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Fast Regulation of the Rolling Sphere: A Motion Planning Approach

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***Abstract*—This paper presents a simple stabilizing control scheme for the rolling sphere. A tracking control problem with respect to a "virtual moving trajectory" is first solved. Then, it is used to achieve practical stability for the regulation problem. The derived controllers have a simple form and can guarantee fast convergence. To verify the effectiveness of the proposed results, an interesting simulation result is presented.**

I. Introduction

OVER the last decades, the stabilization problem of nonholonomic systems has been extensively studied by many researchers [4-8, 12, 14, 16-27]. See the survey paper [14] for numerous references before 1995 and the book [1] for further discussions. According to Brockett's necessary condition, the stabilization problem for such systems usually is not solvable by employing continuous static state feedback controllers [2]. Instead, it has been found that time-varying feedback and discontinuous feedback methods can be employed to address the stabilization problem [5, 21]. However, most of existing literature focused on the nonholonomic systems that can be described or transformed into the chained form systems [7, 8, 11, 12, 18, 19, 20, 22]. For general nonholonomic systems, there are still many important systems the stabilization problems of which deserve further study.

Among these, an interesting example is the rolling sphere. The rolling sphere has important applications to space multi-body systems [6] and multi-fingered manipulation [13]. Recently, it attracts more attentions [4, 6, 17, 23, 24]. Particularly, its controllability was proven in [24]. Based on this fact and due to [5], there exists an implicit controller to stabilize the rolling sphere. Finding an explicit stabilizer of rolling sphere is then the next pursuit target. Generally, the rolling sphere has two special properties such that it is not trivial to find a stabilizer explicitly. One is the non-flatness and another is that it cannot be transformed into a chained form system [17] by employing the condition proposed in the paper [15]. To overcome such obstacle, an iterative control scheme was employed to robustly stabilize the rolling sphere [17]. Based on a discontinuous control strategy, a novel exponential convergence result for some regulation problem of the rolling sphere was established in the paper [6]. In the case studied in that paper, they only need to stabilize four partial state variables rather than all state variables. Quite recently, a time-switching control method was used to guarantee asymptotic stability of the origin [4]. In [26, 27], several more general approaches were proposed.

This paper continues this research line. Our approach is based on a motion planning method to study the regulation problem. A similar ideal was proposed in [20] for mobile robots and demonstrated by an example in the paper [19]. Another interesting approach can be found in [23]. Recently, [12] proposed a more general framework. Based on this framework, we will first study a special tracking control problem with respect to a periodic "virtual moving trajectory" (i.e., to-be tracked signals.) The virtual moving trajectory will be designed such that it has zero value at some points and satisfies a persistent excitation condition. Uniform global asymptotic stability as well as a local exponential convergence result will then be established for the error model. Simultaneously, as soon as the norm of the original state is less than a pre-assigned error bound at some time instant, the controllers will be set as the zero value. In this way, a practical stabilization with fast convergence can be achieved based on our approach. The derived controllers will be very simple but guaranteeing fast convergence. Moreover, an interesting simulation will be presented to verify the effectiveness of the proposed controllers.

II. A Tracking Control Design with Exponential Convergence

In this paper, we study the asymptotic stability problem of the rolling sphere modeled by [3, 4]

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$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z}_1 &= xv \\ \dot{z}_2 &= x^2 v \\ \dot{z}_3 &= xyv \end{aligned} \tag{1}$$

where $(x, y, z_1, z_2, z_3)^T \in \Re^5$ is a state vector and $(u,v)^T \in \Re^2$ is a control vector.

Remark 1. Based on the condition proposed in [15], it can be seen that (1) cannot be transformed into a chained form system [4]. Thus, the existing controllers derived for the nonholonomic chained form systems cannot be used here.■

Remark 2. Equation (1) is only an approximating model of rolling sphere under some proper coordinate transformations. For example, consider the nilpotent approximation model (20) derived in [17] (via the method developed in [25]). Then, equation (20) in that paper can be transformed into (1) by using the following coordinate transformations:

$$(x, y, z_1, z_2, z_3) = (\hat{z}_2, \hat{z}_1, \hat{z}_3, 2(\hat{z}_5 + \hat{z}_2\hat{z}_3), \hat{z}_4 + \hat{z}_3\hat{z}_1 + \hat{z}_3^3/6)$$

and $(v,u) = (w_1, w_2)$.

Although (1) is not an exact model of rolling sphere, it is hoped that the method proposed here can yield further insights into the design of complete model. ■

In this section, we will first consider a special tracking problem. Toward this end, let us recall a class of periodic functions defined in [12] as follows.

Definition 1. For any positive constant M, let S_M be the set of all continuous real-valued periodic functions h defined on $[0,\infty)$ that satisfy the following conditions:

1) $\int_0^T h(t)\,dt = 0$, where T is a period of h.

2) There exist two constants t_0 and s_0 such that $h(t_0)= M$ and $h(s_0)=-M$. ■

Let $u_r, v_r \in S_M$ be the desired control signals for some positive constant M with the same period T, and

$$x_r(t) = \int_0^t u_r(\tau)\,d\tau \ \text{ and } \ y_r(t) = \int_0^t v_r(\tau)\,d\tau, \ \forall t \ge 0 \tag{2}$$

be the (virtual) tracking signals with respect to the state variables x and y, respectively.. By their definitions, we have

$$\dot{x}_r(t) = u_r(t) \ \text{ and } \ \dot{y}_r(t) = v_r(t), \ \forall t \ge 0 \,. \tag{3}$$

Let us further assume that

$$x_r(t) v_r(t) = 0, \ \forall t \ge 0. \tag{4}$$

Notice that, there are many possible choices to match these conditions. For instance, let u_r and v_r be two periodic functions with the same period T and over a period, they can be described as follows:

$$u_r(t) = \begin{cases} M\sin 4\pi t/T, \text{if } 0 \le t \le T/2 \\ 0, \text{if } T/2 \le t \le T \end{cases} \tag{5}$$

$$v_r(t) = \begin{cases} 0, \text{if } 0 \le t \le T/2 \\ M\sin 4\pi t/T, \text{if } T/2 \le t \le T. \end{cases} \tag{6}$$

From (2), it can be checked that

$$x_r(t) = \begin{cases} (1-\cos 4\pi t/T)MT/4\pi, \text{if } 0 \le t \le T/2 \\ 0, \text{if } T/2 \le t \le T \end{cases} \tag{7}$$

$$y_r(t) = \begin{cases} 0, \text{if } 0 \le t \le T/2 \\ (1-\cos 4\pi t/T)MT/4\pi, \text{if } T/2 \le t \le T. \end{cases} \tag{8}$$

In view of (5)-(8), it can be seen that $u_r, v_r \in S_M$ and (4) holds.

Define an error state vector and new control variables as follows:

$$\tilde{x} = (x_e, y_e, z_1, z_2, z_3)^T = (x - x_r, y - y_r, z_1, z_2, z_3)^T \tag{9}$$

$$(\tilde{u}, \tilde{v}) = (u - u_r, v - v_r)\,. \tag{10}$$

By employing (1) and (4), the following error model can be derived:

$$\dot{\tilde{x}} = \begin{pmatrix} 0 \\ 0 \\ v_r \\ xv_r \\ yv_r \end{pmatrix} x_e + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{u} + \begin{pmatrix} 0 \\ 1 \\ x \\ x^2 \\ xy \end{pmatrix} \tilde{v}. \tag{11}$$

Let $V = |\tilde{x}|^2/2$ be a Lyapunov candidate. Using direct computation, it can be checked that

$$\dot{V} = x_e[\tilde{u} + v_r(z_1 + xz_2 + yz_3)] + \tilde{v}[y_e + x(z_1 + xz_2 + yz_3)]. \tag{12}$$

This suggests that the following controllers should be chosen:

$$\tilde{u} = -v_r(z_1 + xz_2 + yz_3) - k_u x_e \tag{13}$$

and

$$\tilde{v} = -k_v[y_e + x(z_1 + xz_2 + yz_3)]\,, \tag{14}$$

with k_u and k_v being any positive constants. According to LaSalle invariance principle [9], the following result can be proposed. It will be used to derive a fast regulator in next section. The proof is postponed to appendix.

Theorem 1. Consider rolling sphere system (1) and its error model (11). With the controllers chosen as (13)-(14), $\tilde{x} = 0$ is uniformly globally asymptotically stable and locally exponentially stable. ■

Remark 3. By using a result from [22], it can be concluded that the origin is globally K-exponentially stable. ■

Remark 4. Equation (4) is a technique condition that was used to simplify the form of error model (11) as well as the design of the controllers. ■

Remark 5. It should be noticed that we just solved a tracking control problem with respect to a special trajectory in Theorem 1. It is interesting but not easy to attack the general tracking control problem. For our purpose, a partial solution is enough to be used in the regulation problem. ■

III. Fast Regulation of the Rolling Sphere

In this section, Theorem 1 is used to derive fast regulation of the rolling sphere by employing the control scheme proposed in [12]. Indeed, by the facts that

$$x_r(kT) = k\int_0^T u_r(\tau)\, d\tau = 0 \text{ and } y_r(kT) = k\int_0^T v_r(\tau)\, d\tau = 0$$

for all $k \in \aleph \cup \{0\}$, it can be seen that

$$\tilde{x}(kT) = (x(kT), y(kT), z_1(kT), z_2(KT), z_3(kT))^T. \quad (15)$$

Since the tracking signals are both bounded (by the periodic and continuity properties), it can also be seen that all solutions $\hat{x} = (x, y, z_1, z_2, z_3)^T$ of the closed-loop system are uniformly globally bounded. Thus, the following result is a consequence of Theorem 1. We refer readers to [12] for more related discussions.

Proposition 1. Consider rolling sphere system (1) where the controllers are chosen as (13)-(14). Then, all solutions $\hat{x} = (x, y, z_1, z_2, z_3)^T$ of the closed-loop system are uniformly globally bounded. Furthermore, for any $\varepsilon > 0$ and $r > 0$, there exists a large $T(\varepsilon, r) > 0$ such that if $|\hat{x}(0)| \le r$, then $|\hat{x}(T)| < \varepsilon$. ■

Notice that, (1) is a driftless system. Thus, the state vector will keep the same value under the condition $u = v = 0$. Based on this observation and Proposition 1, for any given error bound ε, the final stabilizer can be modified as

$$u = \begin{cases} u_r - v_r(z_1 + xz_2 + yz_3) - k_u(x - x_r), if|\hat{x}| \ge \varepsilon \\ 0, if|\hat{x}| < \varepsilon, \end{cases} \quad (16)$$

$$v = \begin{cases} v_r - k_v[y - y_r + x(z_1 + xz_2 + yz_3)], if|\hat{x}| \ge \varepsilon \\ 0, if|\hat{x}| < \varepsilon. \end{cases} \quad (17)$$

with k_u and k_v being any positive constants. Based on Theorem 1 and Proposition 1, a practical and fast stabilization of the rolling sphere can be achieved by employing controllers (16)-(17). Particularly, we have the following result.

Theorem 2. Consider rolling sphere system (1). Let $\varepsilon > 0$ be any given error bound and the controllers be chosen as (16)-(17). Then, all solutions $\hat{x} = (x, y, z_1, z_2, z_3)^T$ of the closed-loop system are uniformly globally bounded and for any r>0, there exists a large $T(\varepsilon, r) > 0$ such that if $|\hat{x}(0)| \le r$, then $|\hat{x}(t)| < \varepsilon, \forall t \ge T$. ■

Remark 6. It can be seen that controllers (16)-(17) take a simple form and is easily implemented. Moreover, a fast convergence result can be guaranteed by employing the exponential convergence result provided in Theorem 1. ■

Remark 7. In theorem 1, exponential convergence was achieved by employing (13)-(14). This fact can be used to guarantee certain robustness results, see [11]-[12] for some related discussions. Based on different approaches, several robustness results were also proposed in present literature [17, 26]. Since this is not main concerned issue in this paper, the detailed comparison and discussion are omitted here. ■

IV. Simulation Results

In this section, simulation results are presented. For simulations, the initial condition is set as $(x, y, z_1, z_2, z_3) = (0, 0, 0, -1, 1)$ that is the same as [4]. Other parameters are chosen as follows:

$$M = 1.5, T = 6, k_u = 3, k_v = 4, \varepsilon = 0.05. \quad (18)$$

The reference signals are given as (5)-(8). Simulation results are presented in Figures 1-3 where Figure 1 shows that a fast convergence result is achieved.

V. Conclusions

In this paper, we have proposed a new solution to the stabilization of the rolling sphere. A key idea behind our controller design is to invoke the framework proposed in [12]. The proposed controllers have a simple form and achieve a fast convergence result. Our future work will be directed at extending the proposed result to more general nonholonomic systems.

Appendix: A Proof of Theorem 1

In view of (12)-(14), the following inequality holds:

$$\dot{V} = -\{k_u x_e^2 + k_v[y_e + x(z_1 + xz_2 + yz_3)]^2\} \le 0. \quad \text{(A1)}$$

Additionally, V is positive and proper. From the standard Lyapunov argument, it is easy to see that the origin is Lyapunov stable and all solutions are uniformly globally bounded [9]. To guarantee uniform global asymptotic stability, it remains to check attractivity. Since all tracking signals are continuous and periodic with the same period, the closed-loop system is also continuous and periodic. Thus, the well-known LaSalle invariance principle can be used to check attractivity. Indeed, let $\dot{V} \equiv 0$. From (A1), it can be seen that

$$x_e \equiv 0 \text{ and } y_e + x(z_1 + xz_2 + yz_3) \equiv 0. \quad \text{(A2)}$$

This results in

$$\tilde{v} \equiv 0 \text{ and } \tilde{u} = \dot{x}_e \equiv 0, \quad \text{(A3)}$$

by (13)-(14). This implies that $\tilde{x}$ is a constant function in view of (11). Again by (13), we also have

$$v_r(z_1 + xz_2 + yz_3) \equiv 0. \quad \text{(A4)}$$

By the choices of v_r and u_r, there exist a $t_v \ge 0$ and $t_u \ge 0$ so that $v_r(t_v) \ne 0$ and $u_r(t_u) \ne 0$. Since v_r and u_r are both continuous, there are also two open intervals $t_v \in I_v$ and $t_u \in I_u$ satisfying

$$v_r(t) \ne 0, \forall t \in I_v, \text{ and } u_r(t) \ne 0, \forall t \in I_u. \quad \text{(A5)}$$

According to (4), this implies

$$x_r(t) = 0, \forall t \in I_v. \quad \text{(A6)}$$

Since $x_r(t) = x_e(t) = 0$ and $v_r(t) \ne 0, \forall t \in I_v$, we have $x(t) = x_e(t) + x_r(t) = 0$, $\forall t \in I_v$, and y_r is not a constant function on I_v. In view of (A2), we also have that $y = y_r + y_e = y_r$ is not a constant function on I_v. Since $\tilde{x}$ is a constant function, it can be derived from (A4) that

$$z_1 = z_3 \equiv 0. \qquad \text{(A7)}$$

Since $\dot{x}_r(t) = u_r(t) \neq 0, \forall t \in I_u$, there exists a non-empty open interval $I_x \subseteq I_u$ such that x_r is not a constant function and takes no zero value on I_x. Again by (4), it can be seen that $\dot{y}_r(t) = v_r(t) = 0, \forall t \in I_x$. Notice that, y_e is a constant function and thus, $y = y_r + y_e$ is also a constant function on I_x. From (A2), it can be seen that for each $t \in I_x$, $x(t) = x_r(t) + x_e(t) = x_r(t)$ is a root of a fixed second order polynomial. Since x_r is continuous and is not a constant function on I_x, this implies that the coefficients of the polynomial are all equal to zero. Particularly,

$$y_e(t) = z_1(t) + y(t)z_3(t) = z_2(t) = 0, \forall t \in I_x.$$

Again using the fact that $\tilde{x}$ is a constant function, it can be seen that

$$y_e = z_2 \equiv 0. \qquad \text{(A8)}$$

From (A2), (A7) and (A8), we conclude that $\tilde{x} \equiv 0$ whenever $\dot{V} \equiv 0$. Thus, the origin is uniformly globally asymptotically stable by LaSalle invariance principle or Krasovskii-LaSalle Theorem [9].

Now, let us show that the origin is locally exponentially stable. Consider the linearized system of the closed-loop system. It is not difficult to see that the linearized system takes the same form as the original system with the variables x and y replaced by x_r and y_r, respectively. Thus, all previous arguments can be repeated with a minor modification to show that the origin of the linearized system is uniformly globally asymptotically stable. A detailed proof is omitted to save space. From this, the origin is locally exponentially stable [9]. This completes the proof of the theorem. ■

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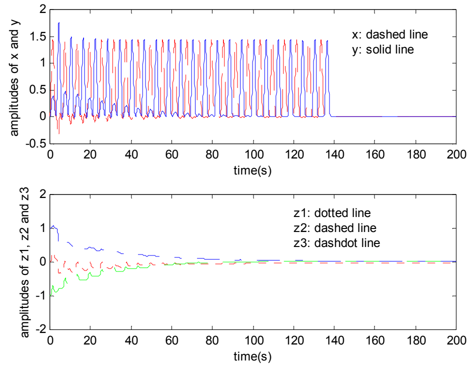

Figure 1: Time history of the state vector.

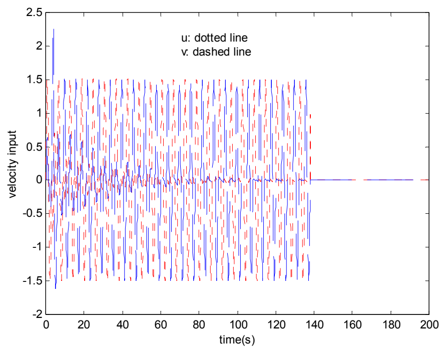

Figure 2: Time history of the velocities.

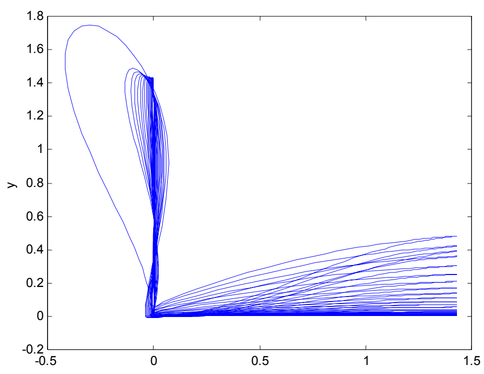

Figure 3: Displacement variations of the rolling sphere.