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Dynamics and control of actuated parallel structures as a constrained optimization problem through Gauss' principle and Appell's equations

Georges Le Vey
IRCCyN-UMR CNRS 6597/Ecole des Mines de Nantes
4, rue A. Kastler, F44307 NANTES Cedex 1 - FRANCE
Email: levey@emn.fr

Abstract— **This paper develops a *purely deductive* approach for dynamic analysis of parallel robots, based on consideration of Gauss' principle through Appell's approach. It follows previous works, along these lines, by the author for multibody systems and continuous media. The approach consists in formulating a constrained optimization problem that simultaneously leads to the inverse and direct dynamical models, thereby exhibiting the links between both models. The Stewart-Gough platform appears as a consequence of algebra, a specific example not limiting the generality of the method.**

I. Introduction

More and more attention has been given to the kinematics of parallel manipulators since the 80's [1], [12], [14], [21], for better and more sophisticated designs. On another hand, due to the importance of dynamical models for simulation and control, several methods have been (and continue to be) developed to carry out dynamical analysis of parallel manipulators, in order to achieve good performances in terms of algorithms complexity. The general schemes for this range from the Newton-Euler method [8], [9], [15], [16], to Lagrange equations [19], [5], [20] or the principle of virtual work [28], [27]. For recent and partial bibliographical review of the subject since the 90's, see [28], [27], [15], [16] and references therein. Usually direct and inverse dynamics of mechanisms are viewed and derived separately, each having different computational complexity, depending on mechanism being serial or parallel. In the following developments, use is made of Gauss' least constraint principle through P. Appell's approach [2], to derive simultaneously both dynamics, and establishing the inherent links between them, by considering a convenient constrained optimization problem. The exposition is purposedly given in a rather pedestrian -although intrinsic- way, using the classical language of vector calculus but screw theory is apparent behind this (see e.g. [13], [26] for recent works on screw theory in mechanisms, [13] using the principle of virtual work for dynamical analysis). One distinctive feature of the approach presented hereafter is that it is based on two main ingredients only : the *acceleration energy* (or Gibbs-Appell function) of the system on one hand, making it a kind of *energy based method* -although the Gibbs-Appell function is not homogeneous to an energy- comparable to Lagrange and virtual work methods. On the other hand the *geometric design constraints* translated to the level of accelerations by elementary computations. There is no need of cutting the closed chains as is usually done ([7] e.g. in a similar context), nor computing jacobian matrices explicitely or expliciting inertia forces and wrenches. Everything -direct and inverse dynamics- comes down in a purely deductive way from the formulation as a constrained optimization problem, with connections between both dynamical models coming as a natural byproduct. A nice feature of the approach is that one does not need to go into details of mechanical nature such as e.g. equilibrium between forces and moments that can be cumbersome for complicated mechanisms : only elementary vector calculus is called for, once the two main ingredients are explicited, namely the acceleration energy and the geometric design constraints. All this makes the method rather straightforward.

The following developments are an extension to parallel mechanisms of previous work by the author on serial multibody chains and continuous Newton-Euler algorithms for distributed parameter actuated systems [18], [17]. Whereas these last systems can be considered as, respectively, multistage linear control systems and continuous linear control systems with independent variable, respectively, the label number or the space dimension, a parallel platform is considered, in the same context, as a linear "one-stage" control system. "Linear" must be understood when considering the accelerations (or the efforts) as the "state" of the considered system. Hence the "dynamical" side that motivated the use of optimal control theory in [18], [17] is simply reduced here to a *parameter optimization problem*. The point is that, when viewed in the same *purely deductive* framework, all these apparently different systems can be connected in a very simple way, once each has been modelled for its own. Notice eventually that the Gough-Stewart platform comes down as a logic, algebraic consequence of the method, when all degrees of freedom of the mobile platform are to be controlled. Thus it is seen here as an illustrative and specific example which does not limit the generality of the method.

Concerning Gauss' principle, as mentionned in [7], [25], it has been far less used for dynamics of mechanical systems, when compared to other more familiar principles of analytical mechanics, such as the virtual work principle, d'Alembert principle, Maupertuis least action principle and so forth. Nevertheless, a renewed interest has shown up in recent works : Bruyninckx, Khatib [7] use it in the context of force-controlled redundant robots and hybrid control paradigm to show that the "natural" way to solve the

redundancy problem is to use, in the subsequent optimization problem, the generalized inverse weighted by the inertia matrix of the manipulator. Redon et alt. [25] use Gauss' principle for deriving computationnally efficient algorithms in a motion-space for dynamical frictionless simulation. Sapio, Khatib [10] build upon the framework of [7] to derive a method for controlling a general class of holonomically constrained multibody systems, the constraints coming from the usual method of "breaking the loops" and deriving the dynamics of the resulting branching structures. Besides these recent works inside the robotics community, and apart from textbooks ([4] e.g.), it is fair to mention some important works, dating back a few years to a few decades ago, on Gauss' principle, apart from the old -but important for the developments hereafter- work of P. Appell [2], [3]. In [22], [23], J.J. Moreau has proven that Gauss' principle is valid not only for bilateral constraints but also for unilateral ones. This and other work of J.J. Moreau were used for further developments on nonsmooth mechanics [6]. This same work ([23], p. 156) has to be related to the recent [25] from the point of view of optimization as a *nearest-point problem* (see [6], chap. 5). Also, [23] adopts a variational formulation and explicitely mentions systems with looseness as a possible application of the general results, having thus to do with contact problems as in [25]. Eventually, the recent book of P.J. Rabier and W.C. Rheinboldt [24] takes Gauss' principle as a point of departure for elegant developments of motion equations, in a geometric setting, for rigid bodies subjected to general holonomic as well as nonholonomic constraints, towards efficient algorithms for the DAE form of the motion equation and illustrated by several nice examples.
The present paper has the following features : 1) it is shown that there is no need to "break the loops" for deriving motion equations for parallel structures, contrary to what is usely done : the geometric design leads in a simple and natural way to second-order constraints for the optimization problem, hence avoids introducing artificial constraint forces. For hybrid structures (parallel+series) currently under study, this will prove fruitful. 2) two classical problems are simultaneously treated, namely the direct and inverse dynamical algorithms, which was not done in that setting before, to the best knowledge of the author. One could even solve other types of problems from the obtained necessary conditions, depending on the choice of the unknowns. 3) the Gough-Stewart platform appears, from linear algebra arguments only, as the logical particular case for which the 6 degrees of freedom of the mobile platform are to be controlled.
The material for these developments is organized as follows : after having fixed the notations in section II, section III recalls the essence of Gauss' least constraint principle, following the approach of P. Appell. Then, the so-called *Appell's function* for a parallel manipulator is computed in section IV, completed, in section IV-C by the derivation of geometric constraints due to the design, that are in a second step translated into second order constraints. In section V, a constrained optimization problem is posed and solved, leading at once to both inverse and direct dynamical models. Conclusions and perspective are drawn in section VI.

II. NOTATIONS

The parallel structure is constituted by a fixed basis, a n-polygon $(A_1A_2\ldots A_n)$, connected to a mobile platform, a n-polygon $(B_1B_2\ldots B_n)$, through n legs, numbered 1 to n[1]. Leg numbered i is connected to the basis (resp. platform) at point A_i (resp. B_i) by a universal joint (resp. a rotoïd joint), according to figure 1. Quantities relevant to the platform itself will receive index '0'. As a consequence, when not otherwise mentioned, an index can refer to any leg or the platform itself : the word *body* then refers to any of these. An orthonormal reference frame R_r is chosen on the fixed basis, say at A_1. The following notations are in order :

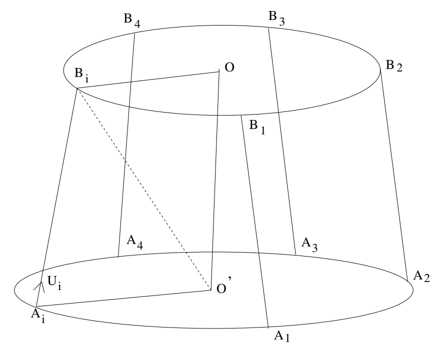

Fig. 1. Geometry of a parallel structure

M_j : Mass of body j ;
G_j : mass center of body j
u_j : unit vector along leg j axis ;
L_j : leg j length.
ω_j : rotation velocity of body j
V_{g_j} : absolute velocity of mass center G_j of body j
I_j : inertia tensor of body j, in a frame parallel to R_r, with origin in A_j
f_j : force, applied along unit vector u_j of leg j, exerted on the mobile platform.
In the sequel, dots over some quantity will indicate differentiation with respect to time. The scalar product of two vectors a and b is noted with a dot as in $a.b$ and their vector product is noted $a \wedge b$.

III. GAUSS' PRINCIPLE AND GIBBS-APPELL EQUATIONS

Gauss' principle is one of the very principles of analytical mechanics, besides the well-known *least action principle* or *Hamilton principle*. All of them cannot be essentially distinct from the principle of virtual velocities together with d'Alembert principle, except possibly for the nature of the bindings. Nevertheless, each can give a different light to given problems. Hence, as Gauss' principle is concerned with accelerations, as shown by P. Appell [3], it appears that there can be an interest in using it for dynamical analysis. For that reason, in what follows, Gauss' least constraint principle, under the formulation of P. Appell, is used to

[1] Purposedly at this stage and until the final derivation of the dynamical models in section V-A, no restriction is put on the number n of legs

pose a constrained optimization problem for convenient data and unknowns of interest for dynamical analysis of parallel manipulators. For the sake of completeness, the approach of [2], [3] (or "Gibbs-Appell" method, although there is no canonical denomination) for deriving motion equations of a mechanical system is quickly recalled here. It is based on the consideration of an "acceleration energy" [11] or "Gibbs-Appell function" instead of kinetic energy that is used classically for deriving Lagrange equations. Notice that the expression "acceleration energy" can be confusing as it is not really homogeneous to an energy such as the kinetic energy. This name, coined for the first time in [11] is likely due to the evident resemblance with the kinetic energy expression. It is worth noting that, as observed by P. Appell, kinetic energy is not sufficient to describe the motion of a mechanical system as e.g. two different such systems, one holonomic and one nonholonomic, can possess the same kinetic energy. On the contrary, the acceleration energy -together with potential energy- of whatever system uniquely determines a specific system. Thus, as Gauss' principle works at the level of accelerations, it is well-suited to dynamical analysis of general mechanical systems, including nonholonomic constraints.

For the time being, in this short introduction, Appell notations [3] are used for the sake of reference to the original papers : S is the acceleration energy of the mechanical system under consideration, q is the configuration parameter (generalized coordinate), Q the vector of applied efforts. Let $\gamma(P,q)$ be the acceleration of particle with mass dm_P located in some point P of the system under consideration. Then : $S = \int \frac{1}{2}|\gamma(P,q)|^2 dm_P$ where the integral extends to the whole system. P. Appell has shown that the motion equations write :

$$\frac{\partial S}{\partial \ddot{q}} = Q \tag{1}$$

But, much more important in the present situation, P. Appell observed that this acceleration energy has strong connections with Gauss' least constraint principle, saying that, "at each time, the motion [of a constrained system] agrees as closely as possible with the free motion ; that is, it occurs under the least constraint where the measure of the constraint, to which the entire system is subjected, is defined as the sum of the products of the mass of each particle with the square of the deviation of that point from its free motion.". More precisely, P. Appell has observed and shown [2] that the motion equations are those obtained when searching for the minimum, with respect to $\ddot{q}$, of what he called the *analytical expression of the constraint* :

$$R = S - Q^T \ddot{q} \tag{2}$$

which is a quadratic function of $\ddot{q}$ and has to be taken in the sense of Gauss' principle. In [23], [17], this function was named *the Appell's function* of the considered system and this denomination will also be used hereafter. The interest in considering the Appell's function is that actuation is easily taken into account through the term $Q^T \ddot{q}$ so that it reduces to S, the acceleration energy, for free systems. In summary, in order to apply Gauss' principle through Appell's approach, one has to compute the Appell's function, i.e. the acceleration energy of the system, on one hand and the contribution of the applied forces on another hand. Last, constraints due to the design are taken into account through the use of Lagrange multipliers.

IV. Appell's function for a parallel manipulator

To compute the Appell's function, one first computes the "acceleration energy", noted E, as simply the sum of the corresponding quantities for each individual body (legs and platform) :

$$E = \sum_{i=0}^{n} E_i \tag{3}$$

and then takes into account the contribution of the applied forces.

A. Acceleration energy of the platform

It is simply computed thanks to a Koenig's theorem for accelerations [3], analogous to the well-known Koenig's theorem for kinetic quantities, as :

$$E_0 = \frac{1}{2}\dot{\omega}_0^T I_0 \dot{\omega}_0 + \frac{1}{2} M_0 \dot{V}_{g_0}^T \dot{V}_{g_0} \tag{4}$$

B. Acceleration energy of the legs

For each leg i, the acceleration $\gamma(D)$ of a point D having abcissa r_i along the unit vector u_i, is easily computed as the second time derivative of $A_i D$:

$$\gamma(D) = \ddot{r}_i u_i + \omega_i \wedge (\dot{r}_i u_i) + \dot{\omega}_i \wedge (r_i u_i) + \omega_i \wedge (\tfrac{d(r_i u_i)}{dt}) \tag{5}$$

Then the acceleration energy of leg i writes :

$$\begin{aligned} E_i &= \textstyle\int_{D \in leg i} \frac{1}{2}\|\gamma(D)\|^2 dm \\ &= \textstyle\int_{D \in leg i} \frac{1}{2}\|\ddot{r}_i u_i + \omega_i \wedge (\dot{r}_i u_i) + \dot{\omega}_i \wedge (r_i u_i) \\ &\quad + \omega_i \wedge (\tfrac{d(r_i u_i)}{dt})\|^2 dm \end{aligned} \tag{6}$$

i.e. a quadratic form in the acceleration variables, which eventually assumes the following form :

$$E_i = \frac{1}{2} M_i \ddot{L}_i^2 + \frac{1}{2}\dot{\omega}_i^T K_i \dot{\omega}_i + \alpha_i \ddot{L}_i + \beta_i^T \dot{\omega}_i + \gamma_i \tag{7}$$

where coefficients $K_i, \alpha_i, \beta_i, \gamma_i$ are not explicited here and depend on the inertia properties of each leg but not on $\ddot{L}_i, \dot{\omega}_i$. Notice also that one has not detailed the way the input forces are introduced but at this stage, this is not needed. Suffice to say that a simple way would be to introduce two colinear bars, sliding along each other, compute their respective acceleration energy and introduce a colinearity constraint (see e.g. [27] for such computations).

C. Taking into account the geometry of the structure

Now, as the quantities of interest in a dynamical analysis are those related to the mobile plaform, it is useful to express variables describing the state of the legs, $\ddot{L}_i, \dot{\omega}_i$, as a function of the accelerations of the plaform, $\dot{V}_{g_0}, \dot{\omega}_0$. This is a matter of standard computation, starting from the geometric closedness conditions for the parallel structure. Looking at figure 1, one has, $\forall i = 1, \ldots n$:

$$\overrightarrow{O'B_i} = \overrightarrow{O'A_i} + \overrightarrow{A_iB_i} = \overrightarrow{O'O} + \overrightarrow{OB_i} \tag{8}$$

A first time differentiation gives the first order kinematical constraints :

$$\dot{L}_i u_i + \omega_i \wedge L_i u_i = V_{g_0} + \omega_0 \wedge b_i \tag{9}$$

and a second one gives the second order kinematical constraints :

$$\begin{aligned} &\ddot{L}_i u_i + 2\dot{L}_i \omega_i \wedge u_i + \dot{\omega}_i \wedge L_i u_i + \omega_i \wedge (\omega_i \wedge L_i u_i) \\ &= \dot{V}_{g_0} + \dot{\omega}_0 \wedge b_i + \omega_0 \wedge (\omega_0 \wedge b_i) \end{aligned} \tag{10}$$

At this stage, one can take two different routes : either explicitly express $\ddot{L}_i, \dot{\omega}_i$ as function of $\dot{V}_{g_0}, \dot{\omega}_0$ (this was done e.g. in [27], using local coordinates) or keep the constraints as such. In the first case one is only interested in the dynamical analysis, i.e. obtaining the forces at the leg as functions of the mobile platform trajectory. In the second case, intermediate quantities such as reaction forces at the legs are of interest, e.g. in a design step. Either situation can be treated as well by the approach. Obviously, this is the question of keeping the constraints explicit or using them to eliminate superfluous variables. In the present work, the choice has been made to keep the constraints explicit and not to substitute for the values of $\ddot{L}_i$ and $\dot{\omega}_i$. A first observation is that, due to the nature of the joint between the fixed basis and the legs (universal joint), the rotation ω_i is orthogonal to the unit vector u_i thus : $\omega_i.u_i = 0$. Then, scalar multiplication of the kinematical constraints by the unit vector u_i gives respectively :

$$\dot{L}_i = u_i.(V_{g_0} + \omega_0 \wedge b_i) \tag{11}$$

at the first order and :

$$\ddot{L}_i = u_i.(\dot{V}_{g_0} + \dot{\omega}_0 \wedge b_i + \omega_0 \wedge (\omega_0 \wedge b_i)) - L_i \omega_i.\omega_i \tag{12}$$

at the second order. In a second step, vector multiplication of these kinematical constraints by u_i gives respectively :

$$L_i \omega_i = u_i \wedge (V_{g_0} + \omega_0 \wedge b_i) \tag{13}$$

and :

$$2\dot{L}_i \omega_i + L_i \dot{\omega}_i = u_i \wedge (\dot{V}_{g_0} + \dot{\omega}_0 \wedge b_i + \omega_0 \wedge (\omega_0 \wedge b_i)) \tag{14}$$

D. Contribution of the applied forces

For the sake of simplicity, gravity is assumed to vanish here. Thus, the only applied efforts are the control forces at the actuated legs. Denoting their intensity by f_i, and noticing that they are directed along the unit vector u_i, their contribution to the Appell's function is $(f_i u_i).(\ddot{L}_i u_i) = f_i \ddot{L}_i$.

E. Appell's function

The Appell's function, R, can now be expressed from the above results :

$$\begin{aligned} R = &\tfrac{1}{2} M_0 \dot{V}_{g_0}^T \dot{V}_{g_0} + \tfrac{1}{2} \dot{\omega}_0^T I_0 \dot{\omega}_0 + \textstyle\sum_{i=1}^n \tfrac{1}{2} M_i \ddot{L}_i^2 \\ &+ \tfrac{1}{2} \dot{\omega}_i^T K_i \dot{\omega}_i + \alpha_i \ddot{L}_i + \beta_i^T \dot{\omega}_i + \gamma_i - f_i \ddot{L}_i \end{aligned} \tag{15}$$

where :

$$\begin{aligned} 2\dot{L}_i \omega_i + L_i \dot{\omega}_i &= u_i \wedge (\dot{V}_{g_0} + \dot{\omega}_0 \wedge b_i + \omega_0 \wedge (\omega_0 \wedge b_i)) \\ \ddot{L}_i &= u_i.(\dot{V}_{g_0} + \dot{\omega}_0 \wedge b_i + \omega_0 \wedge (\omega_0 \wedge b_i)) - L_i \omega_i.\omega_i \end{aligned} \tag{16}$$

are either considered as intermediate quantities, functions of $\dot{V}_{g_0}, \dot{\omega}_0$, or as constraints.

V. Inverse and direct dynamics

Recall that Gauss' least constraint principle, as interpreted by P. Appell [3], stipulates that motion equations of a system of bodies are such that the Appell's function must be minimum with respect to the accelerations. Thus, remembering the computations of the previous section, one is led in a natural way to consider the following optimization problem :

Optimization problem

Find the minimum, with respect to the accelerations, of the quadratic form :

$$\begin{aligned} R = &\tfrac{1}{2} M_0 \dot{V}_{g_0}^T \dot{V}_{g_0} + \tfrac{1}{2} \dot{\omega}_0^T I_0 \dot{\omega}_0 + \textstyle\sum_{i=1}^n \tfrac{1}{2} M_i \ddot{L}_i^2 \\ &+ \tfrac{1}{2} \dot{\omega}_i^T K_i \dot{\omega}_i + \alpha_i \ddot{L}_i + \beta_i^T \dot{\omega}_i + \gamma_i - f_i \ddot{L}_i \end{aligned} \tag{17}$$

subject to the linear constraints :

$$\ddot{L}_i = u_i.(\dot{V}_{g_0} + \dot{\omega}_0 \wedge b_i + \omega_0 \wedge (\omega_0 \wedge b_i)) - L_i \omega_i.\omega_i \tag{18}$$

$$2\dot{L}_i \omega_i + L_i \dot{\omega}_i = u_i \wedge (\dot{V}_{g_0} + \dot{\omega}_0 \wedge b_i + \omega_0 \wedge (\omega_0 \wedge b_i)) \tag{19}$$

As mentioned above, two different routes can be taken to solve this problem : either substitute for $\ddot{L}_i, \dot{\omega}_i$ in R, thanks to equations (18), (19), and write necessary conditions for $\dot{V}_{g_0}$ and $\dot{\omega}_0$; or consider equations (18), (19) as constraints to which all the variables, $\dot{V}_{g_0}$, $\dot{\omega}_0$, $\ddot{L}_i$, $\dot{\omega}_i$ are subjected. In the following, this second approach is taken, as substitution is rather tricky and leads to complicated expressions, whereas the second way keeps things untangled. To this end, first consider the *Lagrange function* of the problem by adjoining the constraints to the Appell's function through the use of

Lagrange multipliers [2], $\lambda_i \in \mathbb{R}$, $\mu_i \in \mathbb{R}^3$:

$$\begin{aligned} H &= \tfrac{1}{2}M_0\dot{V}_{g_0}^T\dot{V}_{g_0} + \tfrac{1}{2}\dot{\omega}_0^T I_0\dot{\omega}_0 + \textstyle\sum_{i=1}^n\{\tfrac{1}{2}M_i\ddot{L}_i^2 \\ &+\tfrac{1}{2}\dot{\omega}_i^T K_i\dot{\omega}_i + \alpha_i\ddot{L}_i + \beta_i.\dot{\omega}_i + \gamma_i - f_i\ddot{L}_i \\ &+\lambda_i(\ddot{L}_i - u_i.(\dot{V}_{g_0} + \dot{\omega}_0\wedge b_i + \omega_0\wedge(\omega_0\wedge b_i)) \\ &+L_i\omega_i.\omega_i) + \mu_i.(2\dot{L}_i\omega_i + L_i\dot{\omega}_i \\ &-u_i\wedge(\dot{V}_{g_0} + \dot{\omega}_0\wedge b_i + \omega_0\wedge(\omega_0\wedge b_i)))\} \end{aligned} \quad (20)$$

Then the necessary conditions, which are also sufficient thanks to the fact that R is quadratic in the accelerations, and positive definite, are (with an evident abuse of notation) :

$$\begin{aligned} &\tfrac{\partial H}{\partial \dot{V}_{g_0}} = \tfrac{\partial H}{\partial \dot{\omega}_0} = 0 \\ &\tfrac{\partial H}{\partial \ddot{L}_i} = \tfrac{\partial H}{\partial \dot{\omega}_i} = \tfrac{\partial H}{\partial \lambda_i} = \tfrac{\partial H}{\partial \mu_i} = 0\,,\ \forall i = 1,\dots,n \end{aligned} \quad (21)$$

which are explicited as :

$$\frac{\partial H}{\partial \dot{V}_{g_0}} = M_0\dot{V}_{g_0} + \sum_{i=1}^n(u_i\wedge\mu_i - \lambda_i u_i) = 0 \quad (22)$$

$$\frac{\partial H}{\partial \dot{\omega}_0} = I_0\dot{\omega}_0 + \sum_{i=1}^n(b_i.\mu_i)u_i - (b_i.u_i)\mu_i + \lambda_i(u_i\wedge b_i) = 0 \quad (23)$$

$$\frac{\partial H}{\partial \ddot{L}_i} = M_i\ddot{L}_i + \alpha_i - f_i + \lambda_i = 0 \quad (24)$$

$$\frac{\partial H}{\partial \dot{\omega}_i} = K_i\dot{\omega}_i + \beta_i + L_i\mu_i = 0 \quad (25)$$

$$\frac{\partial H}{\partial \lambda_i} = \ddot{L}_i - u_i.(\dot{V}_{g_0} + \dot{\omega}_0\wedge b_i + \omega_0\wedge(\omega_0\wedge b_i)) + L_i\omega_i.\omega_i = 0 \quad (26)$$

$$\frac{\partial H}{\partial \mu_i} = 2\dot{L}_i\omega_i + L_i\dot{\omega}_i - u_i\wedge(\dot{V}_{g_0} + \dot{\omega}_0\wedge b_i + \omega_0\wedge(\omega_0\wedge b_i)) = 0 \quad (27)$$

As a rule, the above set of equations, which is linear in the accelerations, is underdetermined (there are $9n+6$ variables and $8n+6$ equations), i.e. some of the variables are free and have to be imposed in some way in order to solve the system (22)-(27). In other words, in these equations, some variables will remain unknown whereas others can be imposed as data. Depending on the choice of the data, the inverse or direct model will be obtained in the following but one can observe that other ways of fixing n variables are at will, e.g. one could imagine to take $\ddot{L}_i$ as the unknowns, possibly for design purposes, and so forth. One will restrict here to the usual cases in dynamical analysis of parallel structures, those of *inverse* and *direct* dynamical models.

[2] For continuous systems one would be lead to define analogously the *hamiltonian*[18]

A. Inverse dynamics

As a first remark, observe that, given the position and attitude of the platform, with its linear and angular velocities and accelerations, one is able to compute successively $u_i, L_i, \dot{L}_i, \omega_i, \ddot{L}_i, \dot{\omega}_i$ for every leg. Then the previous equations lead in a simple way to the inverse dynamical model which writes as follows :

<u>*Algorithm for Inverse dynamics*</u>

Data : $\overrightarrow{O'O}, \overrightarrow{O'A_i}, \overrightarrow{OB_i} = b_i, M_0, I_0, K_i, \alpha_i, \beta_i, \gamma_i,$
$V_{g_0}, \omega_0, \dot{V}_{g_0}, \dot{\omega}_0$

Begin

1) **Compute** u_i, L_i
2) **Compute** $\omega_i, \dot{L}_i, \dot{\omega}_i, \ddot{L}_i$ thanks to equations (11)-(14).
3) **Compute** μ_i thanks to equation (25)
4) *At this stage, no restriction has been put on the number of legs. But now, the only remaining unknowns to compute* f_i *are the* λ_i*. On another side, the only remaining equations are :* $\frac{\partial H}{\partial \dot{V}_{g_0}} = \frac{\partial H}{\partial \dot{\omega}_0} = 0$*, i.e.* 6 *equations. Thus, for the system to be determined, i.e. to have a unique solution, there must be exactly* 6 *unknowns i.e. the number of legs must be equal to* 6*. In other words the parallel structure must be a Stewart-Gough platform. This comes as a natural conclusion from linear algebra, without mechanical consideration.* **Compute** λ_i as the solution of the 6×6 linear system made of equations (22), (23).
5) **Compute** the forces f_i to exert on the legs through the explicit formula (24)

End

The complexity of the inverse algorithm is straightforwardly analyzed as there are only explicit formulae involving $O(n)$ multiplications and additions, and one $6\times n$ linear system to solve. Another point worth noticing is that inverse model for other parallel structures with different numbers of degrees of freedom can easily be derived from the above considerations. Manipulators with a number of degrees of freedom n less than 6 will necessitate exactly n legs for these degrees of freedom to be controllable outside singular configurations. Whenever more legs than degrees of freedom are used, e.g. in order to make singular configurations controllable, the above linear system in the algorithm (step 4) could be solved in the mean square sense.

B. Direct dynamics

The direct dynamics is obtained symmetrically from the necessary (and sufficient as seen above) conditions (22)-(27), the path being just “reversed” in some sense, i.e. with forces given as data. For the reasons exposed above (step 4 of the Inverse Dynamics algorithm), n is supposed to equal 6. The algorithm is the following :

Algorithm for Direct dynamics

Data : $\overrightarrow{O^{'}O}, \overrightarrow{O^{'}A_i}, \overrightarrow{OB_i} = b_i, M_0, I_0, K_i, \alpha_i, \beta_i, \gamma_i,$
V_{g_0}, ω_0, f_i

Begin

1) **Compute** u_i, L_i
2) **Compute** $\dot{L}_i, \omega_i$ thanks to equations (11) and (13).
3) **Substitute** for $\ddot{L}_i$ from (26) into (24)
4) **Substitute** for $\dot{\omega}_i$ from (27) into (25)
5) **Substitute** for λ_i and μ_i from (24) and (25) into (22) and (23)
6) **Compute** $(\dot{V}_{g_0}, \dot{\omega}_0)$ as the unique solution of the 6×6 linear system made of equations (22), (23), obtained after these substitutions.

End

The complexity of the direct algorithm is analogous to that of the inverse algorithm, as the same equations are to be solved.

VI. CONCLUSIONS

Following previous works by the author on serial multibody mechanical systems and continuous media, a novel method for deriving both inverse and direct dynamics of parallel manipulators has been presented, based on Appell's approach to Gauss' least constraint principle. The interest of this principle in dynamical analysis is that it works at the second order, i.e. directly with accelerations. As in dynamical analysis, either the accelerations are known and the forces exerted along the legs are unknown or the converse, use has been made of standard optimization theory in order to solve for the remaining unknowns, considered as unknown parameters. This leads to necessary and sufficient conditions that give at once the inverse and direct dynamics, unifying thus the derivation of both models. It is worth noticing that, contrary to the usual methods for parallel mechanisms, the presented method does not "breaks the loops" hence no supplementary closure constraints are needed. These conditions are also suited to solving other problems related to design. Another observation of interest is that, when considering the spatial dimension as an independent variable, it is clear that the dynamical model of a parallel manipulator such as a Stewart-Gough platform is a *linear one stage control system*, in the sense of evolutionary dynamical systems, whereas serial chains and continuous media, can be viewed, respectively, as *multistage (or "discrete-time") and continuous control systems*. Future work along these lines will procede with hybrid systems, i.e. assemblage of serial and parallel mechanisms, as well as with multibody flexible systems for which other complications intervene. As a conclusion, the *purely deductive* approach used in the present work and previous ones on serial and continuous mechanisms is a desirable property and will be of great help when connecting them all together.

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