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Optimization of multi-product nodes in supply chains

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Abstract— **In this paper, supply chain nodes are considered, with the aim of optimizing the production of parts belonging to different classes (the case of two classes is here taken into account). The production node is modelled by a discrete-event model, being the system state, consisting of the continuous level of input and output inventories, affected by the asynchronous processes relevant to part arrivals and product departures. Moreover, also the resource capacity of the node is represented by a continuous variable (to be shared among the two classes). In the paper, two optimization problems are proposed: the former is only stated due to its intrinsic complexity, whereas the latter (a simplified version of the former) is stated and solved. In the latter problem, an optimization is performed each time an event occurs, in order to determine which the next event is, when the next event will occur, how many raw parts/products will arrive/depart, and which portions of production capacity are assigned to the two classes.**

I. INTRODUCTION

In the last years, a growing attention of different groups of researchers and practitioners has been attracted by issues concerning distributed production systems. In particular, the "supply-chain" model has been widely considered [1], [2]. A supply-chain model is mainly characterized by the presence of several production centers (usually distributed over the territory) which interact with raw material and part suppliers and with logistic service providers, in an integrated environment in which co-ordination aspects as well as competitive issues may take place. Different decisional agents coexist within such a model, each one associated with a particular production (or service) unit and characterized by its own set of information and objectives.

In this work, an innovative model for a single node of a supply chain is proposed together with an optimization approach representing the decisional part of such a node. More specifically, a node relevant to a production plant will be taken into account. Owing to the intrinsic complexity of the whole system, the single production plant has to be analytically represented at a quite aggregate level of detail. A variety of models representing production plants can be found in the literature, and they can be classified into three main modelling categories. In the first class of models, both resources and materials are represented by continuous variables [3]; in the second case, a fluid model is adopted, in which material flow is still represented by continuous variables, whereas resources are separately considered and represented by discrete variables. Finally, in the third class of models, mainly adopted for scheduling problems [4], both resources and materials are represented by discrete variables.

None of the above defined modelling categories actually fits the characteristics of the production plant here considered. What is needed when dealing with complex production networks is, actually, a very aggregate model for the production process, that can be represented as a set of continuous resources to be shared among different product classes. Moreover, specific attention has to be posed on the interactions among the considered node and the rest of the network (as regards, for instance, the arrivals and departures of materials, the issuing of orders, etc.), thus requiring a discrete dynamics for the arrival and departure processes involving the production node. The resulting model which will be detailed in the present paper turns out to be a hybrid model combining continuous dynamics (representing the production process) with discrete–event ones (modelling the arrival/departure processes).

A preliminary version of the present work is reported in [5] and [6]. In such papers, a simplified model for a production node with only one product class has been proposed, together with an approach for the optimal control of the considered production nodes. Both the modelling methodology and the optimization approach has been here extended to consider a wider class of nodes and optimization problems. Specifically, in the model here proposed, the single production node is supposed to manufacture different classes of products by means of a single production resource. The production process for each product class is quite simple and consists of a single operation transforming raw materials into finite products (with no assembly operations). Moreover, the overall available amount of production capacity in the plant is fixed, and the decision variable is relative to the resource share for the various product classes. In this case, the state variables, whose evolution has to be represented in the single plant model, are continuous-valued and represent the inventory levels of raw materials and finished products. Finally, the processes characterizing the flow of parts into the node and the departure of finite products from the plant are sequences of instantaneous and asynchronous events.

In regard to the decision process of the production node, the performance objectives are related to production costs, purchase costs, inventory costs, and costs relevant to the timeliness of the satisfaction of production demand. A suitable optimization problem relevant to the minimization of such costs, with respect to the parameters of the arrival and departure processes and to the production effort dedicated to the different product classes, is stated in the paper. Also, a "one-step" solution approach to the overall optimization problem is proposed.

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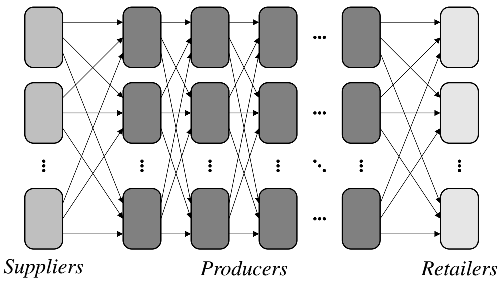

Fig. 1. Schematic representation of a supply chain

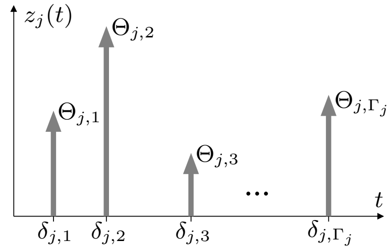

Fig. 3. The arrival process

II. THE SINGLE NODE MODEL

A supply chain is generally defined as a complex production network, characterized by a multiplicity of sites (belonging to one of the three following classes: suppliers, producers, and retailers), which can be modelled as different nodes of the network (Fig. 1). At the operational level, a discrete-event model of the single node is usually adopted. In particular, with reference to producers, relevant events are those concerning the arrival of a raw part to manufacture and those concerning the departure of a product towards other nodes of the network. The state of the node can be represented by means of a set of inventory levels: one for each class of raw parts and one for each class of products. In this paper, a single production node is considered, which is able to manufacture parts belonging to two classes (it is worth noting that the choice of considering two classes is only driven by the need of clarity in writing problems and equations; however, the same approach can be also adopted in the case of J classes). Moreover, the proposed model may be also applied to suppliers and to retailers, as they can be considered as specific instances of producers.

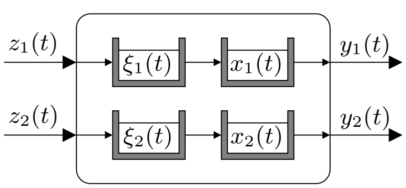

Fig. 2. Model of the single production node

Consider the model of the single production node in Fig. 2. Raw parts arrive from either suppliers or upstream production nodes. One raw part entering the node is processed by a single operation in order to be transformed into one product of the same class (that is, no assembly operation is present in the considered model). Once manufactured, products leave the node towards either downstream production nodes or retailers. In this connection, let $z_j(t)$, $j = 1, 2$, represent the *arrival process* of raw parts, for the j-th class and $y_j(t)$, $j = 1, 2$, represent the *departure process* of products, for the j-th class. Arrival and departure processes are supposed to be independent, and each class of raw parts/products is characterized by its own flows.

The arrival process $z_j(t)$, $j = 1, 2$, is modelled as a finite (and discrete) sequence of arrivals (Fig. 3). An arrival concludes the transportation of a finite amount of raw parts from outside to the production node. In the arrival process of raw parts belonging to class j, Γ_j, $j = 1, 2$, is the number of arrivals within the considered time horizon; $\delta_{j,i}$, $j = 1, 2$, $i = 1, \ldots, \Gamma_j$, is the time instant at which the i-th arrival takes place; $\Theta_{j,i} > 0$, $j = 1, 2$, $i = 1, \ldots, \Gamma_j$, is the amount of raw parts entering the node at time instant $\delta_{j,i}$.

In an analogous way, the departure process $y_j(t)$, $j = 1, 2$, is modelled as a finite and discrete sequence of departures (Fig. 4). A departure has to be intended as the starting time of a transportation of a finite amount of products from the production node to any other node in the network. Such a process is characterized by the following quantities: N_j, $j = 1, 2$, is the number of departures (that is, products requests), for the j-th class of products, within the considered time horizon; $t_{j,i}$, $j = 1, 2$, $i = 1, \ldots, N_j$, is the time instant at which the i-th departure of products of class j occurs, $Q_{j,i} > 0$, $j = 1, 2$, $i = 1, \ldots, N_j$, is the amount of products of class j leaving the system at time instant $t_{j,i}$. Note that arrivals and departures are here modelled as asynchronous processes, in which the time interval between one event and the subsequent one is not constant.

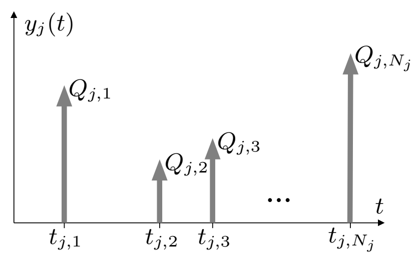

Fig. 4. The departure process

When two classes of products are considered, the model of the production node is mainly characterized by the presence of four inventories: two relevant to the considered classes of raw parts, and two relevant to the classes of products. In this connection, let

- $\xi_j(t)$, $j = 1, 2$, model the inventory level of raw parts of class j (*input inventory*);
- $x_j(t)$, $j = 1, 2$, model the inventory level of products of class j (*output inventory*).

Each class has its own input and output inventories. Raw parts arriving from the outside are initially inserted into the relevant input inventory; then, after the manufacture, the resulting products are inserted into the relevant output inventory (from which they leave the node). Inventories are independent for each class; the only dependence is represented by the whole *production capacity* of the node, that must be appropriately shared. In this connection, let

- K be the overall *work-capacity* of the production node;
- $k_j(t)$, $j = 1, 2$, be the portion of K which is assigned to the production of the j-th class of products at time instant t;
- q be the number of products that the node can manufacture in a time unit (*production rate*).

Of course, it must be:

$$0 \leq k_j(t) \leq K \qquad j = 1, 2 \qquad 0 \leq t \leq t_{j,N_j} \tag{1}$$

$$0 \leq k_1(t) + k_2(t) \leq K \qquad 0 \leq t \leq \min\{t_{1,N_1}, t_{2,N_2}\} \tag{2}$$

The external *demand* of products of class j consists of M_j orders, and each order is characterized by the due-date and the amount of required products. In this connection, let $t^\star_{j,i}$, $j = 1, 2$, $i = 1, \ldots, M_j$, be the due-date of the i-th order of products of class j, and let $Q^\star_{j,i}$, $j = 1, 2$, $i = 1, \ldots, M_j$, be the amount of products required at $t^\star_{j,i}$.

The system state variables are $\xi_j(t)$ and $x_j(t)$, $j = 1, 2$, whereas the decision variables are Γ_j and N_j, $j = 1, 2$, $\delta_{j,i}$ and $\Theta_{j,i}$, $j = 1, 2$, $i = 1, \ldots, \Gamma_j$, $k_j(t)$, $j = 1, 2$, $t_{j,i}$ and $Q_{j,i}$, $j = 1, 2$, $i = 1, \ldots, N_j$. On the basis of the above variables, it is possible to define the state equations for the proposed single production node of a supply chain. In particular, the state equations of the two input inventories for raw parts are:

$$\xi_1(\delta_{1,i+1}) = \xi_1(\delta_{1,i}) - q \int_{\delta_{1,i}}^{\delta_{1,i+1}} k_1(t)\,\mathrm{d}t + \Theta_{1,i+1} \qquad i = 0, \ldots, \Gamma_1 - 1 \tag{3}$$

$$\xi_2(\delta_{2,i+1}) = \xi_2(\delta_{2,i}) - q \int_{\delta_{2,i}}^{\delta_{2,i+1}} k_2(t)\,\mathrm{d}t + \Theta_{2,i+1} \qquad i = 0, \ldots, \Gamma_2 - 1 \tag{4}$$

whereas the state equations of the two output inventories for products are:

$$x_1(t_{1,i+1}) = x_1(t_{1,i}) + q \int_{t_{1,i}}^{t_{1,i+1}} k_1(t)\,\mathrm{d}t - Q_{1,i+1} \qquad i = 0, \ldots, N_1 - 1 \tag{5}$$

$$x_2(t_{2,i+1}) = x_2(t_{1,i}) + q \int_{t_{2,i}}^{t_{2,i+1}} k_2(t)\,\mathrm{d}t - Q_{2,i+1} \qquad i = 0, \ldots, N_2 - 1 \tag{6}$$

being $\xi_j(0)$ and $x_j(0)$, $j = 1, 2$, the initial inventory levels of the j-th class, and $\delta_{j,0} = 0$ and $t_{j,0} = 0$, $j = 1, 2$, the initial time instants (note that, $\delta_{j,0}$ and $t_{j,0}$ do not correspond to an arrival of raw parts or a departure of products and are not decision variables). On the whole, the proposed model is a hybrid model, as it combines some continuous processes, such as the production process, together with discrete-event dynamics, such as the arrivals of raw parts and the departures of products.

Furthermore, in the proposed model, it is assumed that each order is associated with one and only one departure of products. Then, $N_j = M_j$, $j = 1, 2$. Moreover, due-dates are assigned to the departures sequentially, that is, the earliest due-date is assigned to the lot of products which is completed first (and then departs first), the second earliest due-date is assigned to the lot of products which is completed second (and then departs second), and so on. In the literature, this situation is referred to as the case of *generalized due-dates* [7], [8]. Of course, the model is appropriate only when the serviced jobs (of a certain class) have to be assigned to the various due-dates (of that class), i.e., to the various waiting customers, according to a strict due-date ordering, for some technical or commercial/legal reasons.

III. THE OPTIMIZATION PROBLEM

The optimization of the dynamic behaviour of the production site, subject to the external demand, needs the definition of a suitable cost function. An overall optimization problem can be defined taking into account the cost due to the acquisition of raw parts from suppliers/upstream production nodes, inventory costs, production costs, and the cost related to the non-fulfillment of external demand requisites (in terms of tardiness and difference between the actual amount of delivered products and the required one).

The cost due to the acquisition of raw parts from suppliers/upstream production nodes can be stated as:

$$C^{\mathrm{A}} = \sum_{i=1}^{\Gamma_1} \left(c_1^{\mathrm{f}} + c_1^{\mathrm{v}} \cdot \Theta_{1,i}\right) + \sum_{i=1}^{\Gamma_2} \left(c_2^{\mathrm{f}} + c_2^{\mathrm{v}} \cdot \Theta_{2,i}\right) \tag{7}$$

where c_j^{f} and c_j^{v}, $j = 1, 2$, are the fixed and variable unitary order costs, respectively.

The cost due to the inventory occupancy is:

$$C^{\mathrm{I}} = H^{\mathrm{in}} \left(\int_0^{\delta_{1,\Gamma_1}} \xi_1(t)\,\mathrm{d}t + \int_0^{\delta_{2,\Gamma_2}} \xi_2(t)\,\mathrm{d}t\right) + H^{\mathrm{out}} \left(\int_0^{t_{1,M_1}} x_1(t)\,\mathrm{d}t + \int_0^{t_{2,M_2}} x_2(t)\,\mathrm{d}t\right) \tag{8}$$

being H^{in} and H^{out} the unitary inventory costs for raw parts and products, respectively.

The production cost can be stated as:

$$C^{\mathrm{P}} = \gamma \left(\int_0^{t_{1,M_1}} k_1(t)\,\mathrm{d}t + \int_0^{t_{2,M_2}} k_2(t)\,\mathrm{d}t\right) \tag{9}$$

where γ is a suitable coefficient weighting the production cost.

Finally, the cost term relevant to the deviations from the due-dates and from the required products quantities is:

$$C^{\mathrm{D}} = \alpha \left(\sum_{i=1}^{M_1} (t_{1,i} - t^\star_{1,i})^2 + \sum_{i=1}^{M_2} (t_{2,i} - t^\star_{2,i})^2\right) + \beta \left(\sum_{i=1}^{M_1} (Q_{1,i} - Q^\star_{1,i})^2 + \sum_{i=1}^{M_2} (Q_{2,i} - Q^\star_{2,i})^2\right) \tag{10}$$

being α and β suitable weighting coefficients.

The overall optimization problem can be stated as follows.

Problem 1: Given the initial conditions $\delta_{j,0} = 0$, $t_{j,0} = 0$, $\xi_j(0) \geq 0$, and $x_j(0) \geq 0$, $j = 1, 2$, find

$$\min_{\substack{\Gamma_j\,,\, j=1,2 \\ \delta_{j,i}\,,\,\Theta_{j,i}\,,\, j=1,2\,,\, i=1,\dots,\Gamma_j \\ t_{j,i}\,,\, Q_{j,i}\,,\, j=1,2\,,\, i=1,\dots,M_j \\ k_j(t)\,,\, j=1,2\,,\, 0\leq t\leq t_{j,M_j}}} \quad C_1 = C^{\mathrm{A}} + C^{\mathrm{I}} + C^{\mathrm{P}} + C^{\mathrm{D}} \qquad (11)$$

subject to (1), (2), (3), (4), (5), (6), and

$$\Gamma_j \geq 0 \qquad j = 1, 2 \qquad (12)$$

$$\delta_{j,i+1} \geq \delta_{j,i} \qquad j = 1, 2\,,\, i = 0, \dots, \Gamma_j - 1 \qquad (13)$$

$$t_{j,i+1} \geq t_{j,i} \qquad j = 1, 2\,,\, i = 0, \dots, M_j - 1 \qquad (14)$$

$$\xi_j(t) \geq 0 \qquad j = 1, 2\,,\, 0 < t \leq \delta_{j,\Gamma_j} \qquad (15)$$

$$x_j(t) \geq 0 \qquad j = 1, 2\,,\, 0 < t \leq t_{j,M_j} \qquad (16)$$

$$\Theta_{j,i} > 0 \qquad j = 1, 2\,,\, i = 1, \dots, \Gamma_j \qquad (17)$$

$$Q_{j,i} > 0 \qquad j = 1, 2\,,\, i = 1, \dots, M_j \qquad (18)$$

Problem 1 is a functional optimization problem with nonlinear cost function and nonlinear constraints, thus, it is a quite complex problem. Moreover, it is a highly combinatorial problem, as the decision variables $\delta_{j,i}$, $j = 1, 2$, $i = 1, \dots, \Gamma_j$, and $t_{j,i}$, $j = 1, 2$, $i = 1, \dots, M_j$, are not a-priori ordered over the timeline. For these reasons, Problem 1 is not solved here, and a simplified optimization problem will be introduced in the following.

IV. THE SIMPLIFIED "ONE-STEP" OPTIMIZATION PROBLEM

Consider the timeline in Figure 5, and assume that an event, which introduces a perturbation in one or more system state variables, has occurred in τ_h. The possible events are: A_1 (arrival of raw parts of class 1), A_2 (arrival of raw parts of class 2), D_1 (departure of products of class 1), and D_2 (departure of products of class 2). In this connection, let $E = \{A_1, A_2, D_1, D_2\}$ be the *event set*. The task of optimizing the performances of the system from τ_h onward, by considering all the future events, is still a functional, nonlinear, and combinatorial optimization problem. Then, in the proposed simplified problem, the decision to be taken at τ_h is only relevant to the *next event* which will occur. In particular, at τ_h, the following decisions have to be taken:

- which is the next event $e_{h+1} \in E$?
- which is the time instant $\tau_{h+1} > \tau_h$ at which e_{h+1} will occur?
- which is the amount P_{h+1} of raw parts or products which arrive or depart at τ_{h+1}?
- which are the portions of production capacity, $k_1(t)$ and $k_2(t)$, to assign to the production of products of class 1 and class 2, respectively, in the time interval $(\tau_h, \tau_{h+1}]$?

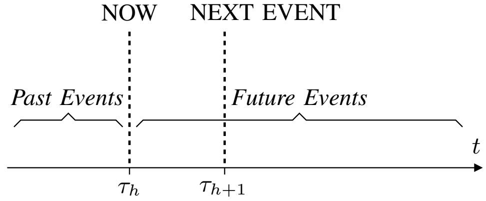

Fig. 5. Events over the timeline

TABLE I

DECISION VARIABLES

e_{h+1}	A_1	A_2	D_1	D_2
τ_{h+1}	$\delta_{1,n_{A_1}+1}$	$\delta_{2,n_{A_2}+1}$	$t_{1,n_{D_1}+1}$	$t_{2,n_{D_2}+1}$
P_{h+1}	$\Theta_{1,n_{A_1}+1}$	$\Theta_{2,n_{A_2}+1}$	$Q_{1,n_{D_1}+1}$	$Q_{2,n_{D_2}+1}$
$k_{1,h}$	$k_{1,h}$	$k_{1,h}$	$k_{1,h}$	$k_{1,h}$
$k_{2,h}$	$k_{2,h}$	$k_{2,h}$	$k_{2,h}$	$k_{2,h}$

Assume that, at time instant τ_h, h events have already occurred. In particular, assume that n_{A_1} events of type A_1, n_{A_2} events of type A_2, n_{D_1} events of type D_1, and n_{D_2} events of type D_2 occurred (obviously, it turns out $n_{A_1} + n_{A_2} + n_{D_1} + n_{D_2} = h$). Moreover, it is assumed that $k_j(t)$, $j = 1, 2$, is constant in the time interval $(\tau_h, \tau_{h+1}]$; in this connection, let $k_{j,h}$ denote the value of $k_j(t)$ within $(\tau_h, \tau_{h+1}]$. Then, on the basis of the choice of the decision variable e_{h+1}, the other decision variables are those in Table I.

Even if the decision to be taken is only relevant to the next event, the simplified problem should also take into account all the future events (in order to avoid "blind" solutions). Then, two kinds of parameters are introduced in the single node model: $\xi^{\star}_{j,h}$, $j = 1, 2$, represents the *reference value of inventory level* of raw parts of class j in the time interval $(\tau_h, \tau_{h+1}]$, whereas $k^{\star}_{j,h}$, $j = 1, 2$, represents the *reference value of portion of production capacity* necessary to manufacture products of class j within $(\tau_h, \tau_{h+1}]$. The meaning of the reference value of inventory level is that of maintaining a certain level of parts within the input inventories in order to be able to satisfy the external demand in the future; the meaning of the reference value of portion of production capacity is that of maintaining a certain rate of production, always with the objective of being able to satisfy the external demand. Then, a new cost term is stated as follows:

$$C^{\mathrm{R}}_h = \sum_{j=1}^{2} \left[\lambda \left(\xi_j(\tau_{h+1}) - \xi^{\star}_{j,h} \right)^2 + \mu \left(k_{j,h} - k^{\star}_{j,h} \right)^2 \right] \qquad (19)$$

being λ and μ suitable weighting coefficients. Note that, the cost term (19) is function of the considered time interval, namely $(\tau_h, \tau_{h+1}]$. In the same way, let C^{A}_h, C^{I}_h, C^{P}_h, and C^{D}_h be the cost due the acquisition of raw parts, the inventory cost, the production cost, and the cost due to the non-fulfillment of external demand, respectively, computed in the time interval $(\tau_h, \tau_{h+1}]$ only (and not over the whole timeline).

The simplified optimization problem to be solved each time an event occurs, is stated as follows.

Problem 2 ("one-step" optimization problem): Given the system state at τ_h, namely,

$$\underline{s}(\tau_h) = \begin{bmatrix} \underline{\sigma}(\tau_h) & \underline{n}(\tau_h) \end{bmatrix}^T \tag{20}$$

being $\underline{\sigma}(\tau_h) = \begin{bmatrix} \xi_1(\tau_h) & \xi_2(\tau_h) & x_1(\tau_h) & x_2(\tau_h) \end{bmatrix}^T$ and $\underline{n}(\tau_h) = \begin{bmatrix} n_{A_1} & n_{A_2} & n_{D_1} & n_{D_2} \end{bmatrix}^T$, find

$$\min_{\substack{e_{h+1}, \tau_{h+1}, P_{h+1} \\ k_{1,h}, k_{2,h}}} C_2 = C_h^{\mathrm{A}} + C_h^{\mathrm{I}} + C_h^{\mathrm{P}} + C_h^{\mathrm{D}} + C_h^{\mathrm{R}} \tag{21}$$

subject to system state equations and

$$e_{h+1} \in E \tag{22}$$

$$\tau_{h+1} \geq \tau_h \tag{23}$$

$$P_{h+1} > 0 \tag{24}$$

$$k_{j,h} \geq 0 \qquad j = 1, 2 \tag{25}$$

$$k_{1,h} + k_{2,h} \leq K \tag{26}$$

$$\xi_j(\tau_{h+1}) \geq 0 \qquad j = 1, 2 \tag{27}$$

$$x_j(\tau_{h+1}) \geq 0 \qquad j = 1, 2 \tag{28}$$

The preliminary determination of the reference values is a very important task, as they are the only link between the "one-step" optimization problem and all the future activities of the system. In this paper, the reference value of portion of production capacity is computed by taking into account the overall external demand still to be satisfied, and considering such a demand as uniformly distributed over the time, that is

$$k_{j,h}^{\star} = \frac{\displaystyle\sum_{i=n_{D_j}+1}^{M_j} Q_{j,i}^{\star}}{t_{j,M_j}^{\star} - \tau_h} \qquad j = 1, 2 \tag{29}$$

Once $k_{j,h}^{\star}$, $j = 1, 2$, has been determined, the reference value of inventory level is computed by considering the average level of a virtual inventory obtained by applying the basic *EOQ* (Economic Order Quantity) model with an external demand equal to the reference value $k_{j,h}^{\star}$, that is

$$\xi_{j,h}^{\star} = \frac{\sqrt{\dfrac{2c_j^{\mathrm{f}} q k_{j,h}^{\star}}{H^{\mathrm{in}}}}}{2} \qquad j = 1, 2 \tag{30}$$

Finally, observe that the two reference values have to be updated each time Problem 2 has to be solved.

Problem 2 is solved by considering the following cases: 1) the next event is A_1; 2) the next event is A_2; 3) the next event is D_1; 4) the next event is D_2.

For each case, the *"conditioned" cost-to-go* $J_h^v\big(\underline{s}(\tau_h) \mid e_{h+1} = v\big)$, $v = A_1, A_2, D_1, D_2$ is computed, which is defined as the cost in the interval $(\tau_h, \tau_{h+1}]$ when the next event at τ_{h+1} is v. In the following, the determination of such a cost is detailed for each of the above cases.

Case 1 – The next event is A_1

In this case, the cost terms become

$$C_h^{\mathrm{A}} = c_1^{\mathrm{f}} + c_1^{\mathrm{v}} \cdot \Theta_{1,n_{A_1}+1} \tag{31}$$

$$\begin{aligned} C_h^{\mathrm{I}} = H^{\mathrm{in}} \Big[& \big(\xi_1(\tau_h) + \xi_2(\tau_h)\big)\big(\delta_{1,n_{A_1}+1} - \tau_h\big) + \\ & - \frac{q}{2}\big(k_{1,h} + k_{2,h}\big)\big(\delta_{1,n_{A_1}+1} - \tau_h\big)^2 \Big] + \\ & + H^{\mathrm{out}} \Big[\big(x_1(\tau_h) + x_2(\tau_h)\big)\big(\delta_{1,n_{A_1}+1} - \tau_h\big) + \\ & + \frac{q}{2}\big(k_{1,h} + k_{2,h}\big)\big(\delta_{1,n_{A_1}+1} - \tau_h\big)^2 \Big] \end{aligned} \tag{32}$$

$$C_h^{\mathrm{P}} = \gamma(k_{1,h} + k_{2,h})(\delta_{1,n_{A_1}+1} - \tau_h) \tag{33}$$

$$C_h^{\mathrm{D}} = 0 \tag{34}$$

and the state equations are

$$\xi_1(\delta_{1,n_{A_1}+1}) = \xi_1(\tau_h) - q k_{1,h}(\delta_{1,n_{A_1}+1} - \tau_h) + \Theta_{1,n_{A_1}+1} \tag{35}$$

$$\xi_2(\delta_{1,n_{A_1}+1}) = \xi_2(\tau_h) - q k_{2,h}(\delta_{1,n_{A_1}+1} - \tau_h) \tag{36}$$

$$x_1(\delta_{1,n_{A_1}+1}) = x_1(\tau_h) + q k_{1,h}(\delta_{1,n_{A_1}+1} - \tau_h) \tag{37}$$

$$x_2(\delta_{1,n_{A_1}+1}) = x_2(\tau_h) + q k_{2,h}(\delta_{1,n_{A_1}+1} - \tau_h) \tag{38}$$

The conditioned cost-to-go, when the next event is A_1, is

$$J_h^{A_1}\big(\underline{s}(\tau_h) \mid e_{h+1} = A_1\big) = C_h^{\mathrm{A}} + C_h^{\mathrm{I}} + C_h^{\mathrm{P}} + C_h^{\mathrm{D}} + C_h^{\mathrm{R}} \tag{39}$$

being C_h^{A}, C_h^{I}, C_h^{P}, C_h^{D}, and C_h^{R} respectively provided by (31), (32), (33), (34), and (19). The optimal value of such a conditioned cost-to-go, is provided by the following optimization problem:

$$\begin{aligned} & J_h^{A_1\,\circ}\big(\underline{s}(\tau_h) \mid e_{h+1} = A_1\big) = \\ & = \min_{\substack{\delta_{1,n_{A_1}+1} \\ \Theta_{1,n_{A_1}+1} \\ k_{1,h}, k_{2,h}}} \Big\{ J_h^{A_1}\big(\underline{s}(\tau_h) \mid e_{h+1} = A_1\big) \Big\} \end{aligned} \tag{40}$$

subject to (35), (36), (37), and (38), and

$$\delta_{1,n_{A_1}+1} \geq \tau_h \tag{41}$$

$$\Theta_{1,n_{A_1}+1} > 0 \tag{42}$$

$$k_{j,h} \geq 0 \qquad j = 1, 2 \tag{43}$$

$$k_{1,h} + k_{2,h} \leq K \tag{44}$$

$$\xi_j(\delta_{1,n_{A_1}+1}) \geq 0 \qquad j = 1, 2 \tag{45}$$

$$x_j(\delta_{1,n_{A_1}+1}) \geq 0 \qquad j = 1, 2 \tag{46}$$

The optimal value of decision variables is indicated with $\delta_{1,n_{A_1}+1}^{\circ}$, $\Theta_{1,n_{A_1}+1}^{\circ}$, $k_{1,h}^{\circ}$, and $k_{2,h}^{\circ}$, respectively.

Case 2 – The next event is A_2

This case is analogous to case 1. Then, in the same way, it is possible to determine the optimal cost-to-go when the next event is A_2, namely $J_h^{A_2\,\circ}\big(\underline{s}(\tau_h) \mid e_{h+1} = A_2\big)$, and the optimal value of decision variables, namely $\delta_{2,n_{A_2}+1}^{\circ}$, $\Theta_{2,n_{A_2}+1}^{\circ}$, $k_{1,h}^{\circ}$, and $k_{2,h}^{\circ}$.

Case 3 – The next event is D_1

In this case, the cost terms become

$$C_h^{\mathrm{A}} = 0 \tag{47}$$

$$\begin{aligned} C_h^{\mathrm{I}} = H^{\mathrm{in}} \Big[&\big(\xi_1(\tau_h) + \xi_2(\tau_h)\big)\big(t_{1,n_{D_1}+1} - \tau_h\big) + \\ &- \frac{q}{2}\big(k_{1,h} + k_{2,h}\big)\big(t_{1,n_{D_1}+1} - \tau_h\big)^2 \Big] + \\ &+ H^{\mathrm{out}} \Big[\big(x_1(\tau_h) + x_2(\tau_h)\big)\big(t_{1,n_{D_1}+1} - \tau_h\big) + \\ &+ \frac{q}{2}\big(k_{1,h} + k_{2,h}\big)\big(t_{1,n_{D_1}+1} - \tau_h\big)^2 \Big] \end{aligned} \tag{48}$$

$$C_h^{\mathrm{P}} = \gamma(k_{1,h} + k_{2,h})(t_{1,n_{D_1}+1} - \tau_h) \tag{49}$$

$$C_h^{\mathrm{D}} = \alpha(t_{1,n_{D_1}+1} - t^{\star}_{1,n_{D_1}+1})^2 + \beta(Q_{1,n_{D_1}+1} - Q^{\star}_{1,n_{D_1}+1})^2 \tag{50}$$

and the state equations are

$$\xi_1(t_{1,n_{D_1}+1}) = \xi_1(\tau_h) - q k_{1,h}(t_{1,n_{D_1}+1} - \tau_h) \tag{51}$$

$$\xi_2(t_{1,n_{D_1}+1}) = \xi_2(\tau_h) - q k_{2,h}(t_{1,n_{D_1}+1} - \tau_h) \tag{52}$$

$$x_1(t_{1,n_{D_1}+1}) = x_1(\tau_h) + q k_{1,h}(t_{1,n_{D_1}+1} - \tau_h) - Q_{1,n_{D_1}+1} \tag{53}$$

$$x_2(t_{1,n_{D_1}+1}) = x_2(\tau_h) + q k_{2,h}(t_{1,n_{D_1}+1} - \tau_h) \tag{54}$$

The conditioned cost-to-go, when the next event is D_1, is

$$J_h^{D_1}\big(\underline{s}(\tau_h) \mid e_{h+1} = D_1\big) = C_h^{\mathrm{A}} + C_h^{\mathrm{I}} + C_h^{\mathrm{P}} + C_h^{\mathrm{D}} + C_h^{\mathrm{R}} \tag{55}$$

being C_h^{A}, C_h^{I}, C_h^{P}, C_h^{D}, and C_h^{R} respectively provided by (47), (48), (49), (50), and (19). The optimal value of such a conditioned cost-to-go, is provided by the following optimization problem:

$$\begin{aligned} &J_h^{D_1\,\circ}\big(\underline{s}(\tau_h) \mid e_{h+1} = D_1\big) = \\ &\min_{\substack{t_{1,n_{D_1}+1} \\ Q_{1,n_{D_1}+1} \\ k_{1,h}\,,\,k_{2,h}}} \Big\{ J_h^{D_1}\big(\underline{s}(\tau_h) \mid e_{h+1} = D_1\big) \Big\} \end{aligned} \tag{56}$$

subject to (51), (52), (53), and (54), and

$$t_{1,n_{D_1}+1} \geq \tau_h \tag{57}$$

$$Q_{1,n_{D_1}+1} > 0 \tag{58}$$

$$k_{j,h} \geq 0 \qquad j = 1, 2 \tag{59}$$

$$k_{1,h} + k_{2,h} \leq K \tag{60}$$

$$\xi_j(t_{1,n_{D_1}+1}) \geq 0 \qquad j = 1, 2 \tag{61}$$

$$x_j(t_{1,n_{D_1}+1}) \geq 0 \qquad j = 1, 2 \tag{62}$$

The optimal value of decision variables is indicated with $t^{\circ}_{1,n_{D_1}+1}$, $Q^{\circ}_{1,n_{D_1}+1}$, $k^{\circ}_{1,h}$, and $k^{\circ}_{2,h}$, respectively.

Case 4 – The next event is D_2

This case is analogous to case 3. Then, in the same way, it is possible to determine the optimal cost-to-go when the next event is D_2, namely $J_h^{D_2\,\circ}\big(\underline{s}(\tau_h) \mid e_{h+1} = D_2\big)$, and the optimal value of decision variables, namely $t^{\circ}_{2,n_{D_2}+1}$, $Q^{\circ}_{2,n_{D_2}+1}$, $k^{\circ}_{1,h}$, and $k^{\circ}_{2,h}$.

Coming back to Problem 2, the next event is provided by

$$e_{h+1} = \mathrm{argmin} \Big\{ J_h^{v\,\circ}\big(\underline{s}(\tau_h) \mid e_{h+1} = v\big), v \in E \Big\} \tag{63}$$

and the values τ_{h+1}, P_{h+1}, $k_{1,h}$, and $k_{2,h}$, are those computed in the relevant conditioned cost-to-go. In particular

$$\tau_{h+1} = \begin{cases} \delta^{\circ}_{1,n_{A_1}+1} & \text{if } e_{h+1} = A_1 \\ \delta^{\circ}_{2,n_{A_2}+1} & \text{if } e_{h+1} = A_2 \\ t^{\circ}_{1,n_{D_1}+1} & \text{if } e_{h+1} = D_1 \\ t^{\circ}_{2,n_{D_2}+1} & \text{if } e_{h+1} = D_2 \end{cases} \tag{64}$$

$$P_{h+1} = \begin{cases} \Theta^{\circ}_{1,n_{A_1}+1} & \text{if } e_{h+1} = A_1 \\ \Theta^{\circ}_{2,n_{A_2}+1} & \text{if } e_{h+1} = A_2 \\ Q^{\circ}_{1,n_{D_1}+1} & \text{if } e_{h+1} = D_1 \\ Q^{\circ}_{2,n_{D_2}+1} & \text{if } e_{h+1} = D_2 \end{cases} \tag{65}$$

V. CONCLUSIONS

In this paper, two optimization problems have been defined in order to optimize, at the operational decision level, the performance of a single production node within a supply chain. The former problem is only stated, as its exact solution can not be determined in a reasonable time; then, a simplified version of the same problem has been proposed and solved. The optimization procedure is performed each time an event occurs, in order to determine which the next event is, when the next event will occur, how many raw parts/products will arrive/depart, and which the portions of production capacity assigned to the two classes are. In the simplified problem, a further cost term has been added, in order to take into account all the future events, thus avoiding "blind" solutions.

It is important to note that the proposed approach is compatible with an *"infinite horizon"* model, that is a model where the external demand of products is not defined at the beginning. The fact is, the reference value of portion of production capacity and the reference value of inventory level are computed every time an event occurs, and thus they can be computed on the basis of the actual set of external orders.

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