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Formation tracking control of unicycle-type mobile robots

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***Abstract*— A constructive method is presented to design cooperative controllers that force a group of N unicycle-type mobile robots with limited sensing ranges to perform desired formation tracking, and guarantee no collisions between the robots. Robot physical dimensions and dynamics are considered in the control design. p times differential bump functions are introduced and incorporated into novel potential functions for control design. Despite the robot limited sensing ranges, no switchings are needed to solve the collision avoidance problem.**

I. Introduction

Research works on formation control often use one or more of leader-following (e.g. [1], [2]), behavioral (e.g. [3], [4]), and use of virtual structures (e.g. [5], [6] approaches in either a centralized or decentralized manner. Centralized control schemes (e.g. [2] and [7]) use a single controller that generates collision free trajectories in the workspace. Although these guarantee a complete solution, centralized schemes require high computational power and are not robust due to the heavy dependence on a single controller. On the other hand, decentralized schemes, see e.g. [8], [9], require less computational effort, and are relatively more scalable to the team size. The decentralized approach usually involves a combination of agent based local potential fields (e.g. [2], [10], [11]). The main problem with the decentralized approach, when collision avoidance is taken into account, is that it is extremely difficult to predict and control the critical points of the controlled systems. Recently, a method based on a different navigation function from [12] provided a centralized formation stabilization control design strategy is proposed in [7]. This work is extended to a decentralized version in [9]. However, the formation is stabilized to any point in workspace instead of being ”tied” to a fixed coordinate frame. In [12], [7] and [9], the tuning constants are extremely difficult to obtain. In most of the above papers, point-robots with simple (single or double integrator) dynamics (e.g. [2], [7], [9], [11], [13]) or fully actuated vehicles [6] (which can be converted to a double integrator dynamics via a feedback linearization) were investigated. Vehicles with nonholonomic constraints were also considered (e.g. [3]). However, the nonholonomic kinematics are transformed to a double integrator dynamics by controlling the hand position instead of the inertial position of the vehicles. Consequently, the vehicle heading is not controlled. In addition, switching control theory [14] is often used to design a decentralized formation control system (e.g. [1]), especially when the vehicles have limited sensing ranges and collision avoidance between vehicles must be considered. In this paper, cooperative controllers are designed to force a group of N unicycle-type mobile robots with limited sensing ranges to perform desired formation tracking, and to guarantee no collisions between the robots. The physical dimensions and dynamics of the robots are also considered. The control development is based on novel potential functions. Moreover, p times differential bump functions are introduced and incorporated into the potential functions to avoid the use of switchings.

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II. Problem statement

A. Robot dynamics

We consider a group of N mobile robots, of which each has the following dynamics [15]:

$$\begin{aligned}\dot{\eta}_i &= J_i(\eta_i)\omega_i, \\ M_i\dot{\omega}_i &= -C_i(\dot{\eta}_i)\omega_i - D_i\omega_i + \tau_i\end{aligned} \tag{1}$$

where $i = 1, ..., N,\ \eta_i = [x_i\ y_i\ \phi_i]^T$ denotes the position (x_i, y_i), the coordinates of the middle point, P_{0i}, between the left and right driving wheels, and heading ϕ_i of the robot i coordinated in the earth-fixed frame OXY, see Fig. 1, $\omega_i = [\omega_{1i}\ \omega_{2i}]^T$ with ω_{1i} and ω_{2i} being the angular velocities of the wheels of the robot i, $\tau_i = [\tau_{1i}\ \tau_{2i}]^T$ with τ_{1i} and τ_{2i} being the control torques applied to the wheels of the robot i. The rotation matrix $J_i(\eta_i)$, mass matrix M_i, Coriolis matrix $C_i(\dot{\eta}_i)$, and damping matrix D_i in (1) are given by

$$J_i(\eta_i) = \frac{r_i}{2}\begin{bmatrix}\cos(\phi_i) & \cos(\phi_i) \\ \sin(\phi_i) & \sin(\phi_i) \\ \frac{1}{b_i} & -\frac{1}{b_i}\end{bmatrix},\ M_i = \begin{bmatrix} m_{11i} & m_{12i} \\ m_{12i} & m_{11i}\end{bmatrix},$$

$$C_i(\dot{\eta}_i) = \begin{bmatrix} 0 & c_i\dot{\phi}_i \\ -c_i\dot{\phi}_i & 0\end{bmatrix},\ D_i = \begin{bmatrix} d_{11i} & 0 \\ 0 & d_{22i}\end{bmatrix} \tag{2}$$

with $\quad I_i = m_{ci}a_i^2 + 2m_{wi}b_i^2 + I_{ci} + 2I_{mi}$

$$\begin{aligned} c_i &= \frac{r_i^2}{2b_i}m_{ci}a_i, m_{11i} = \frac{r_i^2}{4b_i^2}(m_ib_i^2 + I_i), , \\ m_{12i} &= \frac{r_i^2(m_ib_i^2 - I_i)}{4b_i^2}, m_i = m_{ci} + 2m_{wi}\end{aligned} \tag{3}$$

where m_{ci} and m_{wi} are the masses of the body and wheel with a motor; I_{ci}, I_{wi} and I_{mi} are the moments of inertia of the body about the vertical axis through P_{ci} (center of mass), the wheel with a motor about the wheel axis, and the wheel with a motor about the wheel diameter, respectively; r_i, a_i and b_i are defined in Fig. 1; the nonnegative constants d_{11i} and d_{22i} are the damping coefficients. For convenience, we convert the wheel velocities ω_{1i} and ω_{2i} of the robot i to its linear, v_i, and angular, w_i, velocities by the relationship:

$$\varpi_i = B_i^{-1}\omega_i,\ B_i = \frac{1}{r_i}\left[\begin{array}{cc} 1 & b_i \\ 1 & -b_i \end{array}\right] \tag{4}$$

where $\varpi_i = [v_i\ w_i]^T$. It is noted that B_i is invertible since $\det(B_i) = -2b_i/r_i$. With (4), we can write the robot dynamics (1) in the following convenient form:

$$\begin{aligned} \dot{x}_i &= v_i\cos(\phi_i) \\ \dot{y}_i &= v_i\sin(\phi_i) \\ \dot{\phi}_i &= w_i \\ \overline{M}_i\dot{\varpi}_i &= -\overline{C}_i(w_i)\varpi_i - \overline{D}_i\varpi_i + \overline{B}_i\tau_i \end{aligned} \tag{5}$$

$$\text{where}\quad \begin{aligned} &\overline{M}_i = B_i^{-1}M_iB_i,\ \overline{C}_i(w_i) = B_i^{-1}C_i(\dot{\eta}_i)B_i \\ &\overline{D}_i = B_i^{-1}D_iB_i = D_i,\ \overline{B}_i = B_i^{-1}. \end{aligned} \tag{6}$$

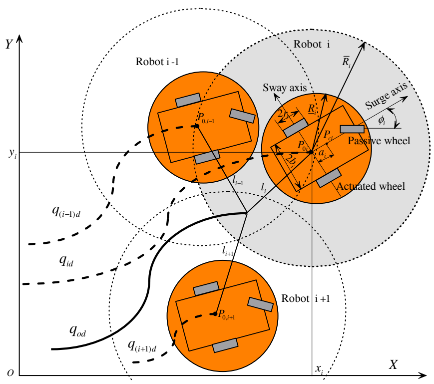

Fig. 1. Illustration of robot parameters and formation setup.

B. Formation control objective

Assumption 1.

1) The robot i has a physical safety circular area, which is centered at the point P_{0i} and has a radius $\underline{R}_i$, and has a circular communication area, which is centered at the point P_{0i} and has a radius $\overline{R}_i$, see Fig. 1. The radius $\overline{R}_i$ is strictly larger than $\underline{R}_i$.
2) The robot i broadcasts its state and reference trajectory in its circular communication area. Moreover, the robot i can receive the states and reference trajectories broadcasted by other robots $j, j = 1, 2, ..., N, j \neq i$ in the group if the points P_{0j} of these robots are in the circular communication area of the robot i.
3) The dimensional terms (r_i, a_i and b_i) of the robot i are known. The terms involved with mass, inertia and damping ($m_{11i}, m_{12i}, d_{11i}, d_{22i}$ and c_i) of the robots are unknown but constant.
4) At the initial time $t_0 \geq 0$, each robot starts at a location that is outside of the safety areas of other robots in the group, i.e. there exists a strictly positive constant ε_1 such that

$$\|q_i(t_0) - q_j(t_0)\| - (\underline{R}_i + \underline{R}_j) \geq \varepsilon_1, \tag{7}$$

for all $(i, j) \in (1, 2..., N),\ i \neq j$, where $q_i = [x_i\ y_i]^T$.

5) The reference trajectory for the robot i is $q_{id} = [x_{id}\ y_{id}]^T$, which is generated by

$$q_{id} = q_{od}(s_{od}) + l_i \tag{8}$$

where $q_{od}(s_{od}) = [x_{od}(s_{od})\ y_{od}(s_{od})]^T$ is referred to as the common reference trajectory with s_{od} being the common trajectory parameter, and l_i is a constant vector. The trajectory q_{od} satisfies the following conditions

$$\begin{aligned} &\lim_{t\to\infty} u_{od}^2(t) \neq 0,\ u_{od} = \sqrt{x_{od}'^2 + y_{od}'^2}\ \dot{s}_{od}, \\ &\sqrt{x_{od}'^2 + y_{od}'^2} > 0,\ |u_{od}(t)| \leq u_{od}^{max} \end{aligned} \tag{9}$$

where $x'_{od} = \frac{\partial x_{od}}{\partial s_{od}}$, $y'_{od} = \frac{\partial y_{od}}{\partial s_{od}}$, and u_{od}^{max} is a strictly positive constant. Moreover, $\dot{u}_{od}, \ddot{u}_{od}$ are also bounded. The constant vectors $l_i, i = 1, 2, ..., N$ satisfy

$$\|l_i - l_j\| - (\underline{R}_i + \underline{R}_j) \geq \varepsilon_2, \tag{10}$$

for all $(i, j) \in (1, 2..., N),\ i \neq j$ where ε_2 is a strictly positive constant.

Formation control objective: Under Assumption 1, design the control input τ_i and update laws for all terms involved mass, inertia and damping for each robot i such that each robot asymptotically tracks its desired reference trajectory q_{id} while avoids collisions with all other robots in the group, i.e. for all $(i, j) \in \{1, 2, ..., N\}, i \neq j, t \geq t_0 \geq 0$

$$\begin{aligned} &\lim_{t\to\infty}(q_i(t) - q_{id}(t)) = 0,\ \lim_{t\to\infty}(\phi_i(t) - \phi_{id}(t)) = 0, \\ &\|q_i(t) - q_j(t)\| - (\underline{R}_i + \underline{R}_j) \geq \epsilon_3 \end{aligned} \tag{11}$$

where $\phi_{id}(t) = \arctan(\frac{y'_{od}}{x'_{od}})$, and ϵ_3 is a positive constant.

III. Preliminaries

We here present one definition and one lemma, which will be used in the control design in the next section.

Definition 1. A scalar function $h(x, a, b)$ is called a p times differential bump function if it enjoys the following properties

1) $h(x, a, b) = 1$ if $0 \leq x \leq a$,
2) $h(x, a, b) = 0$ if $x \geq b$,
3) $0 < h(x, a, b) < 1$ if $a < x < b$,
4) $h(x, a, b)$ is p times differentiable with respect to x

where p is a positive integer, $x \in \mathbb{R}_+$, and a and b are constants such that $0 \leq a < b$.

Lemma 1. Let the scalar function $h(x, a, b)$ be defined as

$$h(x, a, b) = 1 - \frac{\int_a^x f(\tau - a)f(b - \tau)d\tau}{\int_a^b f(\tau - a)f(b - \tau)d\tau} \tag{12}$$

where the function $f(y)$ is defined as follows

$$f(y) = 0 \text{ if } y \leq 0 \text{ and } f(y) = y^p \text{ if } y > 0 \tag{13}$$

with p being a positive integer. Then the function $h(x, a, b)$ is a p times differentiable bump function.

Proof. See http://mech.uwa.edu.au/∼duc/bumpduc.pdf

IV. CONTROL DESIGN

A. Stage I

1) Step I.1: Define

$$\phi_{ie} = \phi_i - \alpha_{\phi_i},\ v_{ie} = v_i - \alpha_{v_i} \tag{14}$$

where α_{ϕ_i} and α_{v_i} are virtual controls of ϕ_i and v_i, respectively. With (14), the first two equations of (5) are read:

$$\dot{q}_i = u_i + \Delta_{1i} + \Delta_{2i} \tag{15}$$

where $q_i = [x_i\ \ y_i]^T$, and

$$\Delta_{1i} = \begin{bmatrix} (\cos(\phi_{ie})-1)\cos(\alpha_{phi_i}) - \sin(\phi_{ie})\sin(\alpha_{\phi_i}) \\ (\sin(\phi_{ie})\cos(\alpha_{\phi_i}) + (\cos(\phi_{ie})-1)\sin(\alpha_{\phi_i}) \end{bmatrix}\alpha_{v_i},$$
$$\Delta_{2i} = \begin{bmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{bmatrix} v_{ie}, \quad u_i = \begin{bmatrix} \cos(\alpha_{\phi_i}) \\ \sin(\alpha_{\phi_i}) \end{bmatrix}\alpha_{v_i}. \tag{16}$$

To fulfill the task of position tracking and collision avoidance, we consider the following potential function

$$\varphi_{I1} = \sum_{i=1}^{N}(\gamma_i + 0.5\beta_i) \tag{17}$$

where γ_i and β_i are the goal and related collision avoidance functions for the robot i specified as follows:
-The goal function is designed such that it puts penalty on the tracking error for the robot, and is equal to zero when the robot is at its desired position. We choose

$$\gamma_i = 0.5\|q_i - q_{id}\|^2. \tag{18}$$

-The related collision function β_i should be chosen such that it is equal to infinity whenever any robots come in contact with the robot i , i.e. a collision occurs, and attains the minimum value when the robot i is at its desired location with respect to other group members belong to the set $\mathbb{N}_i$ robots, where $\mathbb{N}_i$ is the set that contains all the robots in the group except for the robot i. We choose

$$\beta_i = \sum_{j\in\mathbb{N}_i}\beta_{ij} \tag{19}$$

where the function $\beta_{ij} = \beta_{ji}$ is a function of $\|q_{ij}\|^2/2$ with $q_{ij} = q_i - q_j$, and enjoys the following properties:

$$\begin{aligned}
&1)\ \beta_{ij} = 0,\ \beta'_{ij} = 0,\ \beta''_{ij} \geq 0 \quad \text{if} \quad \|q_{ij}\| = \|q_{ijd}\|,\\
&2)\ \beta_{ij} > 0 \quad \text{if} \quad 0 < \|q_{ij}\| < b_{ij},\\
&3)\ \beta_{ij} = 0,\ \beta'_{ij} = 0,\ \beta''_{ij} = 0,\ \beta'''_{ij} = 0 \quad \text{if} \quad \|q_{ij}\| \geq b_{ij},\\
&4)\ \beta_{ij} = \infty \quad \text{if} \quad \big(\|q_{ij}\| - (\underline{R}_i + \underline{R}_j)\big) \leq 0,\\
&5)\ \beta_{ij} \leq \mu_1,\ |\beta'_{ij}| \leq \mu_2, \text{ and } |\beta''_{ij}q_{ij}^T q_{ij}| \leq \mu_3,\\
&\quad\ \forall\, \mu_4 \leq \|q_{ij}\| \leq \mu_5,\\
&6)\ \beta_{ij} \text{ is at least 3 times differentiable with respect to } q_{ij}
\end{aligned} \tag{20}$$

where b_{ij} is a strictly positive constant such that $b_{ij} \leq \min(\overline{R}_i, \overline{R}_j)$, $\beta'_{ij} = \frac{\partial\beta_{ij}}{\partial(\|q_{ij}\|^2/2)}$ and $\beta''_{ij} = \frac{\partial^2\beta_{ij}}{\partial(\|q_{ij}\|^2/2)^2}$, $\beta'''_{ij} = \frac{\partial^3\beta_{ij}}{\partial(\|q_{ij}\|^2/2)^3}$, and $\mu_l, l = 1,...,5$ are positive constants.

There are many functions that satisfy all properties of β_{ij} given in (20). An example is

$$\beta_{ij} = \frac{h_{ij}\big(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\big)}{1 - h_{ij}\big(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\big)} \tag{21}$$

where $h_{ij}\big(\|q_{ij}\|^2/2, a_{ij}^2/2, b_{ij}^2/2\big)$ is a p times differentiable bump function defined in Definition 1 with $p \geq 3$ and $a_{ij} \geq (\underline{R}_i + \underline{R}_j)$, and $b_{ij} \leq \min(\overline{R}_i, \overline{R}_j, \|l_i - l_j\|)$. The time derivative of φ_{I1} along the solutions of (15) satisfies

$$\dot{\varphi}_{I1} = \sum_{i=1}^{N}\Omega_i^T\big(u_i + \Delta_{1i} + \Delta_{2i} - \dot{q}_{od}\big) \tag{22}$$

where we have used $\dot{q}_{id} = \dot{q}_{od}, u_i - u_j = u_i - \dot{q}_{od} - (u_j - \dot{q}_{od}), \forall(i,j) \in (1,2,...,N), i \neq j$, and

$$\Omega_i = q_i - q_{id} + \sum_{j\in\mathbb{N}_i}\beta'_{ij}q_{ij}. \tag{23}$$

From (22), we choose u_i as follows:

$$u_i = -k_0 u_{od}^2\Psi(\Omega_i) + \dot{q}_{od} \tag{24}$$

where $\Psi(\Omega_i)$ denotes a vector of bounded functions of elements of Ω_i in the sense that $\Psi(\Omega_i) = \big[\psi(\Omega_{ix})\ \psi(\Omega_{iy})\big]^T$ with Ω_{ix} and Ω_{iy} the first and second rows of Ω_i, i.e. $\Omega_i = [\Omega_{ix}\ \Omega_{iy}]^T$. The function $\psi(x)$ is a scalar, at least three times differentiable and bounded function with respect to x, and satisfies

$$\begin{aligned}
&1)\quad |\psi(x)| \leq \varrho_1,\\
&2)\quad \psi(x) = 0 \quad \text{if } x = 0, \quad x\psi(x) > 0 \text{ if } x \neq 0,\\
&3)\quad \psi(-x) = -\psi(x), (x-y)[\psi(x) - \psi(y)] \geq 0,\\
&4)\quad |\psi(x)/x| \leq \varrho_2, |\partial\psi(x)/\partial x| \leq \varrho_3, \partial\psi(x)/\partial x|_{x=0} = 1
\end{aligned} \tag{25}$$

for all $x \in \mathbb{R}, y \in \mathbb{R}$, where $\varrho_1, \varrho_2, \varrho_3$ are strictly positive constants. Some functions that satisfy the above properties are $\arctan(x)$ and $\tanh(x)$. The strictly positive constant k_0 is chosen such that

$$k_0 < \frac{1}{2\varrho_1 u_{od}^{max}}\,. \tag{26}$$

The above condition ensures that α_{ϕ_i} and α_{v_i} are solvable from u_i. From (24) and (16), we have

$$\begin{aligned}
\cos(\alpha_{\phi_i})\alpha_{v_i} &= -k_0u_{od}^2\psi(\Omega_{ix}) + \cos(\phi_{od})u_{od},\\
\sin(\alpha_{\phi_i})\alpha_{v_i} &= -k_0u_{od}^2\psi(\Omega_{iy}) + \sin(\phi_{od})u_{od}
\end{aligned} \tag{27}$$

where we have used $\dot{x}_{od} = x'_{od}\dot{s}_{od} = \cos(\phi_{od})u_{od}$ and $\dot{y}_{od} = y'_{od}\dot{s}_{od} = \sin(\phi_{od})u_{od}$ since $\phi_{od} = \arctan(y'_{od}/x'_{od})$ and $\sqrt{x'^2_{od} + y'^2_{od}} > 0$, see Assumption 1. Now solving (27) for α_{ϕ_i} and α_{v_i} gives

$$\begin{aligned}
&\alpha_{\phi_i} = \phi_{od} +\\
&\quad \arctan\left(\frac{-k_0u_{od}\big(-\psi(\Omega_{ix})\sin(\phi_{od}) + \psi(\Omega_{ix})\cos(\phi_{od})\big)}{-k_0u_{od}\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + 1}\right),\\
&\alpha_{v_i} = \cos(\alpha_{\phi_i})\big(-k_0u_{od}^2\psi(\Omega_{ix}) + \cos(\phi_{od})u_{od}\big) +\\
&\quad \sin(\alpha_{\phi_i})\big(-k_0u_{od}^2\psi(\Omega_{iy}) + \sin(\phi_{od})u_{od}\big).
\end{aligned} \tag{28}$$

Note that (28) is well defined since $-k_0u_{od}\big(\psi(\Omega_{ix})\cos(\phi_{od}) + \psi(\Omega_{ix})\sin(\phi_{od})\big) + 1 \geq -2\varrho_1k_0u_{od}^{max} + 1 > 0$ where (26) was used. It is of interest to note that α_{ϕ_i} and α_{v_i} are at least twice differentiable functions of $q_{od}, \phi_{od}, u_{od}, q_i, q_{ij}$ with $j \in \mathbb{N}_i, j \neq i$. Now substituting (24) into (22) results in

$$\dot{\varphi}_{I1} = -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) + \sum_{i=1}^{N} \Omega_i^T (\Delta_{1i} + \Delta_{2i}). \quad (29)$$

2) Step I.2: At this step, we view w_i as an immediate control to stabilize ϕ_{ie} at the origin. As such, we define

$$w_{ie} = w_i - \alpha_{w_i} \quad (30)$$

where α_{w_i} is a virtual control of w_i. To design the virtual control α_{w_i}, we consider the following function

$$\varphi_{I2} = \varphi_{I1} + 0.5 \sum_{i=1}^{N} \phi_{ie}^2 \quad (31)$$

whose derivative along the solutions of (29), (30) and the third equation of (5) satisfies

$$\begin{aligned}\dot{\varphi}_{I2} = & -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) + \sum_{i=1}^{N} \Omega_i^T \Delta_{2i} + \\ & \sum_{i=1}^{N} \phi_{ie} \Bigg(\frac{\Omega_i^T \Delta_{1i}}{\phi_{ie}} + w_{ie} + \alpha_{w_i} - \frac{\partial \alpha_{\phi_i}}{\partial q_{od}} \dot{q}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial \phi_{od}} \dot{\phi}_{od} - \\ & \frac{\partial \alpha_{\phi_i}}{\partial u_{od}} \dot{u}_{od} - \frac{\partial \alpha_{\phi_i}}{\partial q_i} (u_i + \Delta_{1i} + \Delta_{2i}) - \quad (32) \\ & \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (u_i + \Delta_{1i} + \Delta_{2i} - (u_j + \Delta_{1j} + \Delta_{2j})) \Bigg).\end{aligned}$$

It is noted that $\frac{\Delta_{1i}}{\phi_{ie}}$ is well defined since $\frac{\sin(\phi_{ie})}{\phi_{ie}} = \int_0^1 \cos(\lambda \phi_{ie}) d\lambda$ and $\frac{\cos(\phi_{ie})-1}{\phi_{ie}} = \int_1^0 \sin(\lambda \phi_{ie}) d\lambda$ are smooth functions for all $\phi_{ie} \in \mathbb{R}$. From (32), we choose the virtual control α_{w_i} as

$$\begin{aligned}\alpha_{w_i} = & -k_i \phi_{ie} - \frac{\Omega_i^T \Delta_{1i}}{\phi_{ie}} + \frac{\partial \alpha_{\phi_i}}{\partial q_{od}} \dot{q}_{od} + \frac{\partial \alpha_{\phi_i}}{\partial \phi_{od}} \dot{\phi}_{od} + \\ & \frac{\partial \alpha_{\phi_i}}{\partial u_{od}} \dot{u}_{od} + \frac{\partial \alpha_{\phi_i}}{\partial q_i} (u_i + \Delta_{1i}) + \\ & \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (u_i + \Delta_{1i} - (u_j + \Delta_{1j})) \quad (33)\end{aligned}$$

where k_i is a positive constant. Substituting (33) into (32) results in:

$$\begin{aligned}\dot{\varphi}_{I2} = & -k_0 u_{od}^2 \sum_{i=1}^{N} \Omega_i^T \Psi(\Omega_i) - \sum_{i=1}^{N} k_i \phi_{ie}^2 + \sum_{i=1}^{N} \Bigg[\phi_{ie} w_{ie} + \\ & \big(\Omega_i^T - \phi_{ie} \frac{\partial \alpha_{\phi_i}}{\partial q_i} - \sum_{j=1, j\neq i}^{N} \big(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} \phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}} \phi_{je} \big) \big) \Delta_{2i} \Bigg]. \end{aligned} \quad (34)$$

To prepare for the next section, let us compute the term $\overline{M}_i \dot{\varpi}_{ie}$ where $\varpi_{ie} = [v_{ie} \ w_{ie}]^T$. From the second equation of (14), (30), and the last equation of (5), we have

$$\begin{aligned}\overline{M}_i \dot{\varpi}_{ie} & = -\overline{C}_i(w_i)\varpi_i - \overline{D}_i \varpi_i - \overline{M}_i [\dot{\alpha}_{v_i} \ \dot{\alpha}_{w_i}]^T + \overline{B}_i \tau_i \\ & = -\overline{D}_i \varpi_{ie} + \Phi_i \Theta_i + \overline{B}_i \tau_i \end{aligned} \quad (35)$$

where

$$\Phi_i = \begin{bmatrix} w_i^2 & -\alpha_{v_i} & -\dot{\alpha}_{v_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -w_i v_i & -\alpha_{w_i} & -\dot{\alpha}_{w_i} \end{bmatrix}, \quad (36)$$

$$\Theta_i = \begin{bmatrix} b_i c_i & d_{11i} & m_{11i} + m_{12i} & c_i/b_i & d_{22i} & m_{11i} - m_{12i} \end{bmatrix}^T$$

where $\vartheta_i = u_i + \Delta_{1i} + \Delta_{2i}, i = 1, ..., N$.

B. Stage II

At this stage, we design the actual control input vector τ_i and update laws for unknown system parameter vector Θ_i for each robot i. To do so, we consider the following function

$$\varphi_{II} = \varphi_{I2} + \frac{1}{2} \sum_{i=1}^{N} \left(\varpi_{ie}^T \overline{M}_i \varpi_{ie} + \tilde{\Theta}_i^T \Gamma_i^{-1} \tilde{\Theta}_i \right) \quad (37)$$

where $\tilde{\Theta}_i = \Theta_i - \hat{\Theta}_i$ with $\hat{\Theta}_i$ being an estimate of Θ_i, and Γ_i is a symmetric positive definite matrix. Differentiating both sides of (37) along the solutions of (35) and (34) and choosing the actual control τ_i and the update law for $\hat{\Theta}_i$ as

$$\begin{aligned}\tau_i = & \overline{B}_i^{-1} \Bigg(-L_i \varpi_{ie} - \Phi_i \hat{\Theta}_i - \Bigg[\Bigg(\Omega_i^T - \phi_{ie} \frac{\partial \alpha_{\phi_i}}{\partial q_i} - \\ & \sum_{j=1, j\neq i}^{N} \left(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} \phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}} \phi_{je} \right) \Bigg) \overline{\Delta}_{2i} \quad \phi_{ie} \Bigg]^T \Bigg), \\ \dot{\hat{\Theta}}_i = & \Gamma_i \Phi_i^T \varpi_{ie} \end{aligned} \quad (38)$$

where $\overline{\Delta}_{2i} = [\cos(\phi_i) \ \ \sin(\phi_i)]^T$, and L_i is a symmetric positive definite matrix, result in

$$\dot{\varphi}_{II} = -\sum_{i=1}^{N} [k_0 u_{od}^2 \Omega_i^T \Psi(\Omega_i) + k_i \phi_{ie}^2 + \varpi_{ie}^T (\overline{D}_i + L_i) \varpi_{ie}]. \quad (39)$$

From the above control design, we have the closed loop system

$$\begin{aligned}\dot{q}_i = & -k_0 u_{od}^2 \Psi(\Omega_i) + \dot{q}_{od} + \Delta_{1i} + \Delta_{2i}, \\ \dot{\phi}_{ie} = & -k_i \phi_{ie} - \frac{\Omega_i^T \Delta_{1i}}{\phi_{ie}} - \frac{\partial \alpha_{\phi_i}}{\partial q_i} \Delta_{2i} - \\ & \sum_{j=1, j\neq i}^{N} \frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} (\Delta_{2i} - \Delta_{2j}), \quad (40) \\ \overline{M}_i \dot{\varpi}_{ie} = & -(\overline{D}_i + L_i) \varpi_{ie} + \Phi_i \tilde{\Theta}_i - \Bigg[\Bigg(\Omega_i^T - \phi_{ie} \frac{\partial \alpha_{\phi_i}}{\partial q_i} \\ & - \sum_{j=1, j\neq i}^{N} \left(\frac{\partial \alpha_{\phi_i}}{\partial q_{ij}} \phi_{ie} - \frac{\partial \alpha_{\phi_j}}{\partial q_{ji}} \phi_{je} \right) \Bigg) \overline{\Delta}_{2i} \quad \phi_{ie} \Bigg]^T, \\ \dot{\tilde{\Theta}}_i = & -\Gamma_i \Phi_i^T \varpi_{ie} . \end{aligned}$$

Theorem 1. Under Assumption 1, the control τ_i and the update law $\dot{\hat{\Theta}}_i$ given in (38) for the robot i solve the formation control objective. In particular, the closed loop system (40) is forward complete, no collisions between any robots can occur for all $t \geq t_0 \geq 0$ and the position and orientation of the robots track their reference trajectories asymptotically in the sense of (11).

Proof. See Appendix A.

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Appendix A: Proof of Theorem 1. +*Proof of no collisions and complete forwardness of closed loop system*. From (39) and properties of the function ψ , see (25), we have $\dot{\varphi_{II}} \leq 0$, which implies that $\varphi_{II}(t) \leq \varphi_{II}(t_0), \forall t \geq t_0$. With definition of the function φ_{II} in (37) and its components in (31), (17), (18) and (19), we have

$$\varphi_{II}(t) \leq \varphi_{II}(t_0) \tag{41}$$

for all $t \geq t_0 \geq 0$. From the definition of φ_{II}, see 37, the condition specified in item 4) of Assumption 1, and Property 5) of β_{ij}, and definition of ϕ_{ie}, ϖ_{ie}, we have the right hand side of (41) is bounded by a positive constant depending on the initial conditions. Boundedness of the right hand side of (41) implies that the left hand side of (41) must be also bounded. As a result, $\beta_{ij}(t)$ must be smaller than some positive constant depending on the initial conditions for all $t \geq t_0 \geq 0$. From properties of β_{ij}, see (20), $\|q_{ij}(t)\| - (\underline{R}_i + \underline{R}_j)$ must be larger than some positive constant depending on the initial conditions denoted by ϵ_3, i.e. there are no collisions for all $t \geq t_0 \geq 0$. Boundedness of the left hand side of (41) also implies that of $(q_i(t) - q_{id}(t)), \phi_{ie}(t), \varpi_{ie}(t)$ and $\hat{\Theta}_i(t)$ for all $t \geq t_0 \geq 0$. This in turn implies by construction that $x_i(t), y_i(t), \phi_i(t), v_i(t)$ and $w_i(t)$ do not escape to infinity in finite time.

+*Equilibrium points*. An application of Theorem 8.4 in [16] to (39) yields

$$\lim_{t\to\infty}\Big(k_0 u_{od}^2(t)\sum_{i=1}^{N}\Omega_i^T(t)\Psi(\Omega_i(t)) + \sum_{i=1}^{N}k_i\phi_{ie}^2(t) + \sum_{i=1}^{N}\varpi_{ie}^T(t)(\overline{D}_i + L_i)\varpi_{ie}(t)\Big) = 0. \tag{42}$$

By noting that $\lim_{t\to\infty} u_{od}^2(t) \neq 0$ as specified in item 5) of Assumption 1, the limit equation (42) implies that

$$\lim_{t\to\infty}\Omega_i(t) = 0,\ \lim_{t\to\infty}\phi_{ie}(t) = 0,\ \lim_{t\to\infty}\varpi_{ie}(t) = 0.$$

By construction, $\lim_{t\to\infty}\Omega_i(t) = 0$ and $\lim_{t\to\infty}\phi_{ie}(t) = 0$ imply that $\lim_{t\to\infty}(\phi_i(t) - \phi_{od}(t)) = 0$. Moreover, from definition of Ω_i in (23), $\lim_{t\to\infty}\Omega_i(t) = 0$ means

$$\lim_{t\to\infty}\Big(q_i(t) - q_{id}(t) + \sum_{j\in\mathbb{N}_i}\beta'_{ij}q_{ij}(t)\Big) = 0. \tag{43}$$

The limit equation (43) implies that $q(t) = [q_1^T(t)\ q_2^T(t), ..., q_N^T(t)]^T$ can tend to $q_d = [q_{1d}^T\ q_{2d}^T, ..., q_{Nd}^T]^T$ since $\beta'_{ij}|_{\|q_{ij}\|=\|q_{ijd}\|} = 0$ (Property 1) of β_{ij}), or some vector denoted by $q_c = [q_{1c}^T\ q_{2c}^T, ..., q_{Nc}^T]^T$ as the time goes to infinity, i.e. the equilibrium points can be q_d or q_c. It is noted that some elements of q_c can be equal to that of q_d. However, for simplicity we abuse the notation, i.e. we still denote that vector as q_c. Indeed, the vector q_c is such that

$$\Omega_i|_{q=q_c} = \big[q_i - q_{id} + \sum_{j\in\mathbb{N}_i}\beta'_{ij}q_{ij}\big]\Big|_{q=q_c} = 0 \tag{44}$$

for all $i = 1, ..., N$. To investigate properties of the equilibrium points, q_d and q_c, we consider the first equation of the closed loop system (40), i.e.

$$\dot{q}_i = -k_0 u_{od}^2\Psi(\Omega_i) + \dot{q}_{od} + \Delta_{1i} + \Delta_{2i}. \tag{45}$$

Since we have already proved that the closed loop system (40) is forward complete, and $\lim_{t\to\infty}\phi_{ie}(t) = 0$ and $\lim_{t\to\infty}\varpi_{ie}(t) = 0$ imply from the expressions of Δ_{1i} and Δ_{2i}, see (16) that $\lim_{t\to\infty}(\Delta_{1i}(t) + \Delta_{2i}(t)) = 0$, we treat $\Delta_i(t) \triangleq \Delta_{1i}(t) + \Delta_{2i}(t)$ as an input to (45) instead of a state. To investigate properties of the equilibrium points, q_d and q_c, we linearize (45) at these points.

+*Properties of equilibrium points*. The system (45) can be written in a vector form as

$$\dot{q} = -k_0 u_{od}^2\Psi_q(q, q_d) + \text{vec}(\dot{q}_{od}) + \text{vec}(\Delta_i) \tag{46}$$

where $\Psi_q(q, q_d) = [\Psi^T(\Omega_1), ..., \Psi^T(\Omega_i), ..., \Psi^T(\Omega_N)]^T$, $\text{vec}(\dot{q}_{od}) = [\dot{q}_{od}^T, ..., \dot{q}_{od}^T, ..., \dot{q}_{od}^T]^T$, and $\text{vec}(\Delta_i) = [\Delta_1^T, ..., \Delta_i^T, ..., \Delta_N^T]^T$. Therefore, near an equilibrium point q_o, which can be either q_d or q_c, we have

$$\dot{q} = -k_0 u_{od}^2\ \partial\Psi_q(q, q_d)/\partial q\big|_{q=q_o}(q - q_o) + \text{vec}(\dot{q}_{od}) + \text{vec}(\Delta_i) \tag{47}$$

where $\frac{\partial\Psi_q(q,q_d)}{\partial q}=[\bigwedge_{ij}]$ with $\bigwedge_{ij}=\frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\frac{\partial\Omega_i}{\partial q_j},(i,j)\in\mathbb{N}$ denoting the set of all agents. Let $\mathbb{N}^*$ be the set of the agents such that if the agents i and j belong to the set $\mathbb{N}^*$ then $\|q_{ij}\|<b_{ij}$. Next we will show that q_d is asymptotically stable and that q_c is unstable.

-Proof of q_d being asymptotic stable. Using properties of β_{ij} and ψ listed in (20) and (25), we have

$$\begin{aligned}\frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\Big|_{q=q_d}&=I_n,\beta'_{ijd}=0,\\ \frac{\partial\Omega_i}{\partial q_i}\Big|_{q=q_d}&=I_n+\sum_{j\in\mathbb{N}_i}\beta''_{ijd}q_{ijd}q_{ijd}^T,\\ \frac{\partial\Omega_i}{\partial q_j}\Big|_{q=q_d}&=-\beta''_{ijd}q_{ijd}q_{ijd}^T\end{aligned}\tag{48}$$

where $\beta'_{ijd}=\beta'_{ij}\big|_{q_{ij}=q_{ijd}}$ and $\beta''_{ijd}=\beta''_{ij}\big|_{q_{ij}=q_{ijd}}$, with $q_{ijd}=q_{id}-q_{jd}$. We consider the function

$$V_d=\sqrt{1+\|q-q_d\|^2}-1\tag{49}$$

whose derivative along the solutions of (47) with q_o replaced by q_d, using (48), and noting that $\dot q_{od}=\dot q_{id}$ satisfies

$$\dot V_d\le-\frac{k_0u_{od}^2}{\sqrt{1+\|q-q_d\|^2}}\sum_{i=1}^N\|q_i-q_{id}\|^2+\sum_{i=1}^N\|\Delta_i\|\tag{50}$$

since $\beta''_{ijd}\ge0$, see Property 1) in (20). The last inequality of (50) implies that q_d is asymptotically stable because $\lim_{t\to\infty}u_{od}^2(t)\neq0$, and we have already proved that $\lim_{t\to\infty}\Delta_i(t)=0$.

- Proof of q_c being unstable. Again using properties of β_{ij} and ψ in (20) and (25), we have

$$\begin{aligned}\frac{\partial\Psi(\Omega_i)}{\partial\Omega_i}\Big|_{q=q_c}&=I_n,\\ \frac{\partial\Omega_i}{\partial q_i}\Big|_{q=q_c}&=\big(1+\sum_{j\in\mathbb{N}_i}\beta'_{ijc}\big)I_n+\sum_{j\in\mathbb{N}_i}\beta''_{ijc}q_{ijc}q_{ijc}^T,\\ \frac{\partial\Omega_i}{\partial q_j}\Big|_{q=q_c}&=-\beta'_{ijc}-\beta''_{ijc}q_{ijc}q_{ijc}^T\end{aligned}\tag{51}$$

for all $i=1,...,N,i\neq j$, where $q_{ijc}=q_{ic}-q_{jc}$, $\beta'_{ijc}=\beta'_{ij}\big|_{q_{ij}=q_{ijc}}$ and $\beta''_{ijc}=\beta''_{ij}\big|_{q_{ij}=q_{ijc}}$. Since β_i are specified in terms of relative distances between agents, instead of using $V_c=0.5\|q-q_c\|$ to investigate stability of (47) at q_c, we consider the Lyapunov function candidate

$$\bar V_c=\sqrt{1+\|\bar q-\bar q_c\|^2}-1\tag{52}$$

where $\bar q=[q_{12}^T,q_{13}^T,...q_{1N}^T,q_{23}^T,...,q_{2N}^T,...,q_{N-1,N}^T]^T$ and $\bar q_c=[q_{12c}^T,q_{13c}^T,...q_{1Nc}^T,q_{23c}^T,...,q_{2Nc}^T,...,q_{N-1,Nc}^T]^T$. Differentiating both sides of (52) along the solution of (47) with q_o replaced by q_c gives

$$\begin{aligned}\dot{\bar V}_c=&-\frac{k_0u_{od}^2}{\sqrt{1+\|\bar q-\bar q_c\|^2}}\sum_{(i,j)\in\mathbb{N}\setminus\mathbb{N}^*}\|q_{ij}-q_{ijc}\|^2-\\ &\frac{k_0u_{od}^2}{\sqrt{1+\|\bar q-\bar q_c\|^2}}\sum_{(i,j)\in\mathbb{N}^*}(1+N\beta'_{ijc})\|q_{ij}-q_{ijc}\|^2-\\ &\frac{k_0u_{od}^2N}{\sqrt{1+\|\bar q-\bar q_c\|^2}}\sum_{(i,j)\in\mathbb{N}^*}\beta''_{ijc}\big(q_{ijc}^T(q_{ij}-q_{ijc})\big)^2+\\ &\frac{1}{\sqrt{1+\|\bar q-\bar q_c\|^2}}\sum_{(i,j)\in\mathbb{N}}(q_{ij}-q_{ijc})^T(\Delta_i-\Delta_j)\end{aligned}\tag{53}$$

where $i\neq j$ and (51) has been used. To investigate stability properties of $\bar q_c$ based on (53), we will use (44). Define $\Omega_{ijc}=\Omega_{ic}-\Omega_{jc}$, $\forall(i,j)\in\{1,...,N\}$, $i\neq j$ where $\Omega_{ic}=\Omega_i|_{q=q_c}=0$, see (44). Therefore $\Omega_{ijc}=0$. Hence $\sum_{(i,j)\in\mathbb{N}^*}q_{ijc}^T\Omega_{ijc}=0,i\neq j$, which by using (44) is expanded to

$$\begin{aligned}&\sum_{(i,j)\in\mathbb{N}^*}\big(q_{ijc}^T(q_{ijc}-q_{ijd})+N\beta'_{ijc}q_{ijc}^Tq_{ijc}\big)=0\\ &\Rightarrow\sum_{(i,j)\in\mathbb{N}^*}(1+N\beta'_{ijc})q_{ijc}^Tq_{ijc}=\sum_{(i,j)\in\mathbb{N}^*}q_{ijc}^Tq_{ijd}\end{aligned}\tag{54}$$

where $i\neq j$. The sum $\sum_{(i,j)\in\mathbb{N}^*}q_{ijc}^Tq_{ijd}$ is strictly negative since at the point, say F, where $q_{ij}=q_{ijd},\ \forall(i,j)\in\mathbb{N}^*,i\neq j$ all attractive and repulsive forces are equal to zero while at the point, say C, where $q_{ij}=q_{ijc}\ \forall(i,j)\in\mathbb{N}^*,i\neq j$ the sum of attractive and repulsive forces are equal to zero (but attractive and repulsive forces are nonzero). Therefore the point, say O, where $q_{ij}=0,\ \forall(i,j)\in\mathbb{N}^*,i\neq j$ must locate between the points F and C for all $(i,j)\in\mathbb{N}^*,i\neq j$. That is there exists a strictly positive constant b such that $\sum_{(i,j)\in\mathbb{N}^*}q_{ijc}^Tq_{ijd}<-b$, which is substituted into (54) to yield

$$\sum_{(i,j)\in\mathbb{N}^*}(1+N\beta'_{ijc})q_{ijc}^Tq_{ijc}<-b,i\neq j.\tag{55}$$

Since $q_{ijc}^Tq_{ijc}>0,\forall(i,j)\in\mathbb{N}^*,i\neq j$, there exists a nonempty set $\mathbb{N}^{**}\subset\mathbb{N}^*$ such that for all $(i,j)\in\mathbb{N}^{**},i\neq j$, $(1+N\beta'_{ijc})$ is strictly negative, i.e. there exists a strictly positive constant b^{**} such that $(1+N\beta'_{ijc})<-b^{**},\ \forall(i,j)\in\mathbb{N}^{**},i\neq j$. We now define a subspace such that $q_{ij}-q_{ijc}=0,\ \forall(i,j)\in\mathbb{N}\setminus\mathbb{N}^{**}$ and $q_{ijc}^T(q_{ij}-q_{ijc})=0,\ \forall(i,j)\in\mathbb{N}^*,i\neq j$. In this subspace, we have

$$\begin{aligned}\bar V_c&=\sqrt{1+\sum_{(i,j)\in\mathbb{N}^{**}}\|q_{ij}-q_{ijc}\|^2}-1,\\ \dot{\bar V}_c&\ge b^{**}k_0u_{od}^2\bar V_c-\sum_{(i,j)\in\mathbb{N}^{**}}\|\Delta_i-\Delta_j\|.\end{aligned}\tag{56}$$

where we have used $(1+N\beta'_{ijc})<-b^{**},\ \forall(i,j)\in\mathbb{N}^{**},i\neq j$. It is straightforward to show from (56) that q_c is an unstable equilibrium point.