the position vector $\boldsymbol{p}$. With respect to P_f, $P_f \cap F$ is a fan lying on the same side of l. Hence, every force in $P_f \cap F$ generate torques lying not only in the same line but also in the same direction (sign). Since F is a convex set, all P_f yield torques being a convex set in P_p. Hence, T is a fan in this case. The boundary of the fan is the torque generated



(a) (b)

Fig. 6. (a) The shaded regions represent tangential planes (b) The plane perpendicular to $\boldsymbol{n}_f$ and containing the point $\boldsymbol{q}$. The radius of the boundary of the cone is $r = \tan(\theta)(\boldsymbol{p}\cdot\boldsymbol{n}_f)$. The angle ϕ equals to $\arccos(r/|\overline{AB}|)$.

from two particular P_f that tangentially touch F. Formally,T is a fan $H = \{\alpha\boldsymbol{a} + \beta\boldsymbol{b}|\alpha, \beta \geq 0\}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the boundary vectors of the fan H.

We can identify $\boldsymbol{a}$ and $\boldsymbol{b}$ by identifying the vectors $\boldsymbol{f}_a$ and $\boldsymbol{f}_b$ being on the boundary of F such that the plane containing the $\boldsymbol{p}$ and the vector tangentially touch the cone F (Fig. 6(a)). Let $\boldsymbol{n}_f$ be the axis of F, let A be the point having $\boldsymbol{p}$ as its coordinate. We can calculate $\boldsymbol{f}_a$ and $\boldsymbol{f}_b$ by considering the plane Π perpendicular to $\boldsymbol{n}_f$ and containing the point $\boldsymbol{q}$. Let B be the point where the axis of F intersects Π. Fig. 6(b) describes the calculation of $\boldsymbol{f}_a$ and $\boldsymbol{f}_b$.

A. $\mathbb{R}^3$-*Positive Span of Three Torque Sets*

We have established that a torque set of any contact point is either a fan or a plane. Our problem is to check whether the three torque sets positively span $\mathbb{R}^3$. When any of the torque set is a plane, these sets can positively span $\mathbb{R}^3$ only when there exists two vectors that are on the different sides of that torque set plane. Let $\boldsymbol{n}_t$ be the normal vector of the torque set plane. If the other sets are fans, we check whether the dot products of the boundary vectors with $\boldsymbol{n}_t$ have different signs. If the other sets are planes, we check whether the normal vector of the other plane is not parallel to $\boldsymbol{n}_t$.

The remaining case is when all three torque sets are fans. This is done in the same manner as in the case of three force cones (Section IV-B). First, we check whether two of them positively span $\mathbb{R}^3$. If not, we check whether the negative of the remaining fan has non-boundary intersection with the positive span of the previous two fans.

A fan is a positive span of its boundary vectors. To check whether two fans positively span $\mathbb{R}^3$ is equivalent to checking whether their boundary vectors positively span $\mathbb{R}^3$. There are four boundary vectors. By Lemma 3.1, we can directly use the method checking whether a vector lies inside a pyramid of three vectors, as described in the test of four force cone (Section IV-C).

Now, let us assume that we have checked T_1 and T_2 and they do not positively span $\mathbb{R}^3$. We then consider the third cone T_c. Since T_1 and T_2 are fans, their positive span can only be a half space, a larger fan, a plane, or a pyramid. Each case is considered separately. We denote by P_i the plane containing the fan T_i.

1) The Half Space Case: This case can be detected by verifying that one fan, says T_a, has exactly one of its boundary vectors lying on the plane containing the other fan, says T_b, and the negative of that boundary vector lies strictly inside T_b. In such case, the half space is defined by the plane P_b. These three fans positively span $\mathbb{R}^3$ only when T_c has at least one of its boundary vectors lies outside the half space. This can be identified from the sign of the dot products of the boundary vectors of T_c and the normal vector of P_b.

2) The Plane and Fan Case: When both T_1 and T_2 lie on the same plane, their positive span forms either a fan or a plane. It is a plane when the boundary vectors of T_1 and T_2 positively span the plane. This case is verified by checking whether one fan has its negative boundary vectors lie inside the other fans, and vice versa. In this case, the fans positively span $\mathbb{R}^3$ only when the boundary vectors of T_c lie on the different side of P_1. If $\Psi(T_1 \cup T_2)$ is a fan, we can check whether $\Psi(T_1 \cup T_2)$ and T_3 positively span $\mathbb{R}^3$ by the same method for the case of two fans.

3) The Pyramid Case: The only remaining case is that $\Psi(T_1 \cup T_2)$ is a pyramid. In this case, $-T_c$ has non-boundary intersection with the pyramid only when one of the boundary vectors of $-T_c$ lies strictly inside the pyramid or when $-T_c$ intersects with any facet of the pyramid. A pyramid is represented by an intersection of several half spaces, each of which is described by a bounding plane.

We represent the bounding plane of $\Psi(T_1 \cup T_2)$ by a sequence of vectors $S = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$. Each bounding plane is a plane whose normal vector is the cross product of $\boldsymbol{v}_i$ and $\boldsymbol{v}_{i+1}$ (the last facet is defined by $\boldsymbol{v}_n \times \boldsymbol{v}_1$). Since the pyramid is constructed from two fans, i.e., four boundary vectors, the vector in the sequence S must be the boundary vectors of T_1 and T_2. We have to determine which boundary vectors constitute the sequence and in which order. The requirement for the sequence is that, for every facet, all other vectors in the sequence must be on the same side of that facet.

This problem is equivalent to the computation of a convex hull of two segments (four end points), which can be represented by a sequence of points. The boundary of the convex hull are segments joining the adjacent points in the sequence. Each segment has all other points in the sequence lie on the same side. This is analogous to the requirement of the sequence S. Instead of considering the side of end points with respect to a line, we considering the side of the boundary vectors of one fan with respect to the the plane containing the other fan.

Fig. 7 illustrates all possible cases of the side of end points of two segments. We consider the side of the end point of one segment with respect to the line defined by the other segment. In the first case (Fig. 7(a)), both segments has its end points lie in the same side. The convex hull in this case are the end points of one segment follows by the end points of the other segment. In the second case (Fig. 7(b)), end points of one segment lie in the different sides while the end points of the other segment lie on the same side. One end point of the same-sided segment is removed. Finally in the third case (Fig. 7(c)), both segments have their end points lie in the different side. The sequence of end points is the interwoven list of end points of both segments.

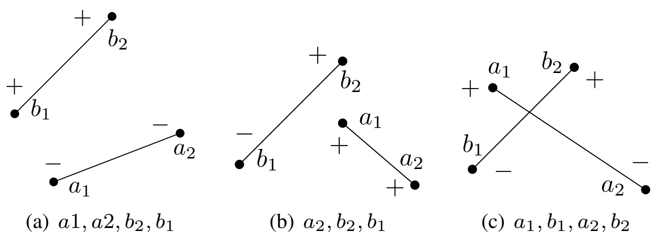


Fig. 7. Calculation of $\Psi(T_1 \cup T_2)$ which is equivalent to the calculation of a 2D convex hull of two lines. Instead of checking whether the other points lie on the same side of the line containing the facet of the 2D convex hull, we check whether the other boundary vectors lie on the same side of the plane containing the facet of a 3D convex hull. The three pictures illustrate all possible configurations of two segments. The caption of each figure is the convex hull of the segments.

VI. NUMERICAL EXAMPLE

The most important concern of any heuristic method is how they compete with a complete method. Our heuristic sacrifices completeness in favor of fast rejection of a negative answer. The obvious question is whether this trade off is beneficial. The complete method which is used as a reference method in our experiment is a point-in-convex-hull assertion algorithm called GJK [7]. To the best of our knowledge, GJK is the fastest way to determine whether the origin is inside the convex hull of the primitive contact wrenches. We use the improved implementation of GJK proposed by [11]. The running time of GJK is believed to be linear in practice. However, it requires the cone be linearized. For GJK, we use a 32-sided pyramids to represent a linear model of a cone. Be noted that our heuristic does not linearize the force cones.

The test suite consists of several sets of 3D grasping configurations. Each grasping configuration is described by four contact positions and their respective inward normal directions. The half angle of a force cone is assumed to be 10 degrees. The grasping configuration is randomly generated from various 3D models illustrated in Fig. 8. For each model, 10^8 grasping configurations are randomly generated. The experiments were run on Pentium Core 2 Duo machine with 1GB memory. The program is implemented in C++.

We compare the actual running time for testing all grasps. The first method use GJK to test all grasps. The other method use our heuristic as a rejection filter. If a grasp is not rejected by our heuristic, we then perform GJK algorithm on the grasp. The result of both experiments are shown in Table I. Our method is labeled as “NEW”. We also count the number the number of false positive solution and the total number of positive answer of our method. The ratios between these number are shown in the last column.

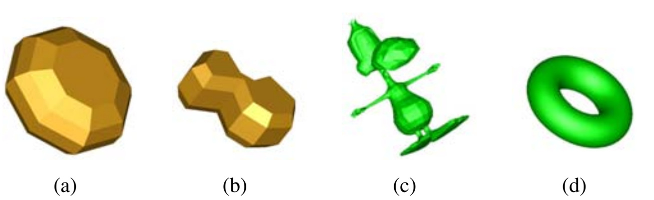


Fig. 8. Test Objects.

TABLE I

RESULT OF THE EXPERIMENT

	Time (s.)			#Solution		
	GJK	NEW	Ratio	Positive	False	Ratio
(a)	161.57	347.98	46.43%	273142	47996	17.57%
(b)	235.34	421.42	55.84%	301762	61751	20.46%
(c)	52.80	187.75	28.12%	48461	26355	54.38%
(d)	216.20	461.59	46.83%	217819	14702	6.74%

VII. CONCLUSIONS

We investigate the nature of the force set and the torque set of a frictional contact point. Specifically, we show that the torque set is either a fan or a plane while the force set is a circular cone. We also provide methods to check whether the force sets of four contact points positively span the force space and whether the torque set of the same contact points positively span the torque space.

The virtue of our method is that it can be computed in a constant time. This speed is obtain from the fact that we consider the actual quadratic force cone without pyramidal linearization. Moreover, we utilize the fact that $\mathbb{R}^n$-positively span is unrelated to the size of the vector. This allow us to derive a method relying on a relative direction of vectors, planes and half space. The computation of such conditions requires only simple calculations.

We propose the use of our method as a fast heuristic to four finger 3D grasping. Our heuristic works as a filter that reject a grasping candidates that cannot positively span $\mathbb{R}^3$. With our heuristic, it is shown that we can reduce the time required to test a large set of grasp configurations.

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